Acta Mathematica Sinica, English Series
Published online: Dec. 22, 2005
DOI: 10.1007/s10114-005-0673-7
Http://www.ActaMath.com

Acta Mathematica Sinica, English Series

Stable Stationary Harmonic Maps to Spheres

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Dedicated to Professor WeiYue Ding on the occasion of his 60th birthday

Abstract For $k \geq 3$, we establish new estimate on Hausdorff dimensions of the singular set of stable-stationary harmonic maps to the sphere $\mathbf{S}^k$. We show that the singular set of stable-stationary harmonic maps from B^5 to $\mathbf{S}^3$ is the union of finitely many isolated singular points and finitely many Hölder continuous curves. We also discuss the minimization problem among continuous maps from B^n to $\mathbf{S}^2$.

Keywords stable stationary harmonic map, Hausdorff dimension, rectifiablity
MR(2000) Subject Classification 58E20, 35J50

1 Introduction

Let $\Omega \subset \mathbb{R}^n$ be a bounded, smooth domain, $\mathbf{S}^k = \{x \in \mathbb{R}^{k+1} : |x| = 1\}$ be a unit sphere in $\mathbb{R}^{k+1}$. The Sobolev space $H^1(\Omega, \mathbf{S}^k) = \{v \in H^1(\Omega, \mathbb{R}^{k+1}) : v(x) \in \mathbf{S}^k \text{ for a.e. } x \in \Omega\}$. Recall that $u \in H^1(\Omega, \mathbf{S}^k)$ is a harmonic map, if

$$\Delta u + |\nabla u|^2 u = 0, \quad \text{in } \mathscr{D}'(\Omega). \tag{1.1}$$

It is well known that Schoen–Uhlenbeck [1, 2] have proved that if $u \in H^1(\Omega, \mathbf{S}^k)$ is an energy minimizing harmonic map, then the Hausdorff dimension of $\mathscr{S}(u)$, the singular set of u, is at most $n - d(k) - 1$, with

$$d(k) = \begin{cases} 2, & k = 2, \\ 3, & k = 3, \\ \min\left\{\left[\dfrac{k}{2}\right] + 1, 6\right\}, & k \geq 4, \end{cases}$$

where [ ] is the integer part.

This theorem has been extended to stable stationary harmonic maps $u \in H^1(\Omega, \mathbf{S}^k)$, $k \geq 3$, by Hong–Wang [3].

One of the purposes of this note is to show that the theorems in [2] and [3] can be improved for $4 \leq k \leq 7$. In order to describe it, we recall

Definition 1 *A harmonic map $u \in H^1(\Omega, \mathbf{S}^k)$ is a stationary harmonic map, if*

$$\int_\Omega \left\{|\nabla u|^2 \mathrm{div}(X) - 2\sum_{i,j=1}^n \langle \nabla_i u, \nabla_j u\rangle \nabla_i X^j\right\} dx = 0, \ \forall X \in C_0^1(\Omega, \mathbb{R}^n). \tag{1.2}$$

Received August 31, 2005, Accepted September 31, 2005
Both authors are partially supported by NSF

Examples of stationary harmonic maps include energy minimizing harmonic maps and smooth harmonic maps. The crucial property of stationary harmonic maps is the energy monotonicity identity (cf. Price [4] or Schoen [5]) : For $x \in \Omega, 0 < r \leq R < \mathrm{d}(x, \partial\Omega)$,

$$R^{2-n} \int_{B_R(x)} |\nabla u|^2 - r^{2-n} \int_{B_r(x)} |\nabla u|^2 = 2 \int_{B_R(x)\setminus B_r(x)} |y-x|^{2-n} \left| \frac{\partial u}{\partial |y-x|} \right|^2. \tag{1.3}$$

Definition 2 *A harmonic map $u \in H^1(\Omega, \mathbf{S}^k)$ is a stable harmonic map, if*

$$\frac{d^2}{dt^2}|_{t=0} \int_\Omega \left| \nabla \left(\frac{u+t\phi}{|u+t\phi|} \right) \right|^2 dx \geq 0, \tag{1.4}$$

or equivalently

$$\int_\Omega \{ |\nabla \phi|^2 - |\nabla u|^2 \phi^2 \} \, dx \geq 0 \tag{1.5}$$

for any $\phi \in C_0^\infty(\Omega, \mathbb{R}^{k+1})$, with $\langle \phi(x), u(x) \rangle = 0$ for a.e. $x \in \Omega$.

It follows from the proof of [2] Theorem 2.7 that for $k \geq 3$, (1.5) implies

$$\int_\Omega \left\{ |\nabla \eta|^2 - \frac{k-2}{k} |\nabla u|^2 \eta^2 \right\} dx \geq 0, \ \forall \eta \in C_0^\infty(\Omega). \tag{1.6}$$

Now we define

$$\hat{d}(k) = \begin{cases} 3, & k = 3 \\ 4, & k = 4 \\ 5, & 5 \leq k \leq 9 \\ 6, & k \geq 10. \end{cases}$$

By direct comparison, we have that $\hat{d}(k) = d(k)$ for $k = 3$ and $k \geq 8$, $\hat{d}(4) = 4 = d(4) + 1$, $\hat{d}(5) = 5 = d(5) + 2$, $\hat{d}(6) = 5 = d(6) + 1$, $\hat{d}(7) = 5 = d(7) + 1$.

Now we are ready to state the first theorem.

Theorem 1 *For $k \geq 3$, let $u \in H^1(\Omega, \mathbf{S}^k)$ be a stable stationary harmonic map. Then the singular set of u, $\mathscr{S}(u)$, has its Hausdorff dimension at most $n - \hat{d}(k) - 1$.*

From the compactness theorem on stable stationary harmonic maps into $\mathbf{S}^k$ ($k \geq 3$) in [3] and the dimension reduction argument in [1, 2], Theorem 1 follows from the nonexistence of stable stationary tangent maps in $\mathbb{R}^m$ for $m \leq \hat{d}(k)$. To show this fact, we need to establish a sharp Kato's type inequality for smooth harmonic maps from $\mathbf{S}^m$ to $\mathbf{S}^k$ (see Proposition 2.3 below). We remark that such an inequality has been proved for harmonic functions (see Yau [6]), and for harmonic maps from $\mathbf{S}^3$ to $\mathbf{S}^3$ recently by Nakajima [7]. Moreover, It has been proved by [7] that any tangent map at an isolated singular point of a stable stationary harmonic map from B^4 to $\mathbf{S}^3$ is, after isometry transformations, of the form $\frac{x}{|x|}$ (see also Theorem 3.1 below). On the basis of this type of classifications, we are able to generalize the theorems by Hardt–Lin [8] (see also Almgren–Lieb [9]) on minimizing harmonic maps from B^3 to $\mathbf{S}^2$ and prove

Theorem 2 *Assume that $u \in H^1(B^4, \mathbf{S}^3)$ is a stable stationary harmonic map. Then for any $0 < \epsilon < 1$, there exist positive constants K_ϵ, L_ϵ such that* (i)

$$r^{2-n} \int_{B_r(a)} |\nabla u|^2 \leq K_\epsilon, \ \forall B_r(a) \subset B_{1-\epsilon}, \tag{1.7}$$

and (ii) *the number of singularities of u in $B_{1-\epsilon}$ is bounded by L_ϵ.*

We are also able to extend the structure theorem on the singular set of minimizing harmonic maps from B^4 to $\mathbf{S}^2$ by Hardt–Lin [10] and obtain

Theorem 3 *Assume that $u \in H^1(B^5, \mathbf{S}^3)$ is a stable stationary harmonic map. Then for any $\epsilon > 0$, the singular set of u inside $B_{1-\epsilon}$ is the union of finitely many isolated singular points* *and a finite collection of bi-Hölder continuous curves with a finite number of intersections. Moreover, the number of isolated singular points, curves, and the intersection points is bounded by a universal constant $M_\epsilon > 0$.*

The paper is written as follows. In §2, we outline the proof of Theorem 1. In §3, we prove Theorems 2 and 3. Since the stability condition for stable harmonice maps into $\mathbf{S}^2$ is void, it remains unknown whether Theorem 1 remains to be true for $k = 2$. In §4, we consider a class of harmonic maps which are weak limits among continuous maps to $\mathbf{S}^2$, and study their regularity properties.

2 Kato's Inequality and the Proof of Theorem 1

In this section, we prove a sharp version of Kato's type inequality and combine it with both the stablity inequality (2.1) and the Bochner formula (2.2) to show Theorem 1. We start with

Proposition 2.1 *For $m \geq 2$ and $k \geq 3$, let $\phi \in H^1(\mathbf{S}^m, \mathbf{S}^k)$ be a harmonic map such that $\bar{\phi}(x) = \phi(\frac{x}{|x|}) : \mathbb{R}^{m+1} \to \mathbf{S}^k$ is a stable harmonic map. Then we have*

$$\int_{\mathbf{S}^m} \left\{ |\nabla \eta|^2 + \frac{(m-1)^2}{4}\eta^2 - \frac{k-2}{k}|\nabla \phi|^2 \eta^2 \right\} \geq 0, \ \forall \, \eta \in C^\infty(\mathbf{S}^m). \tag{2.1}$$

Proof Note that $\bar{\phi}$ is homogeneous of degree zero. For any $\eta_2 \in C^\infty(\mathbf{S}^m)$, we let $\eta(x) = \eta_1(|x|)\eta_2(\frac{x}{|x|})$, where $\eta_1 \in C_0^\infty(\mathbb{R}_+)$ satisfies

$$\frac{\int_0^\infty (\eta_1'(r))^2 r^m \, dr}{\int_0^\infty \eta_1(r)^2 r^{m-2} \, dr} = \inf \left\{ \frac{\int_0^\infty (\eta'(r))^2 r^m \, dr}{\int_0^\infty \eta(r)^2 r^{m-2} \, dr} : \eta \in C_0^\infty((0, +\infty)) \right\} = \frac{(m-1)^2}{4},$$

and substitute it into (1.6). Then a straightforward calculation implies (2.1).

Now we recall the Bochner formula for smooth harmonic maps (see Eells–Lemaire [11]).

Proposition 2.2 *For $m, k \geq 2$, if $\phi \in C^\infty(\mathbf{S}^m, \mathbf{S}^k)$ is a harmonic map, then*

$$\Delta\left(\frac{1}{2}|\nabla \phi|^2\right) = |\nabla^2 \phi|^2 + (m-1)|\nabla \phi|^2 - \sum_{\alpha,\beta=1}^{m} \{|\nabla_{e_\alpha}\phi|^2 |\nabla_{e_\beta}\phi|^2 - \langle \nabla_{e_\alpha}\phi, \nabla_{e_\beta}\phi \rangle^2\}, \tag{2.2}$$

where $\{e_\alpha\}_{\alpha=1}^m$ is any local orthonormal frame of $\mathbf{S}^m$.

The crucial step to improve $d(k)$ is the following Kato's type inequality for harmonic maps (see Nakajima [7] for $m = k = 3$).

Proposition 2.3 *If $\phi \in C^\infty(\mathbf{S}^m, \mathbf{S}^k)$ is a nonconstant harmonic map, then*

$$|\nabla^2 \phi|^2 \geq \frac{m}{m-1}|\nabla |\nabla \phi||^2, \quad x \in \mathbf{S}^m. \tag{2.3}$$

Moreover, the equality holds at a point $x \in \mathbf{S}^m$ iff $\nabla^2 \phi(x) = 0$.

Proof By choosing normal coordinates at $x_0 \in \mathbf{S}^m$ and $\phi(x_0) \in \mathbf{S}^k$, we have

$$|\nabla^2 \phi|^2(x_0) = \sum_{i=1}^{k} \sum_{\alpha,\beta=1}^{m} (\phi^i_{\alpha\beta}(x_0))^2.$$

On the other hand, since ϕ is a harmonic map, we have

$$\sum_{\alpha=1}^{m} \phi^i_{\alpha\alpha}(x_0) = 0, \ \forall \, 1 \leq i \leq k.$$

For any $1 \leq i \leq k$, let $\{\lambda^i_\alpha\}_{1 \leq \alpha \leq m} \subset \mathbb{R}$ be the eigenvalues of $(\phi^i_{\alpha\beta}(x_0))$ such that $|\lambda^i_1| \leq \cdots \leq |\lambda^i_m|$. Then we have

$$|\nabla^2 \phi|^2(x_0) = \sum_{i=1}^{k} \sum_{\alpha=1}^{m} (\lambda^i_\alpha)^2, \tag{2.4}$$

and

$$\sum_{\alpha=1}^{m} \lambda^i_\alpha = 0, \ \forall \, 1 \leq i \leq k. \tag{2.5}$$

On the other hand, by the Cauchy–Schwarz inequality and (2.5), we have

$$\sum_{\alpha=1}^{m-1}(\lambda_\alpha^i)^2 \geq \frac{1}{m-1}\left(\sum_{\alpha=1}^{m-1}\lambda_\alpha^i\right)^2 = \frac{(\lambda_m^i)^2}{m-1}, \quad 1 \leq i \leq k, \tag{2.6}$$

so that

$$\sum_{\alpha=1}^{m}(\lambda_\alpha^i)^2 \geq \frac{m}{m-1}(\lambda_m^i)^2. \tag{2.7}$$

By the Releigh quotient, we have, for $1 \leq i \leq k$,

$$|\lambda_m^i|^2 = \sup_{\{0\neq v\in\mathbb{R}^m\}} \frac{\sum_{\alpha=1}^m(\sum_{\beta=1}^m \phi_{\alpha\beta}^i(x_0)v_\beta)^2}{|v|^2}.$$

Therefore, we have

$$|\nabla\phi^i|^2(x_0)|\nabla^2\phi^i|^2(x_0) \geq \frac{m}{m-1}\sum_{\alpha=1}^m\left[\sum_{\beta=1}^m \phi_{\alpha\beta}^i(x_0)\phi_\beta^i(x_0)\right]^2, \quad 1 \leq i \leq k. \tag{2.8}$$

Taking sum of (2.8) over i and applying the Cauchy–Schwarz inequality and the Minkowski inequality, we have, at x_0,

$$\begin{aligned}
|\nabla\phi|^2|\nabla^2\phi|^2 &= \left(\sum_{i=1}^k|\nabla\phi^i|^2\right)\left(\sum_{i=1}^k|\nabla^2\phi^i|^2\right) \geq \left(\sum_{i=1}^k|\nabla\phi^i||\nabla^2\phi^i|\right)^2\\
&\geq \frac{m}{m-1}\left\{\sum_{i=1}^k\left[\sum_{\alpha=1}^m\left(\sum_{\beta=1}^m\phi_{\alpha\beta}^i\phi_\beta^i\right)^2\right]^{\frac{1}{2}}\right\}^2\\
&\geq \frac{m}{m-1}\sum_{\alpha=1}^m\left[\sum_{i=1}^k\sum_{\beta=1}^m\phi_{\alpha\beta}^i\phi_\beta^i\right]^2 = \frac{m}{m-1}|\langle\nabla^2\phi,\nabla\phi\rangle|^2.
\end{aligned}$$

Since $|\nabla|\nabla\phi||^2 = \frac{|\langle\nabla^2\phi,\nabla\phi\rangle|^2}{|\nabla\phi|^2}$, this yields (1.7).

Observe that the equality in (1.7) holds at $x_0 \in \mathbf{S}^m$ if and only if both the Cauchy–Schwarz inequality and the Minkowski inequality are equalities. This implies:

(i)

$$\lambda_1^i = \cdots = \lambda_{m-1}^i = -\frac{\lambda_m^i}{m-1}, \ \forall\, 1 \leq i \leq k,$$

(ii)

$$\frac{|\nabla^2\phi^1|^2}{|\nabla\phi^1|} = \cdots = \frac{|\nabla^2\phi^k|}{|\nabla\phi^k|},$$

(iii) $\nabla\phi^i(x_0)$ is an eigenfunction of $\nabla^2\phi^i(x_0)$ with the eigenvalue λ_m^i:

$$\sum_{\beta=1}^m\phi_{\alpha\beta}^i(x_0)\phi_\beta^i(x_0) = \lambda_m^i\phi_\alpha^i(x_0), \ \forall\, 1 \leq i \leq k, \ 1 \leq \alpha \leq m,$$

and (iv) $\sum_{\beta=1}^m \phi_{\alpha\beta}^i(x_0)\phi_\beta^i(x_0)(= \lambda_m^i\phi_\alpha^i(x_0))$ is independent of $1 \leq i \leq k$ and $1 \leq \alpha \leq m$.

In particular, we have that $\nabla\phi^i(x_0) = \mu^i(1,\ldots,1)$ for some $\mu^i \neq 0$ for $1 \leq i \leq k$. Note that (i) and (ii) imply

$$\frac{(\lambda_m^1)^2}{(\mu^1)^2} = \cdots = \frac{(\lambda_m^k)^2}{(\mu^k)^2}, \tag{2.9}$$

and (iii) implies

$$\lambda_m^1\mu^1 = \cdots = \lambda_m^k\mu^k. \tag{2.10}$$

(2.9) and (2.10) imply $|\lambda_1^i| = \cdots = |\lambda_m^i|$. Hence we have $\lambda_\alpha^i = 0$ for all $1 \leq \alpha \leq m, 1 \leq i \leq k$, i.e. $\nabla^2\phi(x_0) = 0$. This completes the proof of Proposition 2.3.

Combining Proposition 2.2 with Proposition 2.3, we have

Proposition 2.4 *If $\phi \in C^\infty(\mathbf{S}^m, \mathbf{S}^k)$ is a nonconstant harmonic map, then*

$$\Delta\left(\frac{1}{2}|\nabla\phi|^2\right) \geq \frac{m}{m-1}|\nabla|\nabla\phi||^2 + (m-1)|\nabla\phi|^2 + \frac{1-m}{m}|\nabla\phi|^4. \tag{2.11}$$

Furthermore, the equality holds at a point $x \in \mathbf{S}^m$ iff

$$\nabla^2\phi(x) = 0,\ |\nabla_{e_\alpha}\phi|^2(x) - |\nabla_{e_\beta}\phi|^2 = 0 = \langle\nabla_{e_\alpha}\phi, \nabla_{e_\gamma}\phi\rangle \tag{2.12}$$

for all $1 \leq \alpha, \beta, \gamma \leq m, \alpha \neq \gamma$.

Proof Observe that the lower bound of the last term in (2.2) is given by

$$\begin{aligned} -\sum_{\alpha,\beta=1}^m \{|\nabla e_\alpha\phi|^2|\nabla e_\beta|^2 - \langle\nabla_\alpha\phi, \nabla_\beta\phi\rangle^2\} &\geq \sum_{\alpha=1}^m |\nabla_\alpha\phi|^4 - |\nabla\phi|^4 \\ &\geq \frac{1}{m}\left(\sum_{\alpha=1}^m |\nabla_\alpha\phi|^2\right)^2 - |\nabla\phi|^4 \\ &\geq \frac{1-m}{m}|\nabla\phi|^4. \end{aligned} \tag{2.13}$$

Therefore (2.2) and (2.3) imply (2.11). Moreover, the equality in (2.11) holds iff both (2.3) and (2.13) are equalities and hence (2.12) is true. This proves Proposition 2.4.

Now we are ready to show the obstruction for the existence of stable tangent maps. Recall that a nonconstant map $\phi \in C^\infty(\mathbf{S}^m, \mathbf{S}^k)$ is called a stable tangent map, if ϕ is a harmonic map and its homogeneous degree zero extension $\bar{\phi}(x) = \phi(\frac{x}{|x|}) : \mathbb{R}^{m+1} \to \mathbf{S}^k$ is a stable harmonic map.

Proposition 2.5 *For $m \geq 2$ and $k \geq 3$, if $\phi \in C^\infty(\mathbf{S}^m, \mathbf{S}^k)$ is a stable tangent map, then $\frac{(m-1)^2}{4m} \geq \frac{k-2}{k}$.*

Proof By direct calculations, we have

$$\Delta\left(\frac{1}{2}|\nabla\phi|^2\right) = |\nabla\phi|\Delta|\nabla\phi| + |\nabla|\nabla\phi||^2.$$

This, combined with (2.11), implies

$$\Delta|\nabla\phi| \geq \frac{4}{m-1}|\nabla|\nabla\phi|^{\frac{1}{2}}|^2 + (m-1)\left(|\nabla\phi| - \frac{1}{m}|\nabla\phi|^3\right). \tag{2.14}$$

Integrating (2.14) over $\mathbf{S}^m$, we obtain

$$\int_{\mathbf{S}^m} |\nabla|\nabla\phi|^{\frac{1}{2}}|^2 + \frac{(m-1)^2}{4}|\nabla\phi| \leq \frac{(m-1)^2}{4m}\int_{\mathbf{S}^m} |\nabla\phi|^3. \tag{2.15}$$

On the other hand, the stablity inequality (2.1) implies

$$\int_{\mathbf{S}^m} |\nabla|\nabla\phi|^{\frac{1}{2}}|^2 + \frac{(m-1)^2}{4}|\nabla\phi| \geq \frac{k-2}{k}\int_{\mathbf{S}^m} |\nabla\phi|^3. \tag{2.16}$$

Therefore we have

$$\left(\frac{k-2}{k} - \frac{(m-1)^2}{4m}\right)\int_{\mathbf{S}^m} |\nabla\phi|^3 \leq 0.$$

This implies $\frac{k-2}{k} \leq \frac{(m-1)^2}{4m}$. The proof is complete.

It is easy to check that if $2 \leq m < \hat{d}(k)$ then $\frac{(m-1)^2}{4m} < \frac{k-2}{k}$. Therefore we obtain the following non-existence theorem on stable tangent maps.

Proposition 2.6 *For $k \geq 3$, there exists no stable tangent map in $\mathbb{R}^{m+1}$ for $2 \leq m \leq \hat{d}(k)-1$.*

Completetion of Proof of Theorem 1 By the compactness theorem on stable stationary harmonic maps into $\mathbf{S}^k$ ($k \geq 3$) in [3] and Proposition 2.6, Theorem 1 follows from the Federer dimension reduction argument as in [2].

3 Proof of Theorems 2 and 3

In this section, we first review the classification theorem in [7] on stable tangent maps from $\mathbb{R}^4$ to $\mathbf{S}^3$, which also follows from Propositions 2.4 and 2.5 above, and then we outline proofs of Theorems 2 and 3, which are natural extensions of the corresponding theorems on minimizing harmonic maps from B^3 to $\mathbf{S}^2$ by Hardt–Lin [8] and Almgren–Lieb [9], and from B^4 to $\mathbf{S}^2$ by Hardt–Lin [10].

Theorem 3.1 ([7]) *Suppose that $\phi \in C^\infty(\mathbf{S}^3, \mathbf{S}^3)$ is a nontrivial harmonic map such that its homogeneous of degree zero extension $\bar{\phi}(x) = \phi(\frac{x}{|x|}) : \mathbb{R}^4 \to \mathbf{S}^3$ is a stable harmonic map. Then*

$$\bar{\phi}(x) = Q\Big(\frac{x}{|x|}\Big) \tag{3.1}$$

for some orthogonal rotation $Q \in \mathbf{O}(3)$.

Proof The reader can refer to [7] for the orginal proof. Here we indicate a slightly different argument that follows from Propositions 2.4 and 2.5. In fact, when $m = k = 3$ we have $\frac{k-2}{k} = \frac{(m-1)^2}{4m}$. Therefore both (2.3) and (2.11) must be equalities and (2.12) is satisfied for any $x \in \mathbf{S}^3$. In particular, $\phi : \mathbf{S}^3 \to \mathbf{S}^3$ is a conformal, totally geodesic map. This is possible iff $\phi : \mathbf{S}^3 \to \mathbf{S}^3$ is an isometry.

A direct consequence of Theorem 2.1 is the following fact on stable stationary harmonic maps from $\mathbb{R}^4$ to $\mathbf{S}^3$.

Proposition 3.2 *For a bounded domain $\Omega \subset \mathbb{R}^4$, if $u \in H^1(\Omega, \mathbf{S}^3)$ is a stable stationary harmonic map, then $\mathscr{S}(u)$, the singular set of u, is discrete, and for any $x_0 \in \mathscr{S}$,*

$$\Theta(u, x_0) = \lim_{r\to 0} r^{-2} \int_{B_r(x_0)} |\nabla u|^2 = \frac{3}{2}|\mathbf{S}^3|. \tag{3.2}$$

Moreover, if there exists an $r_0 > 0$ such that $\int_{B_{r_0}(x_0)} |\nabla u|^2 = \frac{3}{2}|\mathbf{S}^3| r_0^2$, then there exists a $Q \in \mathbf{O}(3)$ such that $u(x) = Q\big(\frac{x-x_0}{|x-x_0|}\big)$ for $x \in B_{r_0}(x_0)$.

Proof The discreteness of $\mathscr{S}(u)$ follows from the partial regularity theorem in [3]. Moreover, it follows from the monotonicity identity (1.3) and the compactness theorem in [3] that for any $x_0 \in \mathscr{S}(u)$ and $r_i \to 0$, there exists a nontrivial smooth harmonic map $\phi \in C^\infty(\mathbf{S}^3, \mathbf{S}^3)$ such that, after taking possible subsequences,

$$\lim_{i\to\infty} \Big\| u(x_0 + r_i x) - \phi\Big(\frac{x}{|x|}\Big)\Big\|_{H^1(B_R)} = 0, \ \forall\, R > 0. \tag{3.3}$$

In particular, $\phi(\frac{x}{|x|}) : \mathbb{R}^4 \to \mathbf{S}^3$ is a stable harmonic map. Therefore, Theorem 3.1 implies $\phi = Q$ for some $Q \in \mathbf{O}(3)$. Hence

$$\Theta(u, x_0) = \int_{B_1} \Big|\nabla\Big(Q\Big(\frac{x}{|x|}\Big)\Big)\Big|^2 = \int_{B_1} \Big|\nabla \frac{x}{|x|}\Big|^2 = 3\int_0^1 r\,dr |\mathbf{S}^3| = \frac{3}{2}|\mathbf{S}^3|.$$

This implies (3.2). This, combined with the monotonicity identity (1.3), implies

$$r^{-2} \int_{B_r(x_0)} |\nabla u|^2 = \frac{3}{2}|\mathbf{S}^3| + 2\int_{B_r(x_0)} |y - x_0|^{-2} \Big|\frac{\partial u}{|y - x_0|}\Big|^2, \ \ \forall\, r > 0 \text{ small}. \tag{3.4}$$

If $\int_{B_{r_0}(x_0)} |\nabla u|^2 = \frac{3}{2}|\mathbf{S}^3| r_0^2$ holds for some $r_0 > 0$, then (3.4) implies $\frac{\partial u}{\partial |y-x_0|} = 0$ for a.e. $y \in B_{r_0}(x_0)$. Therefore $u(x) = \phi(\frac{x-x_0}{|x-x_0|})$ for $x \in B_{r_0}(x_0)$, with some nontrivial smooth harmonic map $\phi \in C^\infty(\mathbf{S}^3, \mathbf{S}^3)$. Applying Theorem 3.1 again, we have that there is a $Q \in \mathbf{O}(3)$ such that $u(x) = Q\big(\frac{x-x_0}{|x-x_0|}\big)$ in $B_{r_0}(x_0)$.

Now we are ready to prove Theorem 2.

Proof of Theorem 2 Since $u \in H^1(B^4, \mathbf{S}^3)$ is a stable harmonic map, the stablity inequality (1.6) implies

$$\int_{B^4} |\nabla u|^2 \eta^2 \le \frac{1}{3} \int_{B^4} |\nabla \eta|^2, \quad \forall\, \eta \in C_0^1(B^4). \tag{3.5}$$

This implies, for any $0 < \epsilon < 1$,

$$\int_{B_{1-\epsilon}} |\nabla u|^2 \leq C\epsilon^{-1}. \tag{3.6}$$

Therefore (1.7) follows from (3.6) and the monotonicity identity (1.3).

To show (ii), we may assume for simplicity $\epsilon = \frac{1}{2}$ and prove that the number of singular points of u inside $B_{\frac{1}{2}}$ is uniformly bounded. This follows if we can prove that there exists a universal constant $\delta_0 > 0$ such that the distance between any two singular points of u inside $B_{\frac{1}{2}}$ is at least δ_0. Suppose that this is false. Then we may assume that there exist a sequence of stable stationary harmonic maps $u_i \in H^1(B^4, \mathbf{S}^3)$ and a sequence of points $x_i \in B_{\frac{1}{2}}$ with $|x_i| \to 0$ such that $\{0, x_i\} \subset \mathscr{S}(u_i)$. Now we claim that there exists another universal constant $\delta_1 > 0$ such that

$$\int_{B_{2|x_i|}} |\nabla u_i|^2 \geq \left(\frac{3}{2}|\mathbf{S}^3| + \delta_1\right)(2|x_i|)^2. \tag{3.7}$$

For otherwise, the rescaled maps $v_i(x) = u_i(|x_i|x) : B_2 \to \mathbf{S}^3$ satisfy

$$\lim_{i\to\infty} 2^{-2} \int_{B_2} |\nabla v_i|^2 = \frac{3}{2}|\mathbf{S}^3|,$$

so that the compactness theorem of [3] implies that there exist a stable stationary harmonic map $v_\infty \in H^1(B_4, \mathbf{S}^3)$ and a point $x_\infty \in S^3$ such that $v_i \to v_\infty$ in $H^1(B_2)$, $\{0, x_\infty\} \in \mathscr{S}(v_\infty)$, and

$$2^{-2} \int_{B_2} |\nabla v_\infty|^2 = \frac{3}{2}|\mathbf{S}^3|.$$

This, combined with Proposition 3.2, implies $v_\infty(x) = Q(\frac{x}{|x|})$ in B_2 for some $Q \in \mathbf{O}(3)$. Hence $\mathscr{S}(v_\infty) = \{0\}$. We get the desired contradiction.

Proof of Theorem 3 With the help of Theorem 3.1, Proposition 3.2, the proof of Theorem 3 can be carried out exactly in the same manner as in Hardt–Lin [10]. We omit it here.

We end this section with a remark.

Remark 3.3 For $m, k \geq 3$, if $u \in H^1(B^n, \mathbf{S}^k)$ is a stable stationary harmonic map, then $\mathscr{S}(u)$, the singular set of u, is an $(n - \hat{d}(k) - 1)$-rectifiable set. In fact, since the set containing all stable stationary harmonic maps from B^n to $\mathbf{S}^k$ forms a compact subset of $H^1_{\text{loc}}(B^n, \mathbf{S}^k)$ by the compactness theorem of [3], one can apply Simon's proof [12] to this case to yield the rectifiablity of $\mathscr{S}(u)$.

4 Harmonic Maps into $\mathbf{S}^2$

Since the stability inequality (1.6) for stable harmonic maps into $\mathbf{S}^2$ has no implication on the regularity, we consider a special class of harmonic maps that are obtained from minimizing sequences among continuous maps into $\mathbf{S}^2$. Some general discussions have been presented by Lin [13], but the $\mathbf{S}^2$ case seems of particular interest.

For $n \geq 3$, asssume that $g \in C^0(\partial B^n, \mathbf{S}^2)$ has a continuous, H^1-extension map $G \in C^0(B^n, \mathbf{S}^2) \cap H^1(B^n, \mathbf{S}^2)$. Consider an energy minimizing sequence $\{u_i\} \subset C^0 \cap H^1(B^n, \mathbf{S}^2)$, with $u_i|_{\partial B^n} = g$. After taking possible subsequences, we may assume that $u_i \to u$ weakly in $H^1(B^n, \mathbf{S}^2)$ and there exists a nonnegative Radon measure μ on B^n such that $|\nabla u_i|^2\, dx \to \mu$ as convergence of Radon measures. Moreover, by Fatou's lemma, we know that there is a nonnegative Radon measure ν, called a defect measure, such that $\mu = |\nabla u|^2\, dx + \nu$.

Lemma 4.1 *$u \in H^1(B^n, \mathbf{S}^2)$ is a weakly harmonic map.*

Proof Let $\phi \in C^1_0(B^n, \mathbb{R}^3)$, consider the maps $u_i^t(x) = u_i(x) - t(\phi(x) \wedge u_i(x)) : B^n \to \mathbf{S}^2$ for $|t|$ sufficiently small. Then it is easy to see

$$|u_i^t(x)|^2 = 1 + O(t^2), \quad |\nabla u_i^t(x)|^2 = |\nabla u_i(x)|^2 + 2t\langle \nabla(\phi \wedge u_i)(x), \nabla u_i(x)\rangle + O(t^2).$$

Setting $w_i^t(x) = \frac{u_i^t(x)}{|u_i^t(x)|} : B^n \to \mathbf{S}^2$, we have

$$\lim_{i\to\infty} \int_{B^n} |\nabla w_i^t|^2(x) \geq \lim_{i\to\infty} \int_{B^n} |\nabla u_i|^2(x).$$

Note that

$$\nabla w_i^t(x) = \frac{\nabla u_i^t(x)}{|\nabla u_i^t(x)|} - \frac{\langle \nabla u_i^t(x), u_i^t(x)\rangle u_i^t(x)}{|u_i^t(x)|^3},$$

so that

$$\int_{B^n} |\nabla w_i^t(x)|^2 = \int_{B^n} \frac{|\nabla u_i^t(x)|^2}{|u_i^t(x)|^2} - \frac{|\langle \nabla u_i^t(x), u_i^t(x)\rangle|^2}{|u_i^t(x)|^4}.$$

Since $\langle \nabla u_i^t(x), u_i^t(x)\rangle = O(t)$ we obtain

$$\begin{aligned}\lim_{i\to\infty} \int_{B^n} |\nabla w_i^t(x)|^2 &= \lim_{i\to\infty} \int_{B^n} |\nabla u_i^t(x)|^2 + O(t^2)\\ &= \lim_{i\to\infty} \int_{B^n} |\nabla u_i|^2 + 2t \int_{B^n} \langle \nabla(\phi \wedge u_i)(x), \nabla u_i(x)\rangle + O(t^2).\end{aligned}$$

This implies $\lim_{i\to\infty} \int_{B^n} \langle \nabla(\phi\wedge u_i)(x), \nabla u_i(x)\rangle = 0$. This implies $\Delta u \wedge u = 0$ in the sense of distributions. Since $|u| = 1$, this implies u satisfies $\Delta u + |\nabla u|^2 u = 0$, and hence u is a weakly harmonic map.

For the limit Radon measure μ, we have

Lemma 4.2 *For any $a \in B^n$, $0 < r < d_a = 1 - |a|$, then $\frac{\mu(B_r(a))}{r^{n-2}}$ is monotonically nondecreasing with respect to r. In particular, $\Theta^{n-2}(\mu, a) = \lim_{r\to 0} \frac{\mu(B_r(a))}{r^{n-2}}$ exists for any $a \in B^n$ and is uppersemicontinuous.*

Proof See [13] Lemma 2.2. The basic reason is that a homogeneous of degree zero extension is essentially admissiable, with some replacement near the origin by suitable rescalings of the minimizing sequence.

For the limit Radon measure μ and the limit map u, we have the following Caccioppolli type inequality.

Lemma 4.3 *There is an $\epsilon_0 > 0$ such that for any $\theta \in (0, \frac{1}{2})$ there exists a $C_\theta > 0$ such that if $\frac{\mu(B_{2r}(a))}{(2r)^{n-2}} \leq \epsilon_0^2$ for $B_{2r}(a) \subset B^n$, then*

$$\frac{\mu(B_r(a)}{r^{n-2}} \leq \theta \frac{\mu(B_{2r}(a))}{(2r)^{n-2}} + C_\theta \frac{\int_{B_{2r}(a)} |u - u_{B_{2r}(a)}|^2}{(2r)^n}, \tag{4.1}$$

where $u_{B_{2r}(a)} = \frac{\int_{B_{2r}(a)} u}{|B_{2r}(a)|}$.

Proof See [13] Lemma 2.4 and Lemma 2.6.

Now we state two consequences of the Caccoppolli inequality (4.1).

Proposition 4.4 *There exist $\epsilon_0 > 0$ and $C_0 > 0$ such that if $\frac{\mu(B_{2r}(a))}{(2r)^{n-2}} \leq \epsilon_0^2$ for $B_{2r}(a) \subset B^n$ then*

$$\frac{\mu(B_r(a))}{r^{n-2}} \leq C_0 \frac{\int_{B_{2r}(a)} |u - u_{B_{2r}(a)}|^2}{(2r)^n}. \tag{4.2}$$

Proof (4.2) follows from (4.1) and a covering argument (see Simon [14]).

Proposition 4.5 *There exists an $\epsilon_0 > 0$ such that if $\frac{\mu(B_{2r}(a))}{(2r)^{n-2}} \leq \epsilon_0^2$ for $B_{2r}(a) \subset B^n$, then $u \in C^\infty(B_r(a), \mathbf{S}^2)$.*

Proof First observe that (4.2) implies that if $\frac{\mu(B_{2r}(a))}{(2r)^{n-2}} \leq \epsilon_0^2$ then we have the reverse Hölder inequality:

$$\frac{\int_{B_r(a)} |\nabla u|^2}{r^{n-2}} \leq C_0 \frac{\int_{B_{2r}(a)} |u - u_{B_{2r}(a)}|^2}{(2r)^n}. \tag{4.3}$$

It is well known that a weakly harmonic map $u \in H^1(B^n, \mathbf{S}^2)$ satisfying (4.3) enjoys the partial regularity as above. The reader can also see [13] Theorem 2.3.

Remark 4.6 A direct consequence of (4.2) and Proposiition 4.6 is the following: There exist $\epsilon_0 > 0$ and $C_1 > 0$ such that if for $B_{r_0}(a) \subset B^n$, $\frac{\mu(B_{r_0}(a))}{r_0^{n-2}} \le \epsilon_0^2$ then

$$\frac{\mu(B_r(a))}{r^n} \le C_1 r_0^{-2}, \ \forall\, 0 < r \le \frac{r_0}{2}. \tag{4.4}$$

In particular, there exists a nonnegative $f \in L^\infty(B_{\frac{r_0}{2}}(a))$ such that

$$\nu = f(x)\, dx. \tag{4.5}$$

Proof Proposition 4.5 implies that $u \in C^\infty(B_{\frac{r_0}{2}}(a), S^2)$ and

$$\|\nabla u\|_{C^0(B_{\frac{r_0}{2}}(a))} \le C_1 r_0^{-1}. \tag{4.6}$$

Therefore (4.2) implies

$$\frac{\mu(B_r(a))}{r^{n-2}} \le C_1 \left(\frac{r}{r_0}\right)^2, \ \forall\, 0 < r \le \frac{r_0}{2}. \tag{4.7}$$

This implies (4.4). (4.5) follows from (4.4).

In fact, we have

Proposition 4.7 *There exists an $\epsilon_0 > 0$ such that if $\frac{\mu(B_{r_0}(a))}{r_0^{n-2}} \le \epsilon_0^2$, then $\nu \equiv 0$ in $B_{\frac{r_0}{2}}(a)$.*

Assuming for the moment the conclusion of Proposition 4.7, we have the following theorem.

Theorem 4.8 *Under the above notations, set $\Sigma = \{a \in B^n : \Theta^{n-2}(\mu, a) \ge \epsilon_0^2\}$. Then*

(i) Σ *is a closed subset and $H^{n-2}(\Sigma \cap B_r)$ is finite for any* $0 < r < 1$.

(ii) Σ *is an $(n-2)$-dimensional rectifiable set.*

(iii) $u \in C^\infty(B^n \setminus \Sigma, \mathbf{S}^2)$.

(iv) $\nu \equiv 0$ *in* $B^n \setminus \Sigma$ *and* $u_i \to u$ *in* $H^1_{\text{loc}}(B^n \setminus \Sigma, \mathbf{S}^2)$.

Proof (i) follows from the Vitali's covering Lemma. (iii) follows from Proposition 4.5. (iv) follows from Proposition 4.7. (ii) has been proved by [13] Theorem 3.5 (see also Lin [15]).

Now we return to the proof of Proposition 4.7.

Proof of Proposition 4.7 For $n = 2, 3$, we can refer to the proof by [13, Lemma 3.2]. For $n \ge 4$, the original proof of [13, Lemma 3.2] has a mistake, we outline a proof that is a suitable modification of that of Luckhaus' extension Lemma (see [16]). Since the proof is based on inducation on $n \ge 4$, for simplicity we only consider the $n = 4$ case.

To prove $f \equiv 0$ in $B_{\frac{r_0}{2}}$, it suffices to show that for any $a \in B_{\frac{r_0}{2}}$, there exists $\delta(r) > 0$, with $\lim_{r \to 0} \delta(r) = 0$, such that

$$\nu(B_r(a)) \le \delta(r) r^n, \ \forall\, 0 < r < \frac{r_0}{4}. \tag{4.8}$$

By scalings, we may assume that $a = 0$ and $r_0 = 2$. From Proposition 4.5 and (4.6), we have $u \in C^\infty(B_2, \mathbf{S}^2)$ with

$$\|\nabla u\|_{C^0(B_1)} \le C\epsilon_0. \tag{4.9}$$

By the Fubini's theorem, we may assume that $\{u_i\} \in C^0 \cap H^1(B_2, \mathbf{S}^2)$ is the minimizing sequence such that $u_i \to u$ weakly in $H^1(B_2, \mathbf{S}^2)$ and

$$\int_{\partial B_1} |\nabla u_i|^2 \le C\epsilon_0, \ \forall\, i \ge 1, \ \lim_{i \to \infty} \int_{\partial B_1} |u_i - u|^2 = 0. \tag{4.10}$$

Now we need to prove the following Lemma.

Lemma 4.9 *For any $\delta \in (0,1)$, there exists a sequence of maps $v_i \in C^0 \cap H^1(B_1 \setminus B_{1-\delta}, \mathbf{S}^2)$ such that*

$$v_i|_{\partial B_1} = u_i, v_i|_{\partial B_{1-\delta}} = u\left(\frac{\cdot}{1-\delta}\right), \tag{4.11}$$

and

$$\int_{B_1\setminus B_{1-\delta}}|\nabla v_i|^2\le C\delta\int_{\partial B_1}\Big(|\nabla u_i|^2+\Big|\nabla u\Big(\frac{\cdot}{1-\delta}\Big)\Big|^2\Big)$$
$$+C\delta^{-1}\int_{\partial B_1}\Big|u_i(x)-u\Big(\frac{x}{1-\delta}\Big)\Big|^2+0(i^{-1}). \tag{4.12}$$

Assuming Lemma 4.9 for the moment, we can prove (4.8) as follows. Define another sequence of maps $\bar{u}_i\in C^0\cap H^1(B_1,\mathbf{S}^2)$ by

$$\bar{u}_i(x)=\begin{cases} u\Big(\dfrac{x}{1-\delta}\Big), & x\in B_{1-\delta},\\ v_i(x), & x\in B_1\setminus B_{1-\delta}.\end{cases}$$

Then we have

$$\begin{aligned}\int_{B_1}|\nabla u|^2+\nu(B_1)&=\lim_{i\to\infty}\int_{B_1}|\nabla u_i|^2\le\lim_{i\to\infty}\int_{B_1}|\nabla\bar{u}_i|^2\\ &=\lim_{i\to\infty}\Big((1-\delta)^{n-2}\int_{B_1}|\nabla u|^2+\int_{B_1\setminus B_{1-\delta}}|\nabla v_i|^2\Big)\\ &\le\int_{B_1}|\nabla u|^2+C\delta\Big(\int_{\partial B_1}|\nabla u|^2+\nu(\partial B_1)\Big).\end{aligned}$$

This gives $\nu(B_1)\le C\delta\mu(\partial B_1)$. Rescaling back to the original scales, we prove (4.8). Hence $f\equiv 0$ in $B_{\frac{r_0}{2}}$. The proof of Proposition 4.7 is complete.

Proof of Lemma 4.9 As mentioned before, Lemma 4.9 has been proved by [13] Lemma 3.2 for $n=3$. Now we want to show that Lemma 4.9 also holds for $n=4$.

First we follow [16] to conclude that there exists a triangularization of $\mathbf{S}^3$ by boxes of side lengths δ, $\{\Delta_i\}_{i=1}^L$ with $L\le C\delta^{-3}$, such that

$$\delta\sum_{i=1}^{L}\int_{\partial\Delta_i}\Big(|\nabla u_i|^2+\Big|\nabla u\Big(\frac{\cdot}{1-\delta}\Big)\Big|^2+\Big|u_i-u\Big(\frac{\cdot}{1-\delta}\Big)\Big|^2\Big)$$
$$\le C\int_{S^3}\Big(|\nabla u_i|^2+|\nabla u|^2+\Big|u_i-u\Big(\frac{\cdot}{1-\delta}\Big)\Big|^2\Big). \tag{4.13}$$

By bi-Lipschitz transformation, we may assume that each sub-annual region $F_j=\{(r,\theta): 1-\delta\le r\le 1,\theta\in\Delta_j\}$ can be identified by $G_j=[1-\delta,1]\times\Delta_j$ for $1\le j\le L$. Observe that $\partial G_j=(\{1\}\times\Delta_j)\cup(\{1-\delta\}\times\Delta_j)\cup([1-\delta,1]\times\partial\Delta_j$.

For $1\le j\le L$, applying Lemma 4.9 for $n=3$, we conclude that there exists a map $v_i\in C^0\cap H^1([1-\delta,1]\times\partial\Delta_j,\mathbf{S}^2)$ such that

$$v_i|_{\{1\}\times\partial\Delta_j}=u_i|_{\{1\}\times\partial\Delta_j},\quad v_i|_{\{1-\delta\}\times\partial\Delta_j}=u\Big(\frac{\cdot}{1-\delta}\Big)\Big|_{\{1-\delta\}\times\partial\Delta_j}$$

and

$$\int_{[1-\delta,1]\times\partial\Delta_j}|\nabla v_i|^2\le C\delta\int_{\partial\Delta_j}|\nabla u_i|^2+\Big|\nabla u\Big(\frac{\cdot}{1-\delta}\Big)\Big|^2+C\delta^{-1}\int_{\partial\Delta_j}\Big|u_i-u\Big(\frac{\cdot}{1-\delta}\Big)\Big|^2. \tag{4.14}$$

Now define $w_i\in C^0(\partial G_j,S^2)$ by letting $w_i=u_i$ on $\{1\}\times\Delta_j$, $w_i=u(\frac{\cdot}{1-\delta})$ on $\{1-\delta\}\times\Delta_j$, and $w_i=v_i$ on $[1-\delta,1]\times\partial\Delta_j$.

We now divide it into two cases:

(1) $\alpha=[w_i]\in\Pi_3(\mathbf{S}^2)$ is trivial: In this case we know that w_i has a continuous, H^1-extension $\bar{w}_i: G_j\to\mathbf{S}^2$ such that

$$\int_{G_j}|\nabla\bar{w}_i|^2\le C\Big(\int_{\partial G_j}|\nabla w_i|^2\Big). \tag{4.15}$$

Therefore we can use the same construction as that in the proof of Lemma 4.2 (see also [13, Lemma 2.2]) to find an essentially homogeneous of degree zero extension map $\tilde{w}_i \in C^0 \cap H^1(G_j, \mathbf{S}^2)$ of w_i such that for any $\epsilon > 0$

$$\int_{G_j} |\nabla \tilde{w}_i|^2 \leq C\delta \int_{\partial G_j} |\nabla w_i|^2 + Ci^{-1} \int_{G_j} |\nabla \bar{w}_i|^2 \leq C\delta \int_{\partial G_j} |\nabla w_i|^2 + i^{-1} C\Big(\int_{G_j} |\nabla w_i|^2 \Big). \quad (4.16)$$

(2) $\alpha = [w_i] \in \Pi_3(\mathbf{S}^2)$ is nontrivial: Although w_i doesn't have continuous extension from G_j to $\mathbf{S}^2$, the theorem of White [17] implies that $\alpha^{-1} \in \Pi_3(\mathbf{S}^2)$ can be represented by a map $f_i \in C^1(\mathbf{S}^3, \mathbf{S}^2)$ such that $f_i \equiv p_i$ in $\mathbf{S}^3 \setminus B_{\epsilon_i}$ for some $p_i \in \mathbf{S}^2$, with $\epsilon_i \to 0$, and

$$\int_{B_{\epsilon_i}} |\nabla f_i|^2 \leq i^{-1}. \quad (4.17)$$

Now we first modify w_i on a small ball $B_{2\epsilon_i} \subset [1-\delta, 1] \times \partial\Delta_j$ to obtain $\bar{w}_i$ such that $[\bar{w}_i] = [w_i] = \alpha$, $\bar{w}_i \equiv p_i$ in $B_{\epsilon_i} \subset [1-\delta, 1] \times \partial\Delta_j$, and

$$\int_{\partial G_j} |\nabla \bar{w}_i|^2 = \int_{\partial G_j} |\nabla w_i|^2 + 0(\epsilon_i). \quad (4.18)$$

Next we glue $\bar{w}_i$ with f_i along $\partial B_{\epsilon_i} \subset [1-\delta, 1] \times \partial\Delta_j$ and denote the resulting map by $\hat{w}_i$. It is readily seen that $[\hat{w}_i] \in \Pi_3(S^2)$ is trivial,

$$\hat{w}_i|_{\{1\}\times\Delta_j} = u_i|_{\{1\}\times\Delta_j}, \hat{w}_i|_{\{1-\delta\}\times\Delta_j} = u\Big(\frac{\cdot}{1-\delta}\Big)\Big|_{\{1-\delta\}\times\Delta_j},$$

and

$$\int_{\partial G_j} |\nabla \hat{w}_i|^2 \leq \int_{\partial G_j} |\nabla w_i|^2 + Ci^{-1} + O(\epsilon_i). \quad (4.19)$$

Now we can follow the same construction as in (1) to obtain an extension $\tilde{w}_i \in C^0 \cap H^1(G_j, S^2)$ such that $\tilde{w}_i = u_i$ on $\{1\} \times \Delta_j$, $\tilde{w}_j = u(\frac{\cdot}{1-\delta})$ on $\{1-\delta\} \times \Delta_j$, and

$$\begin{aligned}
\int_{G_j} |\nabla \tilde{w}_i|^2 &\leq C\delta \int_{\partial G_j} |\nabla w_i|^2 + O(i^{-1}, \epsilon_i) \\
&\leq C\delta \Big[\int_{\Delta_j} |\nabla u_i|^2 + \Big|\nabla u\Big(\frac{\cdot}{1-\delta}\Big)\Big|^2 + \int_{[1-\delta,1]\times\partial\Delta_j} |\nabla v_i|^2 \Big] + O(i^{-1}, \epsilon_i) \\
&\leq C\delta \Big[\int_{\Delta_j} |\nabla u_i|^2 + \Big|\nabla u\Big(\frac{\cdot}{1-\delta}\Big)\Big|^2 + \delta \int_{\partial\Delta_j} |\nabla u_i|^2 + \Big|\nabla u\Big(\frac{\cdot}{1-\delta}\Big)\Big|^2 \\
&\quad + \delta^{-1} \int_{\partial\Delta_j} \Big| u_i - u\Big(\frac{\cdot}{1-\delta}\Big)\Big|^2 \Big] + O(i^{-1}, \epsilon_i). \quad (4.20)
\end{aligned}$$

Finally we repeat the above construction over G_j for $1 \leq j \leq L$ to obtain an extension map $\tilde{w}_i \in C^0 \cap H^1(B_1 \setminus B_{1-\delta}, \mathbf{S}^2)$ that satisfies (4.11) and

$$\begin{aligned}
\int_{B_1 \setminus B_{1-\delta}} |\nabla \tilde{w}_i|^2 &\leq C\delta \sum_{i=1}^{L} \int_{\Delta_j} |\nabla u_i|^2 + \Big|\nabla u\Big(\frac{\cdot}{1-\delta}\Big)\Big|^2 \\
&\quad + C\delta \sum_{i=1}^{L} \Big(\delta \int_{\partial\Delta_j} |\nabla u_i|^2 + \Big|\nabla u\Big(\frac{\cdot}{1-\delta}\Big)\Big|^2 \\
&\quad + \delta^{-1} \int_{[1-\delta,1]\times\Delta_j} \Big| u_i - u\Big(\frac{\cdot}{1-\delta}\Big)\Big|^2 \Big) + O(i^{-1}).
\end{aligned}$$

This, combined with (4.13), implies (4.12). Therefore the proof of Lemma 4.9 is complete.

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