Regularity and uniqueness of the heat flow of biharmonic maps

Jay Hineman, Tao Huang, and Changyou Wang

Department of Mathematics
University of Kentucky
Lexington, KY 40506, USA

Abstract

In this paper, we first establish regularity of the heat flow of biharmonic maps into the unit sphere $\mathbb{S}^L \subset \mathbb{R}^{L+1}$ under a smallness condition of renormalized total energy. For the class of such solutions to the heat flow of biharmonic maps, we prove the properties of uniqueness, convexity of hessian energy, and unique limit at $t = \infty$. We establish both regularity and uniqueness for Serrin's (p, q)-solutions to the heat flow of biharmonic maps into any compact Riemannian manifold N without boundary.

1 Introduction

For $n \geq 4$ and $L \geq k \geq 1$, let $\Omega \subset \mathbb{R}^n$ be a bounded smooth domain and $N \subset \mathbb{R}^{L+1}$ be a k-dimensional compact Riemannian manifold without boundary. For $m \geq 1$, $p \geq 1$, the Sobolev space $W^{m,p}(\Omega, N)$ is defined by

$$W^{m,p}(\Omega, N) = \left\{ v \in W^{m,p}(\Omega, \mathbb{R}^{L+1}) \ : \ v(x) \in N \text{ for a.e. } x \in \Omega \right\}.$$

On $W^{2,2}(\Omega, N)$, there are two second order energy functionals:

$$E_2(u) = \int_\Omega |\Delta u|^2 \text{ and } F_2(u) = \int_\Omega |(\Delta u)^T|^2,$$

where $(\Delta u)^T$ is the tangential component of Δu to $T_u N$ at u, which is called the tension field of u ([6]). A map $u \in W^{2,2}(\Omega, N)$ is called an extrinsic (or intrinsic) biharmonic map, if u is a critical point of $E_2(\cdot)$ (or $F_2(\cdot)$ respectively). It is well known that biharmonic maps are higher-order extensions of harmonic maps, which are critical points of the Dirichlet energy $E_1(u) = \int_\Omega |\nabla u|^2$ over $W^{1,2}(\Omega, N)$. Recall that the Euler-Lagrange equation of (extrinsic) biharmonic maps is given by ([41] Lemma 2.1):

$$\Delta^2 u = \mathcal{N}_{\text{bh}}[u] := [\Delta(A(u)(\nabla u, \nabla u)) + 2\nabla \cdot \langle \Delta u, \nabla(P(u))\rangle - \langle \Delta(P(u)), \Delta u\rangle] \perp T_u N, \tag{1.1}$$

where $P(y) : \mathbb{R}^{L+1} \to T_y N$ is the orthogonal projection for $y \in N$, and $A(y)(\cdot, \cdot) = \nabla P(y)(\cdot, \cdot)$ is the second fundamental form of N at $y \in N$. Throughout this paper, we always use $\mathcal{N}_{\text{bh}}[u]$ to denote the nonlinearity in the right hand side of the biharmonic map equation (1.1).

Motivated by earlier studies on regularity of harmonic maps by Schoen-Uhlenbeck [39], Hélein [13], Evans [7], Bethuel [2], Lin [25], Rivière [31], and many others, the study of biharmonic maps has attracted considerable interest in the field and prompted a large number of interesting works by many analysts in the last several years. The regularity of biharmonic maps to $N = \mathbb{S}^L$ – the unit sphere in $\mathbb{R}^{L+1}$ – was first studied by Chang-Wang-Yang [4]. Wang [41, 42, 43] extended the main theorems of [4] to any compact Riemannian manifold N without boundary. It asserts smoothness of biharmonic maps in dimension $n = 4$, and the partial regularity of *stationary* biharmonic maps for $n \geq 5$. Here we mention in passing the interesting works on biharmonic maps by Angelsberg [1], Strzelecki [30], Hong-Wang [16], Lamm-Rivière [23], Struwe [38], Ku [19], Gastel-Scheven [10], Scheven [33, 34], Lamm-Wang [24], Moser [27, 28], Gastel-Zorn [11], Hong-Yin [17], and Gong-Lamm-Wang [12].

The initial and boundary value problem for the heat flow of biharmonic maps is follows. For $0 < T \leq +\infty$, and $u_0 \in W^{2,2}(\Omega, N)$, a map $u \in W_2^{1,2}(\Omega \times [0,T], N)$, i.e. $\partial_t u, \nabla^2 u \in L^2(\Omega \times [0,T])$, is called the heat flow of a biharmonic map, if u satisfies in the sense of distributions

$$\begin{cases} \partial_t u + \Delta^2 u = \mathcal{N}_{\rm bh}[u] & \text{in } \Omega \times (0,T) \\ u = u_0 & \text{on } \partial_p(\Omega \times [0,T]) \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial u_0}{\partial \nu} & \text{on } \partial\Omega \times [0,T), \end{cases} \tag{1.2}$$

where ν denotes the outward unit normal of $\partial\Omega$. Throughout the paper, we denote $\partial_p(\Omega \times [0,T]) = (\Omega \times \{0\}) \cup (\partial\Omega \times (0,T))$ as the parabolic boundary of $\Omega \times [0,T]$.

The formulation of heat flow of biharmonic maps (1.2) remains unchanged, if Ω is replaced by a n-dimensional compact Riemannian manifold M with boundary ∂M. On the other hand, if Ω is replaced by M that is a n-dimensional compact Riemannian manifold without boundary or a complete, non-compact Riemannian manifold without boundary, then the Cauchy problem of heat flow of biharmonic maps is considered. More precisely, if $\partial M = \emptyset$, then (1.2) becomes

$$\begin{cases} \partial_t u + \Delta^2 u = \mathcal{N}_{\rm bh}[u] & \text{in } M \times (0,T) \\ u = u_0 & \text{on } M \times \{0\}. \end{cases} \tag{1.3}$$

The Cauchy problem (1.3) was first studied by Lamm [21], [22] in dimension $n = 4$ for $u_0 \in C^\infty(M, N)$, where the existence of a unique, global smooth solution is established under the condition that $\|u_0\|_{W^{2,2}(M)}$ is sufficiently small. For any $u_0 \in W^{2,2}(M, N)$, the existence of a unique, global weak solution to (1.3), that is smooth away from finitely many times, has been independently proved by Gastel [9] and Wang [44]. The interested readers can verify that with suitable modifications of their proofs, the existence theorem by [9] and [44] remain to hold for (1.2) in any compact 4-dimensional Rimannian manifold M with boundary ∂M, if, in addition, the trace of u_0 on ∂M for $u_0 \in W^{2,2}(M, N)$ satisfies $u_0|_{\partial M} \in W^{\frac{7}{2},2}(\partial M, N)$. Namely, there is a unique, global weak solution $u \in W_2^{1,2}(M \times [0,\infty), N)$ to (1.2) such that

(i) $E_2(u(t))$ is monotone decreasing for $t \geq 0$; and

(ii) there are $T_0 = 0 < T_1 < \ldots < T_k < T_{k+1} = +\infty$ such that

$$u \in \bigcap_{i=0}^{k} C^\infty(M \times (T_i, T_{i+1}), N) \ \text{ and } \ \nabla u \in \bigcap_{i=0}^{k} C^\alpha(\overline{M} \times (T_i, T_{i+1}), N), \ \forall\, \alpha \in (0, 1).$$

For dimensions $n \geq 4$, Wang [45] established the well-posedness of (1.3) on $\mathbb{R}^n$ for any $u_0 : \mathbb{R}^n \to N$ that has sufficiently small BMO norm. Moser [29] showed the existence of global weak solutions $u \in W_2^{1,2}(\Omega \times [0, \infty), N)$ to (1.2) on any bounded smooth domain $\Omega \subset \mathbb{R}^n$ for $n \leq 8$ and $u_0 \in W^{2,2}(\Omega, N)$.

Due to the critical nonlinearity in the evolution equation $(1.2)_1$ of heat flow of biharmonic maps, the question of regularity and uniqueness for weak solutions of (1.2) is very challenging for dimensions $n \geq 4$. There has not been much work done in this direction. This motivates us to study these issues for the equation (1.2) in this paper. Another motivation for us to study these issues comes from our recent work [14] on the these issues for the heat flow of harmonic maps. We are able to obtain several interesting results concerning regularity, uniqueness, convexity, and unique limit at time infinity of the equation (1.2), under the smallness condition of renormalized total energy.

Before stating our main theorems, we need to introduce some notations.

Notations: For $1 \leq p, q \leq +\infty$, $0 < T \leq \infty$, define the Sobolev space

$$W_2^{1,2}(\Omega \times [0, T], N) = \Big\{ v \in L^2([0, T], W^{2,2}(\Omega, N)) : \ \partial_t v \in L^2([0, T], L^2(\Omega)) \Big\},$$

the $L_t^q L_x^p$-space

$$L_t^q L_x^p(\Omega \times [0, T], \mathbb{R}^{L+1}) = \Big\{ f : \Omega \times [0, T] \to \mathbb{R}^{L+1} : \ f \in L^q([0, T], L^p(\Omega)) \Big\},$$

and the Morrey space $M_R^{p,\lambda}$ for $0 \leq \lambda \leq n + 4$, $0 < R \leq \infty$, and $U = U_1 \times U_2 \subset \mathbb{R}^n \times \mathbb{R}$:

$$M_R^{p,\lambda}(U) = \left\{ f \in L^p_{\text{loc}}(U) : \Big\| f \Big\|_{M_R^{p,\lambda}(U)} < +\infty \right\},$$

where

$$\Big\| f \Big\|_{M_R^{p,\lambda}(U)} = \Big(\sup_{(x,t) \in U} \ \sup_{0 < r < \min\{R, d(x, \partial U_1), \sqrt{t}\}} r^{\lambda - n - 4} \int_{P_r(x,t)} |f|^p \Big)^{\frac{1}{p}},$$

and

$$B_r(x) = \{ y \in \mathbb{R}^n : \ |y - x| \leq r \}, \ P_r(x, t) = B_r(x) \times [t - r^4, t], \ d(x, \partial U_1) = \inf_{y \in \partial U_1} |x - y|.$$

Denote B_r (or P_r) for $B_r(0)$ (or $P_r(0)$ respectively), and $M^{p,\lambda}(U) = M_\infty^{p,\lambda}(U)$ for $R = \infty$. We also recall the weak Morrey space for $R = \infty$, $M_*^{p,\lambda}(U)$, that is the set of functions f on U such that

$$\| f \|^p_{M_*^{p,\lambda}(U)} = \sup_{r > 0, (x,t) \in U} \Big\{ r^{\lambda - (n+4)} \| f \|^p_{L^{p,*}(P_r(x,t) \cap U)} \Big\} < +\infty,$$

where $L^{p,*}(P_r(x,t)\cap U)$ is the weak L^p-space, that is the collection of functions v on $P_r(x,t)\cap U$ such that

$$\|v\|^p_{L^{p,*}(P_r(x,t)\cap U)} = \sup_{a>0}\Big\{a^p|\{z\in P_r(x,t)\cap U\ :\ |v(z)|>a\}|\Big\} < +\infty.$$

Recall that if $N=\mathbb{S}^L := \{y\in\mathbb{R}^{L+1}:\ |y|=1\}$, then direct calculations give

$$\mathcal{N}_{\rm bh}[u] = -(|\Delta u|^2 + \Delta(|\nabla u|^2) + 2\langle \nabla u, \nabla\Delta u\rangle)u,$$

so that the heat flow of biharmonic maps to $\mathbb{S}^L$, $(1.2)_1$, can be written as

$$\partial_t u + \Delta^2 u = -(|\Delta u|^2 + \Delta(|\nabla u|^2) + 2\langle \nabla u, \nabla\Delta u\rangle)u. \tag{1.4}$$

The first theorem addresses the regularity of (1.4).

Theorem 1.1 *For $1<p\le 2$ and $0<T<+\infty$, there exists $\epsilon_p>0$ such that if $u\in W_2^{1,2}(\Omega\times[0,T],\mathbb{S}^L)$ is a weak solution to (1.4) and satisfies that, for $z_0=(x_0,t_0)\in\Omega\times(0,T]$ and $0<R_0\le \frac12\min\{d(x_0,\partial\Omega),\sqrt{t_0}\}$,*

$$\|\nabla^2 u\|_{M^{p,2p}_{R_0}(P_{R_0}(z_0))} + \|\partial_t u\|_{M^{p,4p}_{R_0}(P_{R_0}(z_0))} \le \epsilon_p, \tag{1.5}$$

then $u\in C^\infty\left(P_{\frac{R_0}{16}}(z_0),\mathbb{S}^L\right)$, and

$$\Big|\nabla^m u\Big|(z_0) \le \frac{C\epsilon_p}{R_0^m},\ \forall\ m\ge 1. \tag{1.6}$$

Remark 1.2 *It is an open question whether Theorem 1.1 holds true for any compact Riemannian manifold N without boundary (with $p=2$).*

Utilizing this regularity theorem, we have the following uniqueness theorem.

Theorem 1.3 *For $n\ge 4$ and $1<p\le 2$, there exist $\epsilon_0=\epsilon_0(p,n)>0$ and $R_0=R_0(\Omega,\epsilon_0)>0$ such that if $u_1,u_2\in W_2^{1,2}(\Omega\times[0,T],\mathbb{S}^L)$ are weak solutions to (1.2), with the same initial and boundary value $u_0\in W^{2,2}(\Omega,\mathbb{S}^L)$, that satisfy*

$$\max_{i=1,2}\Big[\|\nabla^2 u_i\|_{M^{p,2p}_{R_0}(\Omega\times(0,T))} + \|\partial_t u_i\|_{M^{p,4p}_{R_0}(\Omega\times(0,T))}\Big] \le \epsilon_0, \tag{1.7}$$

then $u_1\equiv u_2$ on $\Omega\times[0,T]$.

There are two main ingredients to prove Theorem 1.3:
(i) The interior regularity of u_i $(i=1,2)$: $u_i\in C^\infty(\Omega\times(0,T),\mathbb{S}^L)$ and

$$\max_{i=1,2}|\nabla^m u_i|(x,t) \lesssim \epsilon_0\left(\frac{1}{R_0^m} + \frac{1}{d^m(x,\partial\Omega)} + \frac{1}{t^{\frac{m}{4}}}\right) \tag{1.8}$$

for any $(x,t)\in\Omega\times(0,T)$ and $m\ge 1$.
(ii) The energy method, with suitable applications of the Poincaré inequality and the higher-order Hardy inequality (see Lemma 3.1 below).

Remark 1.4 (i) *We would like to point out that the novel feature of Theorem 1.3 is that the solutions may have singularities at the parabolic boundary* $\partial_p(\Omega \times [0, T])$. *Thus the standard argument to prove uniqueness for classical solutions is not applicable.*
(ii) *For* $\Omega = \mathbb{R}^n$, *if the initial data* $u_0 : \mathbb{R}^n \to N$ *satisfies that for some* $R_0 > 0$,

$$\sup\left\{r^{4-n}\int_{B_r(x)} |\nabla^2 u_0|^2 : \ x \in \mathbb{R}^n, r \leq R_0\right\} \leq \epsilon_0^2,$$

then by the local well-posedness theorem of Wang [45] there exists $0 < T_0(\approx R_0^4)$ *and a solution* $u \in C^\infty(\mathbb{R}^n \times (0, T_0), N)$ *of (1.3) that satisfies the condition (1.7).*

Prompted by the ideas of proof of Theorem 1.3, we obtain the convexity property of the E_2-energy along the heat flow of biharmonic maps to $\mathbb{S}^L$.

Theorem 1.5 *For* $n \geq 4$, $1 < p \leq 2$, *and* $1 \leq T \leq \infty$, *there exist* $\epsilon_0 = \epsilon_0(p, n) > 0$, $R_0 = R_0(\Omega, \epsilon_0) > 0$, *and* $0 < T_0 = T_0(\epsilon_0) < T$ *such that if* $u \in W_2^{1,2}(\Omega \times [0, T], \mathbb{S}^L)$ *is a weak solution to (1.2), with the initial and boundary value* $u_0 \in W^{2,2}(\Omega, \mathbb{S}^L)$, *satisfying*

$$\|\nabla^2 u\|_{M_{R_0}^{p,2p}(\Omega\times(0,T))} + \|\partial_t u\|_{M_{R_0}^{p,4p}(\Omega\times(0,T))} \leq \epsilon_0, \tag{1.9}$$

then
(i) $E_2(u(t))$ *is monotone decreasing for* $t \geq T_0$; *and*
(ii) for any $t_2 \geq t_1 \geq T_0$,

$$\int_\Omega |\nabla^2(u(t_1) - u(t_2))|^2 \leq C\Big[\int_\Omega |\Delta u(t_1)|^2 - \int_\Omega |\Delta u(t_2)|^2\Big] \tag{1.10}$$

for some $C = C(n, \epsilon_0) > 0$.

A direct consequence of the convexity property of E_2-energy is the unique limit at $t = \infty$ of (1.2).

Corollary 1.6 *For* $n \geq 4$ *and* $1 < p \leq 2$, *there exist* $\epsilon_0 = \epsilon_0(p, n) > 0$, *and* $R_0 = R_0(\Omega, \epsilon_0) > 0$ *such that if* $u \in W_2^{1,2}(\Omega \times [0, \infty), \mathbb{S}^L)$ *is a weak solution to (1.2), with the initial and boundary value* $u_0 \in W^{2,2}(\Omega, \mathbb{S}^L)$, *satisfying the condition (1.9), then there exists a biharmonic map* $u_\infty \in C^\infty \cap W^{2,2}(\Omega, \mathbb{S}^L)$, *with* $(u_\infty, \frac{\partial u_\infty}{\partial \nu}) = (u_0, \frac{\partial u_0}{\partial \nu})$ *on* $\partial\Omega$, *such that*

$$\lim_{t\uparrow\infty} \|u(t) - u_\infty\|_{W^{2,2}(\Omega)} = 0, \tag{1.11}$$

and, for any compact subset $K \subset\subset \Omega$ *and* $m \geq 1$,

$$\lim_{t\uparrow\infty} \|u(t) - u_\infty\|_{C^m(K)} = 0. \tag{1.12}$$

Remark 1.7 (i) *We would like to remark that if Theorem 1.1 has been proved for any compact Riemannian manifold* N *without boundary, then Theorem 1.3, Theorem 1.5, and Corollary 1.6 would be true for any compact Riemannian manifold* N *without boundary.*

(ii) *With slight modifications of the proofs, Theorem 1.1, Theorem 1.3, Theorem 1.5, and Corollary 1.6 remain to be true, if* Ω *is replaced by a compact Riemannian manifold* M *with boundary* ∂M*.*
(iii) *If* Ω *is replaced by a compact or complete, non-compact Riemannian manifold* M *with* $\partial M = \emptyset$ *then Theorem 1.1, Theorem 1.3, Theorem 1.5, and Corollary 1.6 remain to be true for the Cauchy problem (1.3). In fact, the proof is slightly simpler than the one here, since we don't need to use the Hardy inequalities.*
(iv) *Schoen [35] proved convexity of the Dirichlet energy for harmonic maps into* N *with nonpositive sectional curvature. The convexity for harmonic maps into any compact manifold* N *with small renormalized energy was proved by [14]. In §3 below, we will show the convexity for biharmonic maps with small renormalized* E_2*-energy. Theorem 1.5 seems to be the first convexity result for the heat flow of biharmonic maps – a higher-order geometric evolution equation.*
(v) *In general, it is a difficult question to ask whether the unique limit at* $t = \infty$ *holds for geometric evolution equations. Simon in his celebrated work [36] showed the unique limit at* $t = \infty$ *for smooth solutions to the heat flow of harmonic maps into a real analytic manifold* (N, h)*. Corollary 1.6 seems to be first result on the unique limit at time infinity for the heat flow of biharmonic maps.*

A natural class of weak solutions satisfying the smallness condition (1.9) are Serrin's (p, q)-solutions. We say a weak solution $u \in W_2^{1,2}(\Omega \times [0, T], N)$ to (1.2) is a Serrin's (p, q)-solution if, in addition, $\nabla^2 u \in L_t^q L_x^p(\Omega \times [0, T])$ for some $p \geq \frac{n}{2}$ and $q \leq \infty$ satisfying

$$\frac{n}{p} + \frac{4}{q} = 2. \tag{1.13}$$

In §5, we will prove that if u is a Serrin's (p, q)-solution of (1.2), with $p > \frac{n}{2}, q < \infty$ and an initial and boundary data $u_0 \in W^{2,r}(\Omega, N)$ for some $r > \frac{n}{2}$, then u satisfies (1.9) for some $p_0 > 1$. Thus, for $N = \mathbb{S}^L$, the regularity and uniqueness properties for Serrin's (p, q)-solutions to (1.2) with $p > \frac{n}{2}$, $q < \infty$ follow from Theorem 1.1 and Theorem 1.3.

For a compact Riemannian manifold N without boundary, the regularity and uniqueness properties for Serrin's (p, q)-solutions to (1.2) need to be proven by different arguments. We have

Theorem 1.8 *For* $n \geq 4$ *and* $0 < T \leq \infty$*, let* $u_1, u_2 \in W_2^{1,2}(\Omega \times [0, T], N)$ *be weak solutions to (1.2), with the same initial and boundary value* $u_0 \in W^{2,2}(\Omega, N)$*. If, in addition,* $\nabla^2 u_1, \nabla^2 u_2 \in L_t^q L_x^p(\Omega \times [0, T])$ *for some* $p > \frac{n}{2}$ *and* $q < \infty$ *satisfying (1.13), then* $u_1, u_2 \in C^\infty(\Omega \times (0, T), N)$*, and* $u_1 \equiv u_2$ *in* $\Omega \times [0, T]$*.*

Remark 1.9 (i) *It is a very interesting question whether Theorem 1.8 holds for Serrin's* (p, q)*-solutions to (1.2) in the end-point case* $p = \frac{n}{2}$ *and* $q = \infty$*.*
(ii) *If* $u_0 \in W^{2,r}(M, N)$ *for some* $r > \frac{n}{2}$*, then the local existence of Serrin's* (p, q)*-solutions to (1.2), for some* $p > \frac{n}{2}$ *and* $q < \infty$*, can be shown by the fixed point argument (see, e.g. [8] §4).*

In dimension $n = 4$, by applying Theorem 5.2 (with $p = 2$ $(= \frac{n}{2})$ and $q = \infty$) and the second half of the proof of Theorem 1.3, we obtain the following uniqueness result.

Corollary 1.10 *For $n = 4$ and $0 < T \leq \infty$, there exists $\epsilon_1 > 0$ such that if u_1* and *$u_2 \in W_2^{1,2}(\Omega \times [0,T], N)$ are weak solutions of (1.2), under the same initial and boundary value $u_0 \in W^{2,2}(\Omega, N)$, satisfying*

$$\limsup_{t \downarrow t_0^+} E_2(u_i(t)) \leq E_2(u_i(t_0)) + \epsilon_1, \ \forall \ t_0 \in [0, T), \tag{1.14}$$

for $i = 1, 2$. Then $u_1 \equiv u_2$ in $\Omega \times [0, T)$. In particular, the uniqueness holds among the class of weak solutions of (1.2), whose E_2-energy is monotone decreasing for $t \geq 0$.

We would like to point out that for the Cauchy problem (1.3) of heat flow of biharmonic maps on a compact 4-dimensional Riemannian manifold M without boundary, Corollary 1.10 has been recently proven by Rupflin [32] through a different argument.

Concerning convexity and unique limit of (1.2) at $t = \infty$ in dimension $n = 4$, we have

Corollary 1.11 *For $n = 4$, there exist $\epsilon_2 > 0$ and $T_1 > 0$ such that if $u \in W_2^{1,2}(\Omega \times (0, +\infty), N)$ is a weak solution of (1.2), with the initial-boundary value $u_0 \in W^{2,2}(\Omega, N)$, satisfying*

$$E_2(u(t)) \leq \epsilon_2^2, \ \forall \ t \geq 0, \tag{1.15}$$

then (i) $E_2(u(t))$ is monotone decreasing for $t \geq T_1$;
(ii) for $t_2 \geq t_1 \geq T_2$, it holds

$$\int_\Omega |\nabla^2(u(t_1) - u(t_2))|^2 \leq C \left(E_2(u(t_1)) - E_2(u(t_2))\right)$$

for some $C = C(\epsilon_2) > 0$; and
(iii) there exists a biharmonic map $u_\infty \in C^\infty \cap W^{2,2}(\Omega, N)$, with $(u_\infty, \frac{\partial u_\infty}{\partial \nu}) = (u_0, \frac{\partial u_0}{\partial \nu})$ on $\partial\Omega$, such that $\lim_{t\to\infty} \|u(t) - u_\infty\|_{W^{2,2}(\Omega)} = 0$, and for any $m \geq 1$, $K \subset\subset \Omega$, $\lim_{t\to\infty} \|u(t) - u_\infty\|_{C^m(K)} = 0$.

It is easy to see that the condition (1.15) holds for any solution $u \in W_2^{1,2}(\Omega \times [0, \infty), N)$ to (1.2), if $E_2(u(t)) \leq E_2(u_0)$ for $t \geq 0$ (e.g. the solution constructed by [9] and [44]) and $E_2(u_0) \leq \epsilon_2^2$.

The paper is written as follows. In §2, we will prove the ϵ-regularity Theorem 1.1 for weak solutions to (1.2) under the assumption (1.5). In §3, we will show both convexity and uniqueness property for biharmonic maps with small E_2-energy. In §4, we will prove the uniqueness Theorem 1.3, the convexity Theorem 1.5, and the unique limit Theorem 1.6. In §5, we will discuss Serrin's (p, q)-solutions to (1.2). We will prove Theorem 1.8 on both regularity and uniqueness of Serrin's solutions, Corollary 1.10, and Corollary 1.11. In §6 Appendix, we will sketch a proof for the higher-order regularity of heat flow of biharmonic maps.

2 ϵ-regularity

This section is devoted to the proof of Theorem 1.1, i.e. the regularity of heat flow of biharmonic maps to $\mathbb{S}^L$ under the smallness condition (1.5). The idea is motivated by that of [4] on the regularity theorem of stationary biharmonic maps to $\mathbb{S}^L$.

The first step is to rewrite (1.4) into the form where nonlinear terms are of divergence structures, analogous to the equation of biharmonic maps to $\mathbb{S}^L$ discovered by [4]. There are four types of nonlinearities with divergence structures appearing in (1.4):

$$\begin{aligned} T_1 &= \left(u_j^\alpha \Delta u^\beta (u^\beta - c^\beta)\right)_j \text{ or } \left((u^\alpha - c^\alpha) u_i^\beta u_{ij}^\beta\right)_j \\ T_2 &= \Delta\left((u^\alpha - c^\alpha)|\nabla u^\beta|^2\right), \ \Delta\left((u^\beta - c^\beta)\Delta u^\beta\right), \text{ or } \Delta\left(u^\alpha (u^\beta - c^\beta)\Delta u^\beta\right) \\ T_3 &= \left((u^\beta - c^\beta) u_j^\beta\right)_{jii} \\ T_4 &= \left(u^\alpha u_t^\beta - u^\beta u_t^\alpha\right)\left(u^\beta - c^\beta\right), \end{aligned} \tag{2.1}$$

where the upper index $\alpha, \ \beta$, etc, denotes the component of a vector function, the lower index $i, \ j$, etc, denotes the differentiation in the direction $x_i, \ x_j$, etc, and $c^\alpha \in \mathbb{R}^{L+1}$ is a bounded constant.

Lemma 2.1 *The equation of heat flow of biharmonic maps (1.4) is equivalent to*

$$u_t + \Delta^2 u = \mathcal{F}(T_1, T_2, T_3, T_4) := \textit{a linear combination of terms of } T_1, T_2, T_3 \textit{ and } T_4, \tag{2.2}$$

whose coefficients can be bounded independent of u.

Proof. To prove (2.2), we follow [4] Proposition 1.2 closely. First, by Lemma 1.3 of [4], we have that, for every fixed α,

$$c^\alpha \Delta\left(|\nabla u^\beta|^2\right) \text{ and } \left(u_j^\alpha |\nabla u^\beta|^2\right)_j \text{ are linear combination of terms of } T_1, T_2, T_3. \tag{2.3}$$

Set

$$S_1 = u^\alpha |\Delta u^\beta|^2, \ S_2 = 2u^\alpha u_j^\beta \left(\Delta u^\beta\right)_j, \ S_3 = u^\alpha \Delta\left(|\nabla u^\beta|^2\right). \tag{2.4}$$

Differentiation of $|u| = 1$ gives

$$u^\alpha u_j^\alpha = 0, \ u^\beta \Delta u^\beta + |\nabla u^\beta|^2 = 0. \tag{2.5}$$

By (1.2), we have

$$u^\alpha \Delta^2 u^\beta + u^\alpha u_t^\beta = u^\beta \Delta^2 u^\alpha + u^\beta u_t^\alpha. \tag{2.6}$$

It follows from (2.5) and (2.6) that

$$\begin{aligned} \frac{S_2}{2} =& u^\alpha u_j^\beta (\Delta u^\beta)_j \\ =& u_j^\beta \left(u^\alpha (\Delta u^\beta)_j - u^\beta \left(\Delta u^\alpha\right)_j\right) \\ =& u_j^\beta \left(u^\alpha (\Delta u^\beta)_j - u^\beta \left(\Delta u^\alpha\right)_j - u_j^\alpha \Delta u^\beta + u_j^\beta \Delta u^\alpha\right) + u_j^\beta \left(u_j^\alpha \Delta u^\beta - u_j^\beta \Delta u^\alpha\right) \\ =& \left\{\left(u^\beta - c^\beta\right)\left(u^\alpha (\Delta u^\beta)_j - u^\beta \left(\Delta u^\alpha\right)_j - u_j^\alpha \Delta u^\beta + u_j^\beta \Delta u^\alpha\right)\right\}_j \\ &+ \left(u^\beta - c^\beta\right)\left(u^\alpha u_t^\beta - u^\beta u_t^\alpha\right) + u_j^\beta \left(u_j^\alpha \Delta u^\beta - u_j^\beta \Delta u^\alpha\right) \\ =& \left\{\left(u^\beta - c^\beta\right)\left(u^\alpha \Delta u^\beta - u^\beta \Delta u^\alpha\right)\right\}_{jj} - \left\{u_j^\beta \left(u^\alpha \Delta u^\beta - u^\beta \Delta u^\alpha\right)\right\}_j \\ &- 2\left\{\left(u^\beta - c^\beta\right)\left(u_j^\alpha \Delta u^\beta - u_j^\beta \Delta u^\alpha\right)\right\}_j + u_j^\beta \left(u_j^\alpha \Delta u^\beta - u_j^\beta \Delta u^\alpha\right) + T_4 \\ =& -\left\{u_j^\beta \left(u^\alpha \Delta u^\beta - u^\beta \Delta u^\alpha\right)\right\}_j + u_j^\beta \left(u_j^\alpha \Delta u^\beta - u_j^\beta \Delta u^\alpha\right) + T_1 + T_2 + T_4 \text{ terms.} \end{aligned} \tag{2.7}$$

By (2.3) and (2.5), we have

$$\begin{aligned}
S_3 =& \left(u^\alpha - c^\alpha\right) \Delta \left(|\nabla u^\beta|^2\right) + c^\alpha \Delta \left(|\nabla u^\beta|^2\right) \\
=& \Delta \left(\left(u^\alpha - c^\alpha\right) |\nabla u^\beta|^2\right) - 2u_j^\alpha \left(|\nabla u^\beta|^2\right)_j - \Delta u^\alpha |\nabla u^\beta|^2 + \sum_{l=1}^{3} T_l \text{ terms} \\
=& -2\left(u_j^\alpha |\nabla u^\beta|^2\right)_j + \Delta u^\alpha |\nabla u^\beta|^2 + \sum_{l=1}^{3} T_l \text{ terms} \\
=& -\Delta u^\alpha u^\beta \Delta u^\beta + \sum_{l=1}^{3} T_l \text{ terms}.
\end{aligned} \tag{2.8}$$

By (2.8), the definition of S_1, and (2.7), we have

$$\begin{aligned}
S_1 + S_3 =& \left(u^\alpha \Delta u^\beta - u^\beta \Delta u^\alpha\right) \Delta u^\beta + \sum_{l=1}^{3} T_l \text{ terms} \\
=& \left\{\left(u^\alpha \Delta u^\beta - u^\beta \Delta u^\alpha\right) u_j^\beta\right\}_j - \left(u_j^\alpha \Delta u^\beta - u_j^\beta \Delta u^\alpha\right) u_j^\beta \\
& - \left(u^\alpha \Delta u_j^\beta - u^\beta \Delta u_j^\alpha\right) u_j^\beta + \sum_{l=1}^{3} T_l \text{ terms} \\
=& -\frac{S_2}{2} - \frac{S_2}{2} + \sum_{l=1}^{3} T_l \text{ terms}.
\end{aligned} \tag{2.9}$$

Therefore we obtain

$$S_1 + S_2 + S_3 = \text{a linear combination of } T_1, T_2, T_3, T_4 \text{ terms}.$$

This completes the proof. □

Next we recall some basic properties of the heat kernel for Δ^2 in $\mathbb{R}^n$ and the notion of Riesz potentials on $\mathbb{R}^{n+1}$. Let $b(x,t)$ be the fundamental solution of

$$(\partial_t + \Delta^2) v = 0 \text{ in } \mathbb{R}^{n+1}_+.$$

Then we have (see [20] §2.2):

$$b(x,t) = t^{-\frac{n}{4}} g\left(\frac{x}{t^{\frac{1}{4}}}\right),$$

where

$$g(\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{i\xi\eta - |\eta|^4},\ \xi \in \mathbb{R}^n,$$

and the estimates:

$$\left|\nabla^m b(x,t)\right| \le C\left(|t|^{\frac{1}{4}} + |x|\right)^{-n-m},\ \forall\ (x,t) \in \mathbb{R}^{n+1}_+,\ \forall\ m \ge 1. \tag{2.10}$$

To study (1.2), we equip $\mathbb{R}^{n+1}$ with the parabolic distance δ:

$$\delta((x,t),(y,s)) = |t-s|^{\frac{1}{4}} + |x-y|,\ (x,t),\ (y,s) \in \mathbb{R}^{n+1}.$$

We define the Riesz potential of order α in $\mathbb{R}^{n+1}$ with respect to δ, for $0 \le \alpha \le n+4$, by

$$I_\alpha(f)(x,t) = \int_{\mathbb{R}^{n+1}} \left(|t-s|^{\frac{1}{4}} + |x-y|\right)^{\alpha-n-4} |f|(y,s),\ (x,t) \in \mathbb{R}^{n+1}. \tag{2.11}$$

Now we are ready to prove the ϵ-regularity property for heat flow of biharmonic maps to $\mathbb{S}^L$.

Proposition 2.2 *For any $1 < p \le 2$, there exists $\epsilon_p > 0$ such that if $u : P_4 \to \mathbb{S}^L$ is a weak solution of (1.4) and satisfies*

$$\sup_{(x,t)\in P_2, 0<r\le 2} r^{2p-n-4} \int_{P_r(x,t)} \left(|\nabla^2 u|^p + r^{2p}|\partial_t u|^p\right) \le \epsilon_p^p, \tag{2.12}$$

then $u \in C^\infty(P_{\frac{1}{2}}, \mathbb{S}^L)$, and

$$\left\|\nabla^m u\right\|_{C^0(P_{\frac{1}{2}})} \le C(p,n,m),\ \forall\ m \ge 1. \tag{2.13}$$

Proof. We first establish Hölder continuity of u in $P_{\frac{3}{4}}$. It is based on the decay estimate:
Claim. There exist $\epsilon_p > 0$, $q > 1$, and $\theta_0 \in (0, \frac{1}{2})$ such that for any $z_0 = (x_0, t_0) \in P_1$ and $0 < r \le 1$,

$$⨍_{P_{\theta_0 r}(z_0)} |u - u_{P_{\theta_0 r}(z_0)}|^q \le \frac{1}{2} ⨍_{P_r(z_0)} |u - u_{P_r(z_0)}|^q, \tag{2.14}$$

where $u_{P_r(z_0)} = ⨍_{P_r(z_0)} u,\ 0 < r \le 1$, denotes the average of u over $P_r(z_0)$.

By translation and scaling, it suffices to show (2.14) for $z_0 = (0,0)$ and $r = 2$. First, we need to extend u from P_1 to $\mathbb{R}^{n+1}$. Let the extension, still denoted by u, be such that

$$|u| \le 2 \text{ in } \mathbb{R}^{n+1}, \quad u = 0 \text{ outisde } P_2,$$

and

$$\int_{\mathbb{R}^{n+1}} |\nabla^2 u|^p + |\partial_t u|^p \lesssim \int_{P_2} |\nabla^2 u|^p + |\partial_t u|^p.$$

For $1 \le l \le 4$, let $w_l : \mathbb{R}^{n+1}_+ \to \mathbb{R}^{L+1}$ be the solution to the equation:

$$\begin{aligned} \partial_t w_l + \Delta^2 w_l &= T_l \quad \text{in } \ \mathbb{R}^{n+1}_+ \\ w_l &= 0 \quad \text{ on } \ \mathbb{R}^n \times \{0\}. \end{aligned} \tag{2.15}$$

Define $v : P_1 \to \mathbb{R}^{L+1}$ by

$$v = u - \mathcal{F}(w_1, w_2, w_3, w_4).$$

Here $\mathcal{F}$ is the linear combination of $w_1, \ldots, w_4$ given by Lemma 2.1. By (2.2) we have

$$\partial_t v + \Delta^2 v = 0 \ \text{ in } \ P_1. \tag{2.16}$$

It follows from (2.15) and the Duhamel formula that for $1 \le l \le 4$,

$$w_l(x,t) = \int_{\mathbb{R}^n \times [0,t]} b(x-y, t-s) T_l(y,s),\ (x,t) \in \mathbb{R}^{n+1}_+. \tag{2.17}$$

Set $c^\alpha = u^\alpha_{P_2}$ in (2.1). Considering $T_1 = \Big((u^\alpha - u^\alpha_{P_2})u^\beta_i u^\beta_{ij}\Big)_j$ (other forms of T_1 can be handled similarly), we obtain

$$
\begin{aligned}
|w_1(x,t)| =& \left| \int_{\mathbb{R}^n\times[0,t]} \nabla_j b(x-y,t-s)(u^\alpha - u^\alpha_{P_2})u^\beta_i u^\beta_{ij}(y,s) \right| \\
\lesssim& \int_{\mathbb{R}^{n+1}} \Big(|t-s|^{\frac14} + |x-y|\Big)^{-n-1} |u-u_{P_2}||\nabla u||\nabla^2 u|(y,s) \\
\lesssim& I_3\left(\chi_{P_2}|u-u_{P_2}||\nabla u||\nabla^2 u|\right)(x,t),
\end{aligned}
\tag{2.18}
$$

where χ_{P_2} is the characteristic function of P_2.

By the estimates of Riesz potentials in L^q-spaces (see also §5 below), we have that for any $f \in L^q$, $1 < q < +\infty$, $I_\alpha(f) \in L^{\tilde q}$, where $\frac{1}{\tilde q} = \frac1q - \frac{\alpha}{n+4}$. If $\tilde{q_1},\ q_1 > 1$ satisfy

$$
\frac{1}{\tilde{q_1}} = \frac1p + \frac{1}{2p} + \frac{1}{q_1} - \frac{3}{n+4},
$$

then

$$
\Big\|w_1\Big\|_{L^{\tilde{q_1}}(P_2)} \le C\Big\|u-u_{P_2}\Big\|_{L^{q_1}(P_2)}\Big\|\nabla u\Big\|_{L^{2p}(P_2)}\Big\|\nabla^2 u\Big\|_{L^p(P_2)} \le C\epsilon_p\Big\|u-u_{P_2}\Big\|_{L^{q_1}(P_2)}. \tag{2.19}
$$

For $T_2 = \Delta\left((u^\alpha - u^\alpha_{P_2})|\nabla u^\beta|^2\right)$ (other forms of T_2 can be handled similarly), we obtain

$$
\begin{aligned}
|w_2(x,t)| =& \left| \int_{\mathbb{R}^n\times[0,t]} \Delta b(x-y,t-s)(u^\alpha - u^\alpha_{P_2})|\nabla u^\beta|^2(y,s) \right| \\
\lesssim& \int_{\mathbb{R}^{n+1}} \Big(|t-s|^{\frac14} + |x-y|\Big)^{-n-2} |u-u_{P_2}||\nabla u|^2(y,s) \\
\lesssim& I_2\left(\chi_{P_2}|u-u_{P_2}||\nabla u|^2\right)(x,t).
\end{aligned}
\tag{2.20}
$$

If $\tilde{q}_2,\ q_2 > 1$ satisfy

$$
\frac{1}{\tilde{q}_2} = \frac1p + \frac{1}{q_2} - \frac{2}{n+4},
$$

then

$$
\Big\|w_2\Big\|_{L^{\tilde{q}_2}(P_2)} \le C\Big\|u-u_{P_2}\Big\|_{L^{q_2}(P_2)}\Big\||\nabla u|^2\Big\|_{L^p(P_2)} \le C\epsilon_p\Big\|u-u_{P_2}\Big\|_{L^{q_2}(P_2)}. \tag{2.21}
$$

For $T_3 = \Big((u^\beta - u^\beta_{P_2})u^\beta_j\Big)_{jii}$, we obtain

$$
\begin{aligned}
|w_3(x,t)| =& \left| \int_{\mathbb{R}^n\times[0,t]} \Delta b_j(x-y,t-s)(u^\beta - u^\beta_{P_2})u^\beta_j(y,s) \right| \\
\lesssim& \int_{\mathbb{R}^{n+1}} \Big(|t-s|^{\frac14} + |x-y|\Big)^{-n-3} |u-u_{P_2}||\nabla u|(y,s) \\
\lesssim& I_1\left(\chi_{P_2}|u-u_{P_2}||\nabla u|\right).
\end{aligned}
\tag{2.22}
$$

If $\tilde{q}_3,\ q_3 > 1$ satisfy

$$
\frac{1}{\tilde{q}_3} = \frac{1}{2p} + \frac{1}{q_3} - \frac{1}{n+4},
$$

then

$$\Big\|w_3\Big\|_{L^{\tilde q_3}(P_2)} \le C\Big\|u-u_{P_2}\Big\|_{L^{q_3}(P_2)}\Big\|\nabla u\Big\|_{L^{2p}(P_2)} \le C\epsilon_p\Big\|u-u_{P_2}\Big\|_{L^{q_3}(P_2)}. \tag{2.23}$$

For T_4, we have

$$\partial_t w_4 + \Delta^2 w_4 = \left(u^\alpha u_t^\beta - u^\beta u_t^\alpha\right)\left(u^\beta - u_{P_2}^\beta\right). \tag{2.24}$$

Multiplying (2.24) by w_4, integrating over $\mathbb{R}^n$ and using the Hölder inequality, we obtain

$$\sup_{-2^4\le t\le 0}\int_{\mathbb{R}^n}|w_4|^2 + \int_{\mathbb{R}^n\times[-2^4,0]}|\nabla^2 w_4|^2 \le C\Big\|\partial_t u\Big\|_{L^p(P_2)}\Big\|u-u_{P_2}\Big\|_{L^{q_4}(P_2)}\Big\|w_4\Big\|_{L^{\frac{2(n+4)}{n}}(P_2)}, \tag{2.25}$$

where $\dfrac{1}{p}+\dfrac{1}{q_4}+\dfrac{n}{2(n+4)}=1$.
By interpolation and the Sobolev's inequality, we have

$$\Big\|w_4\Big\|^2_{L^{\frac{2(n+4)}{n}}(P_2)} \le C\left(\sup_{-2^4\le t\le 0}\int_{\mathbb{R}^n}|w_4|^2 + \int_{\mathbb{R}^n\times[-2^4,0]}|\nabla^2 w_4|^2\right). \tag{2.26}$$

Combining (2.26) with (2.25), we obtain

$$\Big\|w_4\Big\|_{L^{\frac{2(n+4)}{n}}(P_2)} \le C\Big\|\partial_t u\Big\|_{L^p(P_2)}\Big\|u-u_{P_2}\Big\|_{L^{q_4}(P_2)} \le C\epsilon_p\Big\|u-u_{P_2}\Big\|_{L^{q_4}(P_2)}. \tag{2.27}$$

Let

$$q=\max\{q_1,q_2,q_3,q_4\}>1 \text{ and } \tilde q=\min\{\tilde q_1,\tilde q_2,\tilde q_3,\tilde q_4\}>1.$$

By (2.19), (2.21), (2.23) and (2.27), we have

$$\sum_{1\le l\le 4}\Big\|w_l\Big\|_{L^{\tilde q}(P_2)} \le C\epsilon_p\Big\|u-u_{P_2}\Big\|_{L^q(P_2)}. \tag{2.28}$$

On the other hand, by the standard estimate on v, we have that for any $0<\theta<1$,

$$\left(\fint_{P_\theta}|v-v_{P_\theta}|^{\tilde q}\right)^{\frac{1}{\tilde q}} \le C\theta\left(\fint_{P_1}|v-v_{P_1}|^{\tilde q}\right)^{\frac{1}{\tilde q}} \le C\theta\left(\fint_{P_1}|u-u_{P_2}|^{\tilde q}+\sum_{l=1}^4|w_l|^{\tilde q}\right)^{\frac{1}{\tilde q}}. \tag{2.29}$$

Adding (2.28) and (2.29) together, we obtain

$$\begin{aligned}\left(\fint_{P_\theta}|u-u_{P_\theta}|^{\tilde q}\right)^{\frac{1}{\tilde q}} \le& C\left(\theta^{-\frac{n+4}{\tilde q}}\epsilon_p+\theta\right)\left(\fint_{P_2}|u-u_{P_2}|^q\right)^{\frac{1}{q}}\\ \le&\frac{1}{2}\left(\fint_{P_2}|u-u_{P_2}|^q\right)^{\frac{1}{q}},\end{aligned} \tag{2.30}$$

where we have chosen θ and ϵ_p so that

$$C\left(\theta^{-\frac{n+4}{\tilde q}}\epsilon_p+\theta\right)\le\frac{1}{2}.$$

It follows from (2.12) that $u\in \mathrm{BMO}(P_2)$ and

$$[u]_{\mathrm{BMO}(P_2)} := \left\{\fint_{P_r(z)}|u-u_{P_r(z)}|:\ P_r(z)\subset P_2\right\}\le C\epsilon_p. \tag{2.31}$$

Hence, by the John-Nirenberg inequality (see [18]), we have that for any $0 < r < 1$,

$$\left(\fint_{P_r} |u - u_{P_r}|^q\right)^{\frac{1}{q}} \le C(\epsilon_p, q, \tilde{q}) \left(\fint_{P_r} |u - u_{P_r}|^{\tilde{q}}\right)^{\frac{1}{\tilde{q}}}. \tag{2.32}$$

Combining (2.32) with (2.30) completes the proof of the Claim.

It is standard that iterating (2.14) yields the Hölder continuity of u by using the Campanato theory [3]. The higher-order regularity then follows from the hole-filling type argument and the bootstrap argument, which will be sketched in Proposition 6.1 of §6 Appendix. After this, we have that $u \in C^\infty(P_{\frac{1}{2}}, \mathbb{S}^L)$ and the estimate (2.13) holds. □

Proof of Theorem 1.1. By the definition of Morrey spaces, for $z_0 = (x_0, t_0) \in \Omega \times (0, T)$ and $R_0 \le \frac{1}{2} \min\{d(x_0, \partial\Omega), \sqrt{t_0}\}$, we have

$$\sup_{z=(x,t)\in P_{\frac{R_0}{2}}(z_0),\ r\le\frac{R_0}{2}} r^{2p-(n+4)} \int_{P_r(z)} (|\nabla^2 u|^p + r^{2p}|\partial_t u|^p) \le \epsilon_p^p. \tag{2.33}$$

Consider $v(x, t) = u(x_0 + \frac{R_0}{8}x, t_0 + (\frac{R_0}{8})^4 t) : P_4 \to \mathbb{S}^L$. It is easy to check that v is a weak solution of (1.4) and satisfies (2.12). Hence Proposition 2.2 implies that $v \in C^\infty(P_{\frac{1}{2}}, \mathbb{S}^L)$ and satisfies (2.13). After rescaling, we see that $u \in C^\infty(P_{\frac{R_0}{16}}(z_0), \mathbb{S}^L)$ and the estimate (1.6) holds. □

Since biharmonic maps are steady solutions of the heat flow of biharmonic maps, as a direct consequence of Theorem 1.1 we have the following ϵ-regularity for biharmonic maps to $\mathbb{S}^L$.

Theorem 2.3 *For $1 < p \le 2$, there exist $\epsilon_p > 0$ and $r_0 > 0$ such that if $u \in W^{2,p}(\Omega, \mathbb{S}^L)$ is a weak solution of (1.1) and satisfies*

$$\sup_{x\in\Omega} \sup_{0<r\le\min\{r_0, d(x,\partial\Omega)\}} r^{2p-n} \int_{B_r(x)} |\nabla^2 u|^p \le \epsilon_p^p, \tag{2.34}$$

then $u \in C^\infty(\Omega, \mathbb{S}^L)$, and

$$|\nabla^m u(x)| \le C\epsilon_p\Big(\frac{1}{r_0^m} + \frac{1}{d^m(x, \partial\Omega)}\Big),\ \forall\ m \ge 1. \tag{2.35}$$

Remark 2.4 *For $p = 2$, Theorem 2.3 was first proved by Chang-Wang-Yang [4]. For biharmonic maps into any compact Riemannian manifold N without boundary, Theorem 2.3 was proved by [41, 43] for $p = 2$.*

3 Convexity and uniqueness of biharmonic maps

We will outline the convexity and uniqueness properties for biharmonic maps with small energy, which are the second-order extensions of the corresponding theorems on harmonic maps with small energy by Struwe [37], Moser [26], and Huang-Wang [14].

Consider the Dirichlet boundary value problem for a biharmonic map $u \in W^{2,2}(\Omega, N)$:

$$\begin{cases} \Delta^2 u = \mathcal{N}_{\mathrm{bh}}[u] & \text{in} \quad \Omega \\ \left(u, \frac{\partial u}{\partial \nu}\right) = \left(u_0, \frac{\partial u_0}{\partial \nu}\right) & \text{on} \quad \partial\Omega. \end{cases} \tag{3.1}$$

where $u_0 \in W^{2,2}(\Omega, N)$ given.

We recall the higher-order Hardy inequality.

Lemma 3.1 *There is $C > 0$ depending only on n and Ω such that if $f \in W_0^{2,2}(\Omega)$, then*

$$\int_\Omega \frac{|f(x)|^2}{d^4(x, \partial\Omega)} \le C \int_\Omega |\nabla^2 f(x)|^2. \tag{3.2}$$

Proof. For simplicity, we indicate a proof for the case $\Omega = B_1$ – the unit ball in $\mathbb{R}^n$. The interested readers can refer to [5] for a proof of general domains. By approximation, we may assume $f \in C_0^\infty(B_1)$. Writing the left side of (3.2) in spherical coordinates, integrating over B_1, and using the Hölder inequality, we obtain

$$\begin{aligned} \int_{B_1} \frac{|f(x)|^2}{(1-|x|)^4} &= \int_0^1 \int_{\mathbb{S}^{n-1}} \frac{|f|^2(r,\theta)}{(1-r)^4} r^{n-1}\, dH^{n-1}(\theta) dr \\ &= -\int_0^1 \int_{\mathbb{S}^{n-1}} \frac{1}{3(1-r)^3} \left(2 f f_r r^{n-1} + |f|^2 (n-1) r^{n-2}\right) dH^{n-1}(\theta) dr \\ &\le -\int_0^1 \int_{\mathbb{S}^{n-1}} \frac{2}{3(1-r)^3} f f_r r^{n-1}\, dH^{n-1}(\theta) dr \\ &\le C \int_0^1 \int_{\mathbb{S}^{n-1}} \frac{|f||f_r| r^{n-1}}{(1-r)^3}\, dH^{n-1}(\theta) dr \\ &\le C \int_{B_1} \frac{|f(x)||\nabla f(x)|}{(1-|x|)^3} \\ &\le C \left(\int_{B_1} \frac{|f(x)|^2}{(1-|x|)^4}\right)^{\frac{1}{2}} \left(\int_{B_1} \frac{|\nabla f(x)|^2}{(1-|x|)^2}\right)^{\frac{1}{2}}. \end{aligned} \tag{3.3}$$

Thus, by using the first-order Hardy inequality, we obtain

$$\int_{B_1} \frac{|f(x)|^2}{(1-|x|)^4} \le C \int_{B_1} \frac{|\nabla f(x)|^2}{(1-|x|)^2} \le C \int_{B_1} |\nabla^2 f(x)|^2. \tag{3.4}$$

This yields (3.2). □

Now we introduce the Morrey spaces in $\mathbb{R}^n$. For $1 \le l < +\infty$, $0 < \lambda \le n$, and $0 < R \le +\infty$, $f \in M_R^{l,\lambda}(\Omega)$ if and only if $f \in L^l_{\mathrm{loc}}(\Omega)$ satisfies

$$\|f\|^l_{M_R^{l,\lambda}(\Omega)} := \sup_{x \in \Omega} \sup_{0 < r \le \min\{R, d(x,\partial\Omega)\}} \left\{ r^{\lambda - n} \int_{B_r(x)} |f|^l \right\} < +\infty.$$

We have the convexity property of biharmonic maps with small energy.

Theorem 3.2 *For $n \geq 4$, $\delta \in (0,1)$, and $1 < p \leq 2$, there exist $\epsilon_p = \epsilon(p,\delta) > 0$ and $R_p = R(p,\delta) > 0$ such that if $u \in W^{2,2}(\Omega, N)$ is a biharmonic map satisfying either*
(i) $\|\nabla^2 u\|_{M^{2,4}_{R_2}(\Omega)} \leq \epsilon_2$, when N is a compact Riemannian manifold without boundary, or
(ii) $\|\nabla^2 u\|_{M^{p,2p}_{R_p}(\Omega)} \leq \epsilon_p$, when $N = \mathbb{S}^L$,
then

$$\int_\Omega |\Delta v|^2 \geq \int_\Omega |\Delta u|^2 + (1-\delta) \int_\Omega |\nabla^2 (v-u)|^2 \tag{3.5}$$

holds for any $v \in W^{2,2}(\Omega, N)$ with $\left(v, \frac{\partial v}{\partial \nu}\right) = \left(u, \frac{\partial u}{\partial \nu}\right)$ on $\partial\Omega$.

Proof. First, it follows from Theorem 2.3 for $N = \mathbb{S}^L$ or the regularity theorem by Wang [43] that if $\epsilon_p > 0$ is sufficiently small then $u \in C^\infty(\Omega, N)$, and

$$|\nabla^m u(x)| \leq C\epsilon_p \left(\frac{1}{R_p^m} + \frac{1}{d^m(x, \partial\Omega)}\right), \ \forall\, x \in \Omega, \ \forall\, m \geq 1. \tag{3.6}$$

For $y \in N$, let $P^\perp(y) : \mathbb{R}^{L+1} \to (T_y N)^\perp$ denote the orthogonal projection map from $\mathbb{R}^{L+1}$ to the normal space of N at y. Since N is compact, a simple geometric argument implies that there exists $C > 0$ depending on N such that

$$\left|P^\perp(y)(z-y)\right| \leq C|z-y|^2, \ \forall z \in N. \tag{3.7}$$

Since

$$\mathcal{N}_{\mathrm{bh}}[u] \perp T_u N,$$

it follows from (3.7) that multiplying (1.1) by $(u-v)$ and integrating over Ω yields

$$\begin{aligned} \int_\Omega \Delta u \cdot \Delta(u-v) &= \int_\Omega \mathcal{N}_{\mathrm{bh}}[u] \cdot (u-v) \\ &\lesssim \int_\Omega [|\nabla u|^2 |\nabla^2 u| + |\nabla^2 u|^2 + |\nabla u||\nabla^3 u|]|u-v|^2 \\ &\lesssim \epsilon_p^4 \int_\Omega \frac{|u-v|^2}{R_p^4} + \frac{|u-v|^2}{d^4(x,\partial\Omega)} \\ &\lesssim \epsilon_p \int_\Omega |\nabla^2(u-v)|^2, \end{aligned} \tag{3.8}$$

where we choose $R_p \geq \epsilon_p$, and use (3.6), the Poincaré inequality, and the Hardy inequality (3.2) during the last two steps.

It follows from (3.8) that

$$\int_\Omega |\Delta v|^2 - \int_\Omega |\Delta u|^2 - \int_\Omega |\Delta u - \Delta v|^2 = 2\int_\Omega \Delta u \cdot \Delta(v-u) \geq -C\epsilon_p \int_\Omega |\nabla^2(u-v)|^2. \tag{3.9}$$

Since $(u-v) \in W_0^{2,2}(\Omega)$, we have

$$\int_\Omega |\Delta u - \Delta v|^2 = \int_\Omega |\nabla^2(u-v)|^2,$$

and hence

$$\int_\Omega |\Delta v|^2 - \int_\Omega |\Delta u|^2 \ge (1 - C\epsilon_p) \int_\Omega |\nabla^2(u-v)|^2.$$

This yields (3.5), if $\epsilon_p > 0$ is chosen so that $C\epsilon_p \le \delta$. □

Corollary 3.3 *For $n \ge 2$ and $1 < p \le 2$, there exist $\epsilon_p > 0$ and $R_p > 0$ such that if $u_1, u_2 \in W^{2,2}(\Omega, N)$ are biharmonic maps, with $u_1 - u_2 \in W_0^{2,2}(\Omega, \mathbb{R}^{L+1})$, satisfying either*
(i) $\max_{i=1,2} \|\nabla^2 u_i\|_{M^{2,4}_{R_2}(\Omega)} \le \epsilon_2$, when N is a compact Riemannian manifold without boundary, or
(ii) $\max_{i=1,2} \|\nabla^2 u_i\|_{M^{p,2p}_{R_p}(\Omega)} \le \epsilon_p$, when $N = \mathbb{S}^L$,
then $u_1 \equiv u_2$ in Ω.

Proof. Choose $\delta = \frac{1}{2}$, apply Theorem 3.2 to u_1 and u_2 by choosing sufficiently small $\epsilon_p > 0$ and $R_p > 0$. We have

$$\int_\Omega |\Delta u_2|^2 \ge \int_\Omega |\Delta u_1|^2 + \frac{1}{2}\int_\Omega |\nabla^2(u_2 - u_1)|^2,$$

and

$$\int_\Omega |\Delta u_1|^2 \ge \int_\Omega |\Delta u_2|^2 + \frac{1}{2}\int_\Omega |\nabla^2(u_1 - u_2)|^2.$$

Adding these two inequalities together yields $\int_\Omega |\nabla^2(u_1 - u_2)|^2 = 0$. This, combined with $u_1 - u_2 \in W_0^{2,2}(\Omega)$, implies $u_1 \equiv u_2$ in Ω. □

4 Uniqueness and convexity of heat flow of biharmonic maps

This section is devoted to the proof of uniqueness, convexity, and unique limit at $t = \infty$ for (1.2) of the heat flow of biharmonic maps, i.e. Theorem 1.3, Theorem 1.5, and Corollary 1.6.

Proof of Theorem 1.3. First, by Theorem 1.1, we have that for $i = 1, 2$, $u_i \in C^\infty(\Omega \times (0, T), \mathbb{S}^L)$, and

$$\left|\nabla^m u_i(x,t)\right| \le C\epsilon_p \left(\frac{1}{R_p^m} + \frac{1}{d^m(x, \partial\Omega)} + \frac{1}{t^{\frac{m}{4}}}\right), \ \forall (x,t) \in \Omega \times (0,T), \ \forall\, m \ge 1. \tag{4.1}$$

Set $w = u_1 - u_2$. Then w satisfies

$$\begin{cases} \partial_t w + \Delta^2 w = \mathcal{N}_{\mathrm{bh}}[u_1] - \mathcal{N}_{\mathrm{bh}}[u_2] & \text{in } \Omega \times (0,T) \\ w = 0 & \text{on } \partial_p(\Omega \times (0,T)) \\ \frac{\partial w}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0,T). \end{cases} \tag{4.2}$$

Multiplying (4.2) by w and integrating over Ω, we get, by (3.7), (4.1), the Poincaré inequality and the Hardy inequality (3.2), that

$$\begin{aligned}\frac{d}{dt}\int_\Omega |w|^2 + 2\int_\Omega |\nabla^2 w|^2 &= 2\int_\Omega (\mathcal{N}_{\rm bh}[u_1] - \mathcal{N}_{\rm bh}[u_2])\cdot w \\ &\lesssim \sum_{i=1}^{2}\int_\Omega (|\nabla u_i|^2|\nabla^2 u_i| + |\nabla^2 u_i|^2 + |\nabla u_i||\nabla^3 u_i|)|w|^2 \\ &\lesssim \epsilon_p^4\int_\Omega \frac{|w(x,t)|^2}{R_p^4} + \frac{|w(x,t)|^2}{d^4(x,\partial\Omega)} + \frac{|w(x,t)|^2}{t} \\ &\lesssim \epsilon_p\int_\Omega |\nabla^2 w|^2 + \frac{\epsilon_p}{t}\int_\Omega |w|^2.\end{aligned}$$

If we choose $\epsilon_p > 0$ sufficiently small and $R_p \geq \epsilon_p$, then it holds

$$\frac{d}{dt}\int_\Omega |w|^2 \leq \frac{C\epsilon_p}{t}\int_\Omega |w|^2. \tag{4.3}$$

It follows from (4.3 that

$$\begin{aligned}\frac{d}{dt}\Big(t^{-\frac{1}{2}}\int_\Omega |w|^2\Big) &= t^{-\frac{1}{2}}\frac{d}{dt}\int_\Omega |w|^2 - \frac{1}{2}t^{-\frac{3}{2}}\int_\Omega |w|^2 \\ &\leq (C\epsilon - \frac{1}{2})t^{-\frac{3}{2}}\int_\Omega |w|^2 \leq 0. \qquad (4.4)\end{aligned}$$

Integrating this inequality from 0 to t yields

$$t^{-\frac{1}{2}}\int_\Omega |w|^2 \leq \lim_{t\downarrow 0^+} t^{-\frac{1}{2}}\int_\Omega |w|^2. \tag{4.5}$$

Since $w(\cdot,0)=0$, we have

$$w(x,t) = \int_0^t w_t(x,\tau)\,d\tau, \ \text{ a.e. } x\in\Omega,$$

and so, by the Hölder inequality,

$$t^{-\frac{1}{2}}\int_\Omega |w(x,t)|^2 \leq t^{\frac{1}{2}}\int_0^t\int_\Omega |w_t|^2(x,\tau)\,dxd\tau \leq Ct^{\frac{1}{2}} \to 0, \ \text{ as } t\downarrow 0^+.$$

This, combined with (4.5), implies $w\equiv 0$ in $\Omega\times[0,T]$. Hence $u_1\equiv u_2$. The proof is complete. □

Now we want to prove Theorem 1.5 and Corollary 1.6. To do so, we need:

Lemma 4.1 *Under the same assumptions as in Theorem 1.5, there exists $T_0 > 0$ such that $\int_\Omega |\partial_t u(t)|^2$ is monotone decreasing for $t \geq T_0$:*

$$\int_\Omega |\partial_t u|^2(t_2) + C\int_{\Omega\times[t_1,t_2]} |\nabla^2\partial_t u|^2 \leq \int_\Omega |\partial_t u|^2(t_1), \ T_0\leq t_1\leq t_2\leq T. \tag{4.6}$$

Proof. For any sufficiently small $h>0$, set

$$u^h(x,t) = \frac{u(x,t+h)-u(x,t)}{h}, \ (x,t)\in\Omega\times(0,T-h).$$

Then $u^h \in L^2([0,T-h], W_0^{2,2}(\Omega))$, $\partial_t u \in L^2(\Omega\times[0,T-h])$ and $\lim_{h\downarrow 0^+} \|u^h - \partial_t u\|_{L^2(\Omega\times[0,T-h])} = 0$. Since u satisfies (1.2), we obtain

$$\partial_t u^h + \Delta^2 u^h = \frac{1}{h}\Big(\mathcal{N}_{\rm bh}[u(t+h)] - \mathcal{N}_{\rm bh}[u(t)]\Big). \tag{4.7}$$

Multiplying (4.7) by u^h, applying (3.7), integrating over Ω, and applying (4.1), we have

$$\begin{aligned}
\frac{d}{dt}\int_\Omega |u^h|^2 + 2\int_\Omega |\Delta u^h|^2 \quad &\lesssim \quad \int_\Omega \left(|\mathcal{N}_{\rm bh}[u(t+h)]| + |\mathcal{N}_{\rm bh}[u(t)]|\right)|u^h|^2 \\
&\lesssim \quad \int_\Omega \left(|\nabla^2 u|^2 + |\nabla u||\nabla^3 u| + |\nabla u|^2|\nabla^2 u||\right)(t+h)|u^h|^2 \\
&\qquad + \int_\Omega \left(|\nabla^2 u|^2 + |\nabla u||\nabla^3 u| + |\nabla u|^2|\nabla^2 u||\right)(t)|u^h|^2 \\
&\lesssim \quad \epsilon_p^4 \int_\Omega \frac{|u^h|^2}{R_p^4} + \frac{|u^h|^2}{d^4(x,\partial\Omega)} + \frac{|u^h|^2}{T_0} \\
&\lesssim \quad \epsilon_p \int_\Omega |\nabla^2 u^h|^2
\end{aligned}$$

provided that we choose $R_p \ge \epsilon_p$ and $T_0 \ge \epsilon_p$. Since

$$\int_\Omega |\nabla^2 u^h|^2 = \int_\Omega |\Delta u^h|^2,$$

this implies

$$\frac{d}{dt}\int_\Omega |u^h|^2 + 2\int_\Omega |\nabla^2 u^h|^2 \le \left(\frac{1}{2} + C\epsilon_p\right)\int_\Omega |\nabla^2 u^h|^2. \tag{4.8}$$

Choosing $\epsilon_p > 0$ so that $C\epsilon_p \le \frac{1}{2}$, integrating on $T_0 \le t_1 \le t_2 \le T$, we have

$$\int_\Omega |u^h|^2(t_2) + C\int_{t_1}^{t_2}\int_\Omega |\nabla^2 u^h|^2 \le \int_\Omega |u^h|^2(t_1). \tag{4.9}$$

Sending $h \to 0$ yields (4.6). □

Now we can show the monotonicity of E_2-energy for heat flow of biharmonic maps for $t \ge T_0$.

Lemma 4.2 *Under the same assumptions as in Theorem 1.5, there is $T_0 > 0$ such that $\int_\Omega |\Delta u(t)|^2$ is monotone decreasing for $t \ge T_0$:*

$$\int_\Omega |\Delta u|^2(t_2) + 2\int_{\Omega\times[t_1,t_2]} |\partial_t u|^2 \le \int_\Omega |\Delta u|^2(t_1),\ T_0 \le t_1 \le t_2 \le T. \tag{4.10}$$

Proof. For $\delta > 0$, let $\eta_\delta \in C_0^\infty(\Omega)$ be such that

$$0 \le \eta_\delta \le 1,\ \eta_\delta \equiv 1 \text{ for } x \in \Omega \setminus \Omega_\delta,\ \text{and } |\nabla^m \eta_\delta| \le C\delta^{-m}.$$

Here $\Omega_\delta = \{x \in \Omega :\ d(x,\partial\Omega) \le \delta\}$. Multiplying (1.2) by $\partial_t u \eta_\delta^2$ and integrating over $\Omega\times[t_1,t_2]$, we obtain

$$\begin{aligned}
&\int_\Omega |\Delta u(t_2)|^2\eta_\delta^2 - \int_\Omega |\Delta u(t_1)|^2\eta_\delta^2 + 2\int_{t_1}^{t_2}\int_\Omega |\partial_t u|^2\eta_\delta^2 \\
=& -4\int_{t_1}^{t_2}\int_\Omega \Delta u\cdot\partial_t u\left(|\nabla\eta_\delta|^2 + \eta_\delta\Delta\eta_\delta\right) - 8\int_{t_1}^{t_2}\int_\Omega \Delta u\cdot\nabla\partial_t u\eta_\delta\nabla\eta_\delta.
\end{aligned} \tag{4.11}$$

It suffices to show the right-hand side of the above identity tends to 0 as $\delta \to 0^+$. By Lemma 4.1, we have that $\partial_t u \in L^2([T_0, T], W_0^{2,2}(\Omega))$ so that

$$
\begin{aligned}
&\int_{t_1}^{t_2}\int_\Omega |\nabla \partial_t u|^2 |\nabla \eta_\delta|^2 + |\partial_t u|^2 \left(|\nabla \eta_\delta|^4 + |\Delta \eta_\delta|^2\right) \\
\lesssim &\delta^{-2}\int_{t_1}^{t_2}\int_{\Omega_\delta} |\nabla \partial_t u|^2 + \delta^{-2}|\partial_t u|^2 \\
\lesssim &\int_{t_1}^{t_2}\int_{\Omega_\delta} |\nabla^2 \partial_t u|^2 \to 0, \text{ as } \delta \to 0.
\end{aligned} \tag{4.12}
$$

This, combined with the Hölder inequality, implies that for $t_2 \geq t_1 \geq T_0$,

$$
-4\int_{t_1}^{t_2}\int_\Omega \Delta u \cdot \partial_t u \left(|\nabla \eta_\delta|^2 + \eta_\delta \Delta \eta_\delta\right) - 8\int_{t_1}^{t_2}\int_\Omega \Delta u \cdot \nabla \partial_t u \eta_\delta \nabla \eta_\delta \to 0, \text{ as } \delta \to 0^+.
$$

Thus (4.10) holds and the proof is complete. □

Proof of Theorem 1.5. First, by Theorem 1.1, we have that $u \in C^\infty(\Omega \times (0,T], \mathbb{S}^L)$, and

$$
\left|\nabla^m u(x,t)\right| \leq C\epsilon_p \left(\frac{1}{R_p^m} + \frac{1}{d^m(x,\partial\Omega)} + \frac{1}{t^{\frac{m}{4}}}\right), \ \forall\ (x,t) \in \Omega \times (0,T),\ \forall\ m \geq 1. \tag{4.13}
$$

For $t_2 > t_1 \geq T_0$, we have

$$
\begin{aligned}
&\int_\Omega |\Delta u(t_1)|^2 - \int_\Omega |\Delta u(t_2)|^2 - \int_\Omega |\Delta u(t_1) - \Delta u(t_2)|^2 \\
=&2\int_\Omega \left(\Delta u(t_1) - \Delta u(t_2)\right)\Delta u(t_2) \\
=&-2\int_\Omega \left(u(t_1) - u(t_2)\right) u_t(t_2) \\
&+\int_\Omega \mathcal{N}_{\text{bh}}[u(t_2)] \cdot (u(t_1) - u(t_2)) \\
=&I + II.
\end{aligned} \tag{4.14}
$$

For II, applying (3.7), we obtain

$$
|\mathcal{N}_{\text{bh}}[u(t_2)] \cdot (u(t_1) - u(t_2))| \lesssim |\mathcal{N}_{\text{bh}}[u(t_2)]||u(t_1) - u(t_2)|^2.
$$

Hence, by (4.13), the Hardy inequality and the Poincaré inequality, we have

$$
\begin{aligned}
|II| \lesssim &\epsilon_p^4 \int_\Omega \left(\frac{1}{R_p^4} + \frac{1}{d^4(x,\partial\Omega)} + \frac{1}{T_0}\right)|u(t_1) - u(t_2)|^2 \\
\leq &C\epsilon_p \int_\Omega |\nabla^2(u(t_1) - u(t_2))|^2.
\end{aligned} \tag{4.15}
$$

For I, by Lemma 4.1, we have

$$
\left\|\partial_t u(t_2)\right\|_{L^2(\Omega)}^2 \leq \frac{1}{t_2 - t_1}\int_{t_1}^{t_2}\int_\Omega |\partial_t u|^2. \tag{4.16}
$$

By the Hölder inequality and (4.10), this implies

$$
\begin{aligned}
|I| &\lesssim \int_\Omega |\partial_t u(t_2)||u(t_1)-u(t_2)| \\
&\lesssim \|\partial_t u(t_2)\|_{L^2(\Omega)} \|u(t_1)-u(t_2)\|_{L^2(\Omega)} \\
&\le \sqrt{t_2-t_1}\, \|\partial_t u(t_2)\|_{L^2(\Omega)} \left(\int_{\Omega\times[t_1,t_2]} |\partial_t u|^2\right)^{\frac{1}{2}} \\
&\le \int_{\Omega\times[t_1,t_2]} |\partial_t u|^2 \le \frac{1}{2}\left[\int_\Omega |\Delta u(t_1)|^2 - \int_\Omega |\Delta u(t_2)|^2\right].
\end{aligned}
\tag{4.17}
$$

Putting (4.17) and (4.15) into (4.14) implies (1.10). This completes the proof. □

Proof of Corollary 1.6. It follows from Lemma 4.2 that $\int_\Omega |\Delta u(t)|^2$ is monotone decreasing for $t \ge T_0$. Hence

$$
c = \lim_{t\to\infty} \int_\Omega |\Delta u(t)|^2
$$

exists and is finite. Let $\{t_i\}$ be any increasing sequence such that $\lim_{i\to\infty} t_i = +\infty$. Then (1.10) implies that

$$
\int_\Omega \left|\nabla^2 (u(t_{i+j}) - u(t_i))\right|^2 \le C\Big[\int_\Omega |\Delta u(t_{i+j})|^2 - \int_\Omega |\Delta u(t_i)|^2\Big] \to 0, \text{ as } i\to\infty,
$$

for all $j \ge 1$. Thus there exists a map $u_\infty \in W^{2,2}(\Omega, \mathbb{S}^L)$, with $(u_\infty, \frac{\partial u_\infty}{\partial \nu}) = (u_0, \frac{\partial u_0}{\partial \nu})$ on $\partial\Omega$, such that

$$
\lim_{t\to\infty} \left\|u(t) - u_\infty\right\|_{W^{2,2}(\Omega)} = 0.
$$

Since (4.10) implies that there exists a sequence $t_i \to \infty$, such that

$$
\lim_{i\to\infty} \left\|\partial_t u(t_i)\right\|_{W^{2,2}(\Omega)} = 0.
$$

Thus $u_\infty \in W^{2,2}(\Omega, \mathbb{S}^L)$ is a biharmonic map. Since it also holds, for any $m \ge 1$, and any compact subset $K \subset\subset \Omega$, that

$$
\left\|u(t)\right\|_{C^m(K)} \le C(n, m, K), \ \forall t \ge 1,
$$

we conclude

$$
\lim_{t\to\infty} \left\|u(t) - u_\infty\right\|_{C^m(K)} = 0,
$$

and $u_\infty \in C^\infty(\Omega, \mathbb{S}^L)$. This completes the proof. □

5 Regularity and uniqueness of Serrin's (p, q)-solutions

In this section, we will prove Theorem 1.8 that asserts both smoothness and uniqueness for Serrin's (p, q)-solution to (1.2). First, we would like to verify

Proposition 5.1 *For $n \geq 4$, $0 < T < +\infty$, suppose $u \in W_2^{1,2}(\Omega \times [0,T], N)$ is a weak solution to (1.2), with the initial and boundary value $u_0 \in W^{2,r}(\Omega, N)$ for some $\frac{n}{2} < r < +\infty$, such that $\nabla^2 u \in L_t^q L_x^p(M \times [0,T])$ for some $p > \frac{n}{2}$ and $q < \infty$ satisfying (1.13). Then*
(i) $\partial_t u \in L_t^{\frac{q}{2}} L_x^{\frac{p}{2}}(\Omega \times [0,T])$; and
(ii) for any $\epsilon > 0$, there exists $R = R(u,\epsilon) > 0$ such that for any $1 < s < \min\{\frac{p}{2}, \frac{q}{2}\}$,

$$\sup\Big\{r^{2s-(n+4)} \int_{P_r(x,t)\cap(\Omega\times[0,T])} (|\nabla^2 u|^s + r^{2s}|\partial_t u|^s) \mid (x,t) \in \Omega \times [0,T],\ 0 < r \leq R\Big\} \leq \epsilon^s. \quad (5.1)$$

Proof. For simplicity, we will sketch the proof for $\Omega = \mathbb{R}^n$. By the Duhamel's formula, we have that $u(x,t) = u_1(x,t) + u_2(x,t)$, where

$$u_1(x,t) = \int_{\mathbb{R}^n} b(x-y,t)u_0(y), \quad (5.2)$$

$$\begin{aligned} u_2(x,t) &= \int_0^t \int_{\mathbb{R}^n} b(x-y,t-s)\mathcal{N}_{\mathrm{bh}}[u](y,s) \\ &= \int_0^t \int_{\mathbb{R}^n} b(x-y,t-s)[\nabla\cdot(\nabla(A(u)(\nabla u,\nabla u)) + 2\Delta u\cdot\nabla(P(u))) - \Delta u\cdot\Delta(P(u))](y,s). \end{aligned} \quad (5.3)$$

We proceed with two claims.
Claim 1. $\nabla^3 u \in L_t^{\frac{2q}{3}} L_x^{\frac{2p}{3}}(\mathbb{R}^n \times [0,T])$.
For u_1, we have

$$\nabla^3 u_1(x,t) = \int_{\mathbb{R}^n} \nabla_x b(x-y,t)\nabla^2 u_0(y). \quad (5.4)$$

Direct calculations, using the property of the kernel function b, yield

$$\Big\|\nabla^3 u\Big\|_{L_t^{\frac{2q}{3}} L_x^{\frac{2p}{3}}(\mathbb{R}^n\times[0,T])} \lesssim T^{\frac{1}{4}(2-\frac{n}{r})}\Big\|\nabla^2 u_0\Big\|_{L^r(\mathbb{R}^n)}. \quad (5.5)$$

For u_2, we have

$$\begin{aligned} \nabla^3 u_2(x,t) &= \int_0^t \int_{\mathbb{R}^n} \nabla_x^4 b(x-y,t-s)\Big[\nabla(A(u)(\nabla u,\nabla u)) + 2\Delta u\cdot\nabla(P(u))\Big] \\ &\quad - \int_0^t \int_{\mathbb{R}^n} \nabla_x^3 b(x-y,t-s)\Delta u\cdot\Delta(P(u))(y,s) \\ &= M_1 + M_2. \end{aligned} \quad (5.6)$$

By the Nirenberg interpolation inequality, we have $\nabla u \in L_t^{2q} L_x^{2p}(\mathbb{R}^n \times [0,T])$. By the Hölder inequality, we then have $\nabla(A(u)(\nabla u,\nabla u)) + 2\Delta u\cdot\nabla(P(u))) \in L_t^{\frac{3q}{2}} L_x^{\frac{3p}{2}}(\mathbb{R}^n \times [0,T])$. Hence, by the Calderon-Zygmund $L_t^{\tilde{q}} L_x^{\tilde{p}}$-theory, we have

$$\begin{aligned} \Big\|M_1\Big\|_{L_t^{\frac{2p}{3}} L_x^{\frac{2q}{3}}(\mathbb{R}^n\times[0,T])} &\lesssim \Big\|\nabla(A(u)(\nabla u,\nabla u)) + 2\Delta u\cdot\nabla(P(u))\Big\|_{L_t^{\frac{2p}{3}} L_x^{\frac{2q}{3}}(\mathbb{R}^n\times[0,T])} \\ &\lesssim \Big\|\nabla u\Big\|_{L_t^{2p} L_x^{2q}(\mathbb{R}^n\times[0,T])}\Big\|\nabla^2 u\Big\|_{L_t^p L_x^q(\mathbb{R}^n\times[0,T])} \\ &\lesssim 1 + \Big\|\nabla^2 u\Big\|^2_{L_t^p L_x^q(\mathbb{R}^n\times[0,T])}. \end{aligned} \quad (5.7)$$

For M_2, we have

$$|M_2|(x,t) \lesssim I_1\Big(|\nabla^2 u|^2 + |\nabla u|^4\Big)(x,t),\ (x,t) \in \mathbb{R}^n \times [0,T].$$

Recall the following estimate of $I_1(\cdot)$ (see, for example, [8] §4):

$$\Big\|I_1(f)\Big\|_{L_t^{s_2} L_x^{r_2}(\mathbb{R}^n \times [0,T])} \lesssim \Big\|f\Big\|_{L_t^{s_1} L_x^{r_1}(\mathbb{R}^n \times [0,T])}, \tag{5.8}$$

where $s_2 \geq s_1$ and $r_2 \geq r_1$ satisfy

$$\frac{n}{r_1} + \frac{4}{s_1} \leq \frac{n}{r_2} + \frac{4}{s_2} + 1. \tag{5.9}$$

Applying (5.8) to M_2, we see that $M_2 \in L_t^{\frac{2p}{3}} L_x^{\frac{2q}{3}}(\mathbb{R}^n \times [0,T])$, and

$$\Big\|M_2\Big\|_{L_t^{\frac{2p}{3}} L_x^{\frac{2q}{3}}(\mathbb{R}^n \times [0,T])} \lesssim 1 + \Big\|\nabla^2 u\Big\|^2_{L_t^p L_x^q(\mathbb{R}^n \times [0,T])}. \tag{5.10}$$

Combining these estimates on $\nabla^3 u_1, M_1$, and M_2 yields *Claim* 1.

Claim 2. $\nabla^4 u \in L_t^{\frac{q}{2}} L_x^{\frac{p}{2}}(\mathbb{R}^n \times [0,T])$. It follows from *Claim* 1 that

$$\mathcal{N}_{\text{bh}}[u] = [\Delta(A(u)(\nabla u, \nabla u)) + 2\Delta u \cdot \nabla(P(u))) - \Delta u \cdot \Delta(P(u))] \in L_t^{\frac{q}{2}} L_x^{\frac{p}{2}}(\mathbb{R}^n \times [0,T]).$$

Since

$$\nabla^4 u_2(x,t) = \int_0^t \int_{\mathbb{R}^n} \nabla_x^4 b(x-y, t-s)\mathcal{N}_{\text{bh}}[u](y,s),$$

we can apply the Calderon-Zygmund $L_t^{\tilde{q}} L_x^{\tilde{p}}$-theory again to conclude that $\nabla^4 u_2 \in L_t^{\frac{q}{2}} L_x^{\frac{p}{2}}(\mathbb{R}^n \times [0,T])$. For u_1, we have

$$\nabla^4 u_1(x,t) = \int_{\mathbb{R}^n} \nabla_x^2 b(x-y,t)\nabla^2 u_0(y).$$

Hence, by direct calculations, we have

$$\Big\|\nabla^4 u_1\Big\|_{L_t^{\frac{q}{2}} L_x^{\frac{p}{2}}(\mathbb{R}^n \times [0,T])} \lesssim T^{\frac{1}{4}(2-\frac{n}{r})}\Big\|\nabla^2 u_0\Big\|_{L^r(\mathbb{R}^n)}.$$

Combining these two estimates yields *Claim* 2.

By (1.2), it is easy to see that $\partial_t u \in L_t^{\frac{q}{2}} L_x^{\frac{p}{2}}(\mathbb{R}^n \times [0,T])$. In fact, we have

$$\begin{aligned}\Big\|\partial_t u\Big\|_{L_t^{\frac{p}{2}} L_x^{\frac{q}{2}}(\mathbb{R}^n \times [0,T])} \lesssim& \Big\|\mathcal{N}_{\text{bh}}[u] - \Delta^2 u\Big\|_{L_t^{\frac{p}{2}} L_x^{\frac{q}{2}}(\mathbb{R}^n \times [0,T])} \\ \lesssim& 1 + \Big\|\nabla^2 u\Big\|^2_{L_t^p L_x^q(\mathbb{R}^n \times [0,T])} + T^{\frac{1}{4}(2-\frac{n}{r})}\Big\|\nabla^2 u_0\Big\|_{L^r(\mathbb{R}^n)}.\end{aligned} \tag{5.11}$$

This implies (i).

(ii) follows from (i) and the Hölder inequality. In fact, for any $1 < s < \min\{\frac{p}{2}, \frac{q}{2}\}$, it holds

$$\Big(r^{2s-(n+4)} \int_{P_r(x,t)\cap(\Omega\times[0,T])} |\nabla^2 u|^s\Big)^{\frac{1}{s}} \leq \Big\|\nabla^2 u\Big\|_{L_t^q L_x^p(P_r(x,t)\cap(\Omega\times[0,T]))},$$

and

$$\left(r^{4s-(n+4)}\int_{P_r(x,t)\cap(M\times[0,T])}|\partial_t u|^s\right)^{\frac{1}{s}}\leq\left\|\partial_t u\right\|_{L_t^{\frac{q}{2}}L_x^{\frac{p}{2}}(P_r(x,t)\cap(\Omega\times[0,T]))}.$$

These two inequalities clearly imply (5.1), provided that $R=R(u,\epsilon)>0$ is chosen sufficiently small. □

Now we prove the ϵ-regularity property for Serrin's (p,q)-solutions to (1.2).

Theorem 5.2 *There exists $\epsilon_0>0$ such that if $u\in W_2^{1,2}(P_1,N)$, with $\nabla^2 u\in L_t^qL_x^p(P_1)$ for some $q\geq\frac{n}{2}$ and $p\leq\infty$ satisfying (1.13), is a weak solution of (1.2) and satisfies*

$$\left\|\nabla^2 u\right\|_{L_t^qL_x^p(P_1)}\leq\epsilon_0, \tag{5.12}$$

then $u\in C^\infty(P_{\frac{1}{2}},N)$ and

$$\|\nabla^m u\|_{C^0(P_{\frac{1}{2}})}\leq C(m,p,q,n)\|\nabla^2 u\|_{L_t^qL_x^p(P_1)},\ \forall\ m\geq 1. \tag{5.13}$$

Before proving this theorem, we recall the Serrin type inequalities and the Adams' type estimates of Riesz potential between Morrey spaces in $(\mathbb{R}^{n+1},\delta)$.

Lemma 5.3 *Assume $p\geq\frac{n}{2}$ and $q\leq\infty$ satisfies (1.13). For any $f\in L_t^qL_x^p(\Omega\times[0,T])$, $g\in L_t^2W_x^{2,2}(\Omega\times[0,T])$, and $h\in L_t^2W_x^{1,2}(\Omega\times[0,T])$, we have*

$$\int_{\Omega\times[0,T]}|f||g||h|\lesssim\|h\|_{L^2(\Omega\times[0,T])}\|g\|_{L_t^2W_x^{2,2}(\Omega\times[0,T])}^{\frac{n}{2p}}\left(\int_0^T\|f\|_{L^p(\Omega)}^q\|g\|_{L^2(\Omega)}^2\right)^{\frac{1}{q}}, \tag{5.14}$$

and

$$\int_{\Omega\times[0,T]}|f||\nabla g||h|\lesssim\|h\|_{L_t^2W_x^{1,2}(\Omega\times[0,T])}\|g\|_{L_t^2W_x^{2,2}(\Omega\times[0,T])}^{\frac{n}{2p}}\left(\int_0^T\|f\|_{L^p(\Omega)}^q\|g\|_{L^2(\Omega)}^2\right)^{\frac{1}{q}}. \tag{5.15}$$

Proof. For convenience, we sketch the proof here. By the Hölder inequality, we have

$$\int_\Omega|f||g||h|\leq\|f\|_{L^p(\Omega)}\|g\|_{L^r(\Omega)}\|h\|_{L^2(\Omega)}, \tag{5.16}$$

where $\dfrac{1}{p}+\dfrac{1}{r}=\dfrac{1}{2}$. It follows from (1.13) that $2\leq r\leq\frac{2n}{n-4}$. Hence by the Sobolev inequality we have

$$\|g\|_{L^r(\Omega)}\leq\|g\|_{L^2(\Omega)}^{\frac{2}{q}}\|g\|_{L^{\frac{2n}{n-4}}(\Omega)}^{\frac{2n}{p}}\lesssim\|g\|_{L^2(\Omega)}^{\frac{2}{q}}\|g\|_{W^{2,2}(\Omega)}^{\frac{n}{2p}}. \tag{5.17}$$

Putting (5.17) into (5.16) yields

$$\int_\Omega|f||g||h|\lesssim\|f\|_{L^p(\Omega)}\|g\|_{L^2(\Omega)}^{\frac{2}{q}}\|g\|_{W^{2,2}(\Omega)}^{\frac{n}{2p}}\|h\|_{L^2(\Omega)}. \tag{5.18}$$

Since $\dfrac{1}{q}+\dfrac{n}{4p}+\dfrac{1}{2}=1$, (5.14) follows by integrating on $[0,T]$ and the Hölder inequality.

To see (5.15), note that the Hölder inequality implies

$$\int_\Omega |f||\nabla g||h| \le \|f\|_{L^p(\Omega)}\|\nabla g\|_{L^s(\Omega)}\|h\|_{L^{\frac{2n}{n-2}}(\Omega)} \tag{5.19}$$

where $\dfrac{1}{p}+\dfrac{1}{s}+\dfrac{n-2}{2n}=1$.

Since $\dfrac{1}{s}=\dfrac{1}{n}+\dfrac{n}{2p}\left(\dfrac{1}{2}-\dfrac{2}{n}\right)+\left(1-\dfrac{n}{2p}\right)\dfrac{1}{2}$, the Nirenberg interpolation inequality implies

$$\|\nabla g\|_{L^s(\Omega)} \lesssim \|g\|_{L^2(\Omega)}^{\frac{2}{q}}\|g\|_{W^{2,2}(\Omega)}^{\frac{n}{2p}}. \tag{5.20}$$

Putting (5.20) into (5.19) and using the Sobolev inequality, we obtain

$$\int_\Omega |f||\nabla g||h| \lesssim \|f\|_{L^p(\Omega)}\|g\|_{L^2(\Omega)}^{\frac{2}{q}}\|g\|_{W^{2,2}(\Omega)}^{\frac{n}{2p}}\|h\|_{W^{1,2}(\Omega)}. \tag{5.21}$$

Since $\dfrac{1}{q}+\dfrac{n}{4p}+\dfrac{1}{2}=1$, (5.15) follows by integration on $[0,T]$ and the Hölder inequality. $\square$

Now we state the Adams' estimate for the Riesz potentials on $(\mathbb{R}^{n+1},\delta)$. Since its proof can be done exactly by the same argument as in Huang-Wang ([15] Theorem 3.1), we skip it here.

Proposition 5.4 *(i) For any $\beta>0$, $0<\lambda\le n+4$, $1<p<\frac{\lambda}{\beta}$, if $f\in L^p(\mathbb{R}^{n+1})\cap M^{p,\lambda}(\mathbb{R}^{n+1})$, then $I_\beta(f)\in L^{\tilde p}(\mathbb{R}^{n+1})\cap M^{\tilde p,\lambda}(\mathbb{R}^{n+1})$, where $\tilde p=\frac{p\lambda}{\lambda-p\beta}$. Moreover,*

$$\|I_\beta(f)\|_{L^{\tilde p}(\mathbb{R}^{n+1})} \le C\|f\|_{M^{p,\lambda}(\mathbb{R}^{n+1})}^{\frac{\beta p}{\lambda}}\|f\|_{L^p(\mathbb{R}^{n+1})}^{1-\frac{\beta p}{\lambda}} \tag{5.22}$$

$$\|I_\beta(f)\|_{M^{\tilde p,\lambda}(\mathbb{R}^{n+1})} \le C\|f\|_{M^{p,\lambda}(\mathbb{R}^{n+1})}. \tag{5.23}$$

(ii) For any $0<\beta<\lambda\le n+4$, if $f\in L^1(\mathbb{R}^{n+1})\cap M^{1,\lambda}(\mathbb{R}^{n+1})$, then $f\in L^{\frac{\lambda}{\lambda-\beta},*}(\mathbb{R}^{n+1})\cap M_*^{\frac{\lambda}{\lambda-\beta},\lambda}(\mathbb{R}^{n+1})$. Moreover,

$$\|I_\beta(f)\|_{L^{\frac{\lambda}{\lambda-\beta},*}(\mathbb{R}^{n+1})} \le C\|f\|_{M^{1,\lambda}(\mathbb{R}^{n+1})}^{\frac{\beta}{\lambda}}\|f\|_{L^1(\mathbb{R}^{n+1})}^{1-\frac{\beta}{\lambda}} \tag{5.24}$$

$$\|I_\beta(f)\|_{M_*^{\frac{\lambda}{\lambda-\beta},\lambda}(\mathbb{R}^{n+1})} \le C\|f\|_{M^{1,\lambda}(\mathbb{R}^{n+1})}. \tag{5.25}$$

Proof of Theorem 5.2. The proof is based on three claims.

Claim 1. For any $0<\alpha<1$, we have that $\nabla^2 u\in M^{2,4-4\alpha}(P_{\frac{3}{4}})$, and

$$\left\|\nabla^2 u\right\|_{M^{2,4-4\alpha}(P_{\frac{3}{4}})} \le C\left\|\nabla^2 u\right\|_{L_t^qL_x^p(P_1)}. \tag{5.26}$$

For any $0<r\le\frac{1}{4}$ and $z_0=(x_0,t_0)\in P_{\frac{3}{4}}$, by (5.12) we have

$$\|\nabla^2 u\|_{L_t^qL_x^p(P_r(z_0))} \le \epsilon. \tag{5.27}$$

Let $v : P_r(z_0) \to \mathbb{R}^{L+1}$ solve

$$\begin{cases} v_t + \Delta^2 v = 0 & \text{in } P_r(z_0) \\ v = u & \text{on } \partial_p P_r(z_0) \\ \dfrac{\partial v}{\partial \nu} = \dfrac{\partial u}{\partial \nu} & \text{on } \partial B_r(x_0) \times (t_0 - r^4, t_0]. \end{cases} \tag{5.28}$$

Set $w = u - v$. Multiplying (5.28) and (1.2) by w, subtracting the resulting equations and integrating over $P_r(z_0)$, we obtain

$$\begin{aligned} &\sup_{t_0 - r^4 \le t \le t_0} \int_{B_r(x_0)} |w|^2(t) + 2\int_{P_r(z_0)} |\nabla^2 w|^2 \\ =& \Big|\int_{P_r(z_0)} \mathcal{N}_{\mathrm{bh}}[u] \cdot w\Big| \\ =& \Big|\int_{P_r(z_0)} -\nabla(A(u)(\nabla u, \nabla u))\nabla w - \langle \Delta u, \Delta(P(u))\rangle w - 2\langle \Delta u, \nabla(P(u))\rangle \nabla w\Big| \\ \lesssim& \int_{P_r(z_0)} |\nabla^2 u|^2 |w| + \int_{P_r(z_0)} |\nabla u||\nabla^2 u||\nabla w| \\ =& I + II. \end{aligned} \tag{5.29}$$

For I, we can apply (5.14) to get

$$|I| \lesssim \|\nabla^2 u\|_{L^2(P_r(z_0))} \|w\|^{\frac{n}{2p}}_{L^2_t W^{2,2}_x(P_r(z_0))} \left(\int_{t_0 - r^4}^{t_0} \|\nabla^2 u\|^q_{L^p(B_r(x_0))} \|w\|^2_{L^2(B_r(x_0))}\right)^{\frac{1}{q}}. \tag{5.30}$$

For II, by (5.15), we have

$$|II| \lesssim \|\nabla u\|_{L^2_t W^{1,2}_x(P_r(z_0))} \|w\|^{\frac{n}{2p}}_{L^2_t W^{2,2}_x(P_r(z_0))} \left(\int_{t_0 - r^4}^{t_0} \|\nabla^2 u\|^q_{L^p(B_r(x))} \|w\|^2_{L^2(B_r(x_0))}\right)^{\frac{1}{q}}. \tag{5.31}$$

Putting (5.30) and (5.31) into (5.29) and applying the Poincaré inequality, we obtain

$$\begin{aligned} &\sup_{t_0 - r^4 \le t \le t_0} \int_{B_r(x_0)} |w|^2(t) + 2\int_{P_r(z_0)} |\nabla^2 w|^2 \\ \lesssim& \begin{cases} \|\nabla u\|_{L^2_t W^{1,2}_x(P_r(z_0))} \|\nabla^2 w\|^{\frac{n}{2p}}_{L^2(P_r(z_0))} \left(\int_{t_0 - r^4}^{t_0} \|\nabla^2 u\|^q_{L^p(B_r(x_0))} \|w\|^2_{L^2(B_r(x_0))}\right)^{\frac{1}{q}}, & q < \infty, \\ \|\nabla u\|_{L^2_t W^{1,2}_x(P_r(z_0))} \|\nabla^2 w\|_{L^2(P_r(z_0))} \|\nabla^2 u\|_{L^\infty_t L^{\frac{n}{2}}_x(B_r(x_0))}, & q = \infty. \end{cases} \end{aligned} \tag{5.32}$$

Since $\|\nabla^2 u\|_{L^q_t L^p_x(P_r(z_0))} \le \epsilon$, we obtain, by the Young inequality,

$$\begin{aligned} &\sup_{t_0 - r^4 \le t \le t_0} \int_{B_r(x_0)} |w|^2(t) + 2\int_{P_r(z_0)} |\nabla^2 w|^2 \\ \le& \begin{cases} \|\nabla^2 w\|^2_{L^2(P_r(z_0))} + \epsilon \|\nabla u\|^2_{L^2_t W^{1,2}_x(P_r(z_0))} + C\epsilon^{\frac{p}{2}} \sup\limits_{t_0 - r^4 \le t \le t_0} \|w\|^2_{L^2(B_r(x_0))}, & q < \infty, \\ \|\nabla^2 w\|^2_{L^2(P_r(z_0))} + C\|\nabla^2 u\|^2_{L^\infty_t L^{\frac{n}{2}}_x(B_r(x_0))} \|\nabla u\|^2_{L^2_t W^{1,2}_x(P_r(z_0))}, & q = \infty. \end{cases} \end{aligned} \tag{5.33}$$

By choosing $\epsilon > 0$ sufficiently small, this implies

$$\int_{P_r(z_0)} |\nabla^2 w|^2 \lesssim \epsilon \int_{P_r(z_0)} |\nabla u|^2 + |\nabla^2 u|^2. \tag{5.34}$$

Since N is compact and u maps into N, $|u| \le C_N$. Hence, by the Nirenberg interpolation inequality, we have

$$\int_{P_r(z_0)} |\nabla u|^2 \lesssim \int_{P_r(z_0)} |\nabla^2 u|^2 + r^{n+4}. \tag{5.35}$$

Combining (5.35) with (5.34), we have

$$\int_{P_r(z_0)} |\nabla^2 w|^2 \lesssim \epsilon \int_{P_r(z_0)} |\nabla^2 u|^2 + \epsilon r^{n+4}. \tag{5.36}$$

By the standard estimate on v, we have

$$(\theta r)^{-n} \int_{P_{\theta r}(z_0)} |\nabla^2 v|^2 \lesssim \theta^4 r^{-n} \int_{P_r(z_0)} |\nabla^2 v|^2, \ \forall\ \theta \in (0,1). \tag{5.37}$$

Combining (5.36) with (5.37), we obtain

$$(\theta r)^{-n} \int_{P_{\theta r}(z_0)} |\nabla^2 u|^2 \le C\left(\theta^4 + \theta^{-n}\epsilon\right) r^{-n} \int_{P_r(z_0} |\nabla^2 u|^2 + C\epsilon\theta^{-n} r^4, \ \forall\ \theta \in (0,1). \tag{5.38}$$

For any $0 < \alpha < 1$, choose $0 < \theta < 1$ and ϵ such that

$$C\theta^4 \le \frac{1}{2}\theta^{4\alpha} \text{ and } \epsilon \le \min\left\{\left(\frac{1}{2C}\right)^{\frac{2}{p}}, \frac{\theta^{4\alpha+n}}{2C}\right\}.$$

Therefore, for any $(z_0) \in P_{\frac{3}{4}}$ and $0 < r \le \frac{1}{4}$,

$$(\theta r)^{-n} \int_{P_{\theta r}(x,t)} |\nabla^2 u|^2 \le \theta^{4\alpha} r^{-n} \int_{P_r(x,t)} |\nabla^2 u|^2 + \theta^{4\alpha} r^4. \tag{5.39}$$

It is standard that iterating (5.39) implies

$$r^{-n} \int_{P_r(z_0)} |\nabla^2 u|^2 \le C r^{4\alpha} \left(\int_{P_1} |\nabla^2 u|^2 + 1\right) \tag{5.40}$$

for any $z_0 \in P_{\frac{3}{4}}$ and $0 < r \le \frac{1}{4}$. (5.40) implies that $\nabla^2 u \in M^{2,4-4\alpha}(P_{\frac{3}{4}})$, and (5.26) holds. This proves *Claim* 1.

Claim 2. For any $1 < \beta < +\infty$, $\nabla^2 u \in L^\beta(P_{\frac{9}{16}})$, and

$$\left\|\nabla^2 u\right\|_{L^\beta(P_{\frac{9}{16}})} \lesssim \left\|\nabla^2 u\right\|^2_{L^q_t L^p_x(P_1)}. \tag{5.41}$$

This can be proven by utilizing estimates of Riesz potentials between Morrey spaces. To do so, let $\eta \in C_0^\infty(P_1)$ be such that

$$0 \le \eta \le 1, \ \eta \equiv 1 \text{ in } P_{\frac{5}{8}}, \ |\eta_t| + \sum_{m=1}^{4} |\nabla^m \eta| \le C.$$

Let $Q : \mathbb{R}^n \times [-1, \infty] \to \mathbb{R}^{L+1}$ solve

$$\begin{aligned} \partial_t Q + \Delta^2 Q &= \nabla \cdot \Big(\eta^2 \nabla(A(u)(\nabla u, \nabla u)) + 2\eta^2 \langle \Delta u, \nabla(P(u))\rangle\Big) - \eta^2 \langle \Delta u, \Delta(P(u))\rangle \\ Q\Big|_{t=-1} &= 0. \end{aligned} \tag{5.42}$$

Set

$$J_1 = \nabla \cdot \Big(\eta^2 \nabla(A(u)(\nabla u, \nabla u)) + 2\eta^2 \langle \Delta u, \nabla(P(u))\rangle\Big) \ \text{ and } \ J_2 = -\eta^2 \langle \Delta u, \Delta(P(u))\rangle.$$

By the Duhamel formula, we have, for $(x, t) \in \mathbb{R}^n \times (-1, \infty)$,

$$\begin{aligned} \nabla^2 Q(x,t) &= \int_{\mathbb{R}^n \times [-1,t]} \nabla_x^2 b(x-y, t-s) \, (J_1 + J_2)\,(y, s) \\ &= \int_{\mathbb{R}^n \times [-1,t]} \nabla_x^3 b(x-y, t-s) \Big(\eta^2 \nabla(A(u)(\nabla u, \nabla u)) + 2\eta^2 \langle \Delta u, \nabla(P(u))\rangle\Big)(y, s) \\ &\quad - \int_{\mathbb{R}^n \times [-1,t]} \nabla_x^2 b(x-y, t-s) \eta^2 \langle \Delta u, \Delta(P(u))\rangle(y, s) \\ &= K_1(x,t) + K_2(x,t). \end{aligned} \tag{5.43}$$

It is clear that for $(x, t) \in \mathbb{R}^n \times (-1, \infty)$,

$$|K_1|(x,t) \lesssim I_1\Big(\eta^2(|\nabla u|^3 + |\nabla u||\nabla^2 u|)\Big)(x,t), \ |K_2|(x,t) \le I_2\Big(\eta^2(|\nabla^2 u|^2 + |\nabla u|^4)\Big)(x,t).$$

It follows from (5.26) and the Nirenberg interpolation inequality that $\nabla u \in M^{4,4-4\alpha}(P_{\frac{3}{4}})$ and

$$\Big\|\nabla u\Big\|_{M^{4,4-4\alpha}(P_{\frac{3}{4}})} \lesssim \Big\|\nabla^2 u\Big\|_{L_t^q L_x^p(P_1)}. \tag{5.44}$$

Hence, by the Hölder inequality, we have that for any $0 < \alpha_1, \alpha_2 < 1$,

$$\eta^2(|\nabla u|^3 + |\nabla u||\nabla^2 u|) \in M^{\frac{4}{3}, 4-4\alpha_1}(\mathbb{R}^{n+1}) \text{ and } \eta^2(|\nabla^2 u|^2 + |\nabla u|^4) \in M^{1, 4-4\alpha_2}(\mathbb{R}^{n+1}),$$

and

$$\begin{aligned} \Big\|\eta^2(|\nabla u|^3 + |\nabla u||\nabla^2 u|)\Big\|_{M^{\frac{4}{3}, 4-4\alpha_1}(\mathbb{R}^{n+1})} &\lesssim \Big\|\nabla u\Big\|_{M^{4,4-4\alpha_1}(P_{\frac{3}{4}})} \Big\|\nabla^2 u\Big\|_{M^{2,4-4\alpha_1}(P_{\frac{3}{4}})} \\ &\lesssim \Big\|\nabla^2 u\Big\|^2_{L_t^q L_x^p(P_1)}, \end{aligned} \tag{5.45}$$

$$\begin{aligned} \Big\|\eta^2(|\nabla^2 u|^2 + |\nabla u|^4)\Big\|_{M^{1, 4-4\alpha_2}(\mathbb{R}^{n+1})} &\lesssim \Big\|\nabla u\Big\|_{M^{4,4-4\alpha_2}(P_{\frac{3}{4}})} + \Big\|\nabla^2 u\Big\|_{M^{2,4-4\alpha_2}(P_{\frac{3}{4}})} \\ &\lesssim \Big\|\nabla^2 u\Big\|^2_{L_t^q L_x^p(P_1)}. \end{aligned} \tag{5.46}$$

Now applying Proposition 5.4, we conclude that

$$K_1 \in M^{\frac{4-4\alpha_1}{2-3\alpha_1}, 4-4\alpha_1} \cap L^{\frac{4-4\alpha_1}{2-3\alpha_1}}(\mathbb{R}^{n+1}), \quad K_2 \in M_*^{\frac{2-2\alpha_2}{1-2\alpha_2}, 4-4\alpha_2} \cap L^{\frac{2-2\alpha_2}{1-2\alpha_2}, *}(\mathbb{R}^{n+1}),$$

and

$$\left\|K_1\right\|_{M^{\frac{4-4\alpha_1}{2-3\alpha_1},4-4\alpha_1}(\mathbb{R}^{n+1})}+\left\|K_2\right\|_{M_*^{\frac{2-2\alpha_2}{1-2\alpha_2},4-4\alpha_2}(\mathbb{R}^{n+1})}\lesssim\left\|\nabla^2 u\right\|^2_{L^q_tL^p_x(P_1)}. \tag{5.47}$$

Sending $\alpha_1\uparrow\frac{2}{3}$ and $\alpha_2\uparrow\frac{1}{2}$, we obtain that for any $1<\beta<+\infty$, $K_1,K_2\in L^\beta(\mathbb{R}^{n+1})$, and

$$\|K_1\|_{L^\beta(\mathbb{R}^{n+1})}+\|K_2\|_{L^\beta(\mathbb{R}^{n+1})}\lesssim\left\|\nabla^2 u\right\|^2_{L^q_tL^p_x(P_1)}. \tag{5.48}$$

This implies that for any $1<\beta<+\infty$, $\nabla^2 Q\in L^\beta(\mathbb{R}^{n+1})$, and

$$\left\|\nabla^2 Q\right\|_{L^\beta(\mathbb{R}^{n+1})}\lesssim\left\|\nabla^2 u\right\|^2_{L^q_tL^p_x(P_1)}. \tag{5.49}$$

Since $(u-Q)$ solves

$$\left(\partial_t+\Delta^2\right)(u-Q)=0 \text{ in } P_{\frac{5}{8}},$$

it follows that for any $1<\beta<+\infty$, $\nabla^2 u\in L^\beta(P_{\frac{9}{16}})$, and

$$\left\|\nabla^2 u\right\|_{L^\beta(P_{\frac{9}{16}})}\lesssim\left\|\nabla^2 u\right\|^2_{L^q_tL^p_x(P_1)}. \tag{5.50}$$

This implies (5.49) and *Claim* 2 is proven.

Claim 3. $u\in C^\infty(P_{\frac{1}{2}},N)$ and (5.13) holds. It follows from (5.49) that for any $1<\beta<+\infty$, there exist $f,g\in L^\beta(P_{\frac{9}{16}})$ such that (1.2) can be written as

$$(\partial_t+\Delta^2)u=\nabla\cdot f+g.$$

Thus, by the L^p-theory of higher-order parabolic equations, we conclude that $\nabla^3 u\in L^\beta(P_{\frac{17}{32}})$. Applying the L^p-theory again, we would obtain that $\partial_t u,\nabla^4 u\in L^\beta(P_{\frac{33}{64}})$. Taking derivatives of the equation (1.2) and repeating this argument, we can conclude that $u\in C^\infty(P_{\frac{1}{2}},N)$, and the estimate (5.13) holds. Combining together these three claims, the proof of Theorem 5.2 is complete. □

Proof of Theorem 1.8. Let $\epsilon_0>0$ be given by Theorem 5.2. Since $p>\frac{n}{2}$ and $q<\infty$, there exists $T_0>0$ such that

$$\max_{i=1,2}\|\nabla^2 u_i\|_{L^q_tL^p_x(\Omega\times[0,T_0])}\le\epsilon_0. \tag{5.51}$$

This implies that for any $x_0\in\Omega$ and $0<t_0\le T_0$, if $R_0=\min\{d(x_0,\partial\Omega),t_0^{\frac{1}{4}}\}>0$, then (5.51) implies

$$\max_{i=1,2}\|\nabla^2 u_i\|_{L^q_tL^p_x(P_{R_0}(z_0))}\le\epsilon_0. \tag{5.52}$$

Hence by suitable scalings of the estimate of Theorem 5.2, we have that for $i=1,2$, $u_i\in C^\infty(P_{\frac{R_0}{2}}(z_0),N)$ and

$$\left|\nabla^m u_i\right|(x_0,t_0)\lesssim\epsilon_0\left(\frac{1}{d^m(x_0,\partial\Omega)}+\frac{1}{t_0^{\frac{m}{4}}}\right). \tag{5.53}$$

Using (5.53), the same proof of Theorem 1.3 implies that $u_1\equiv u_2$ in $\Omega\times[0,T_0]$. Repeating this argument on the interval $[T_0,T]$, we can show $u_1\equiv u_2$ in $\Omega\times[0,T]$. □

Proof of Corollary 1.10. Let $\epsilon_0 > 0$ be given by Theorem 5.2. Since $u_0 \in W^{2,2}(\Omega, N)$, by the absolute continuity of $\displaystyle\int |\nabla^2 u_0|^2$ there exists $r_0 > 0$ such that

$$\max_{x\in\Omega} \int_{B_{r_0}(x)\cap\Omega} |\nabla^2 u_0|^2 \leq \frac{\epsilon_0^2}{2}. \tag{5.54}$$

Choosing $\epsilon_1 \leq \frac{\epsilon_0^2}{2}$ and applying (1.14), we conclude that there exists $0 < t_0 \leq r_0^4$ such that

$$\max_{x\in\Omega, 0\leq t\leq t_0} \int_{B_{r_0}(x)\cap\Omega} |\nabla^2 u_i(t)|^2 \leq \epsilon_0^2, \quad \text{for} \quad i = 1, 2. \tag{5.55}$$

Set $R_0 = \min\{r_0, t_0^{\frac{1}{4}}\} = t_0^{\frac{1}{4}} > 0$. Then it is easy to see that (5.55) gives

$$\max_{z=(x,t)\in\Omega\times[0,t_0]} \left\|\nabla^2 u_i\right\|_{L_t^\infty L_x^2(P_{R_0}(z)\cap(\Omega\times[0,t_0]))} \leq \epsilon_0, \quad \text{for} \quad i = 1, 2. \tag{5.56}$$

This implies that u_1 and u_2 satisfy (5.12) of Theorem 5.2 (with $p = 2$ and $q = \infty$) on $P_r(z)$, for any $z \in \Omega \times [0, t_0]$ and $r = \min\{R_0, d(x, \partial\Omega), t^{\frac{1}{4}}\} > 0$. Hence by suitable scalings of the estimate of Theorem 5.2, we have

$$\max_{i,2} \left|\nabla^m u_i(x,t)\right| \lesssim \epsilon_0 \left(\frac{1}{R_0^m} + \frac{1}{d^m(x,\partial\Omega)} + \frac{1}{t^{\frac{m}{4}}}\right) \lesssim \epsilon_0 \left(\frac{1}{d^m(x,\partial\Omega)} + \frac{1}{t^{\frac{m}{4}}}\right), \ \forall\, m \geq 1, \tag{5.57}$$

for any $(x, t) \in \Omega \times [0, t_0]$. Here we have used $R_0 \geq t^{\frac{1}{4}}$ in the last inequality. Applying (5.57) and the proof of Theorem 1.3, we can conclude that $u_1 \equiv u_2$ in $\Omega \times [0, t_0]$. Continuing this argument on the interval $[t_0, T]$ shows $u_1 \equiv u_2$ in $\Omega \times [0, T]$. □

Proof of Corollary 1.11. Let $\epsilon_2 > 0$ be given by Theorem 5.2. Then (1.15) yields

$$\left\|\nabla^2 u\right\|_{L_t^\infty L_x^2(\Omega\times[0,\infty))} \leq \epsilon_2. \tag{5.58}$$

Hence by suitable scalings of the estimate of Theorem 5.2, we have $u \in C^\infty(\Omega \times (0, \infty), N)$ and there exists $T_1 > 0$ such that

$$\left|\nabla^m u(x,t)\right| \lesssim \epsilon_2 \left(\frac{1}{d^m(x,\partial\Omega)} + \frac{1}{t^{\frac{m}{4}}}\right), \ \forall\, m \geq 1, \tag{5.59}$$

holds for all $x \in \Omega$ and $t \geq T_1$. Now we can apply the same arguments as in the proof of Theorem 1.5 and Corollary 1.6 to prove the conclusions of Corollary 1.11. □

6 Appendix: Higher-order regularity

It is known, at least to experts, that higher-order regularity holds for any Hölder continuous solution to (1.2) of the heat flow of biharmonic maps . However, we can't find a complete proof of this fact in the literature. For the completeness, we will sketch, in this appendix, a proof that is based on the parabolic-type hole-filling argument.

Proposition 6.1 *For $0 < \alpha < 1$, if $u \in W_2^{1,2} \cap C^\alpha(P_2, N)$ is a weak solution to (1.2), then $u \in C^\infty(P_1, N)$, and*

$$\left\|\nabla^m u\right\|_{C^0(P_1)} \lesssim \left[u\right]_{C^\alpha(P_2)} + \left\|u\right\|_{L_t^2 W_x^{2,2}(P_2)}, \ \forall\ m \geq 1. \tag{6.1}$$

Proof. By *Claim* 2 and *Claim* 3 in the proof of Theorem 5.2, it suffices to establish that $\nabla^2 u \in M^{2,4-4\tilde{\alpha}}(P_{\frac{3}{2}})$ for some $\frac{2}{3} < \tilde{\alpha} < 1$, and

$$\left\|\nabla^2 u\right\|_{M^{2,4-4\tilde{\alpha}}(P_{\frac{3}{2}})} \lesssim \left[u\right]_{C^\alpha(P_2)} + \left\|\nabla^2 u\right\|_{L^2(P_2)}. \tag{6.2}$$

This will be achieved by the following hole-filling argument. For any fixed $z_0 = (x_0, t_0) \in P_{\frac{3}{2}}$ and $0 < r \leq \frac{1}{4}$, let $\phi \in C_0^\infty(\mathbb{R}^n)$ be a cut-off function of $B_r(x_0)$, i.e.

$$0 \leq \phi \leq 1,\ \phi \equiv 1 \text{ in } B_r(x_0),\ \phi \equiv 0 \text{ outside } B_{2r}(x_0),\ |\nabla^m \phi| \leq C r^{-m},\ \forall\ m \geq 1.$$

Set $c := \fint_{P_r(z_0)} u \in \mathbb{R}^{L+1}$. Multiplying (1.2) by $(u-c)\phi^4$ and integrating over $\mathbb{R}^n$, we obtain

$$\begin{aligned} &\frac{d}{dt}\int_{\mathbb{R}^n} |u-c|^2\phi^4 + 2\int_{\mathbb{R}^n} \Delta(u-c)\cdot\Delta((u-c)\phi^4) = 2\int_{\mathbb{R}^n} \mathcal{N}_{\rm bh}[u]\cdot(u-c)\phi^4 \\ \lesssim\ & \int_{\mathbb{R}^n} |\nabla^2 u|^2|u-c|\phi^4 + \int_{\mathbb{R}^n} |\nabla u||\nabla^2 u||\nabla((u-c)\phi^4)|. \end{aligned} \tag{6.3}$$

For the second term in the left hand side of (6.3), we have

$$\begin{aligned} &2\int_{\mathbb{R}^n} \Delta(u-c)\cdot\Delta((u-c)\phi^4) = 2\int_{\mathbb{R}^n} \nabla^2(u-c)\cdot\nabla^2((u-c)\phi^4) \\ \geq\ & 2\int_{B_r(z_0)} |\nabla^2 u|^2 - C\int_{\mathbb{R}^n} |u-c|^2(|\nabla^2\phi|^2 + |\nabla\phi|^4) + \phi^2|\nabla\phi|^2|\nabla u|^2. \end{aligned} \tag{6.4}$$

Substituting (6.4) into (6.3) and integrating over $t \in [t_0 - r^4, t_0]$, we obtain

$$\begin{aligned} \int_{P_r(z_0)} |\nabla^2 u|^2 \leq & \int_{B_{2r}(x_0)\times\{t_0 - r^4\}} |u-c|^2 + \left(2^{-(n+4)} + C\text{osc}_{P_{2r}(z_0)} u\right)\int_{P_{2r}(z_0)} |\nabla^2 u|^2 \\ & + Cr^n\left(\text{osc}_{P_{2r}(z_0)} u\right)^2 + C\left[1 + (\text{osc}_{P_{2r}(z_0)} u)^2\right] r^{-2}\int_{P_{2r}(z_0)} \phi^2|\nabla u|^2 \\ & + C\int_{P_{2r}(z_0)} |\nabla u|^4\phi^4 \end{aligned} \tag{6.5}$$

By integration by parts and the Hölder inequality, we have

$$r^{-2}\int_{P_{2r}(z_0)} \phi^2|\nabla u|^2 \leq Cr^{-2}\left(\text{osc}_{P_{2r}(z_0)} u\right)\int_{P_{2r}(z_0)} |\nabla^2 u| + Cr^n\left(\text{osc}_{P_{2r}(z_0)} u\right)^2,$$

and

$$C\int_{P_{2r}(z_0)} \phi^4|\nabla u|^4 \leq 2^{-(n+4)}\int_{P_{2r}(z_0)} |\nabla^2 u|^2 + Cr^n\left(\text{osc}_{P_{2r}(z_0)} u\right)^4 + C\left(\text{osc}_{P_{2r}(z_0)} u\right)^2\int_{P_{2r}(z_0)} |\nabla^2 u|^2.$$

Putting these two inequalities into (6.5) and using

$$\mathrm{osc}_{P_{2r}(z_0)} u \le C r^\alpha,$$

we get

$$\begin{aligned}\int_{P_r(z_0)} |\nabla^2 u|^2 &\le \left(2^{-(n+3)} + C r^\alpha\right) \int_{P_{2r}(z_0)} |\nabla^2 u|^2 + C r^{n+2\alpha} + C(1+r^{2\alpha}) r^{\alpha-2} \int_{P_{2r}(z_0)} |\nabla^2 u| \\ &\le \left(2^{-(n+2)} + C r^\alpha\right) \int_{P_{2r}(z_0)} |\nabla^2 u|^2 + C r^{n+2\alpha}, \end{aligned} \tag{6.6}$$

where we have used the following inequality in the last step:

$$C(1+r^{2\alpha}) r^{\alpha-2} \int_{P_{2r}(z_0)} |\nabla^2 u| \le 2^{-(n+3)} \int_{P_{2r}(z_0)} |\nabla^2 u|^2 + C r^{n+2\alpha}.$$

Choosing $r > 0$ so small that $C r^\alpha \le 2^{-(n+3)}$, we see that (6.6) implies

$$r^{-n} \int_{P_r(z_0)} |\nabla^2 u|^2 \le \frac{1}{2} (2r)^{-n} \int_{P_{2r}(z_0)} |\nabla^2 u|^2 + C r^{2\alpha}. \tag{6.7}$$

It is clear that iterating (6.7) implies that there is $\alpha_0 \in (0,1)$ such that $\nabla^2 u \in M^{2,4-2\alpha_0}(P_{\frac{3}{2}})$ and

$$\left\|\nabla^2 u\right\|_{M^{2,4-2\alpha_0}(P_{\frac{3}{2}})} \lesssim \left[u\right]_{C^\alpha(P_2)} + \left\|\nabla^2 u\right\|_{L^2(P_2)}. \tag{6.8}$$

We can apply the estimate (6.8) and repeat the above argument to show that $\nabla^2 u \in M^{2,4-4\alpha_0}(P_{\frac{3}{2}})$ and (6.8) holds with α_0 replaced by $2\alpha_0$. Repeating these argument again and again until there exists $\tilde\alpha \in (\frac{2}{3}, 1)$ such that $\nabla^2 u \in M^{2,4-4\tilde\alpha}(P_{\frac{3}{2}})$ and the estimate (6.2) holds. The remaining parts of the proof can be done by following the same arguments as *Claim* 2 and *Claim* 3 of the proof of Theorem 5.2. This completes the proof. □

Acknowledgements. The authors are partially supported by NSF grant 1001115. The third author is also partially supported by NSFC grant 11128102.

References

[1] G. Angelsberg, *A monotonicity formula for stationary biharmonic maps.* Math. Z. **252**, 287-293 (2006).

[2] F. Bethuel, *On the singular set of stationary harmonic maps.* Manuscripta Math. **78** (4), 417-443 (1993).

[3] S. Campanato, Equazioni ellittiche del II^0 ordine espazi $\mathcal{L}^{(2,\lambda)}$. Ann. Mat. Pura Appl., **69** (4), 321-381 (1965).

[4] S. Y. A. Chang, L. H. Wang, P. C. Yang. *A regularity theory of biharmonic maps*, Comm. Pure Appl. Math., **52** (9), 1113-1137 (1999).

[5] D. Edmunds, J. Rákosník, *On a higher-order Hardy inequality.* Math. Bohem. **124**, no. 2-3, 113-121 (1999).

[6] J. Eells, L. Lemaire, *Another report on harmonic maps.* Bull. London Math. Soc., **20** (5), 385-524 (1988).

[7] L. Evans, *Partial regularity for stationary harmonic maps into spheres.* Arch. Rational Mech. Anal. **116** (2), 101-113 (1991).

[8] E. Fabes, F. Jones, N. Riviere, *The initial value problem for the Navier-Stokes equations with date in L^p.* Arch. Rational Mech. Anal., **45**, 222-240 (1972).

[9] A. Gastel, *The extrinsic polyharmonic map heat flow in the critical dimension.* Adv. Geom., **6** (4), 501-521 (2006).

[10] A. Gastel, C. Scheven, *Regularity of polyharmonic maps in the critical dimension.* Comm. Anal. Geom. **17**, no. 2, 185-226 (2009).

[11] A. Gastel, F. Zorn, *Biharmonic maps of cohomogeneity one between spheres.* J. Math. Anal. Appl. **387**, no. 1, 384-399 (2012).

[12] H. J. Gong, T. Lamm, C. W. Wang, *Boundary partial regularity for a class of biharmonic maps.* Calc. Var. Partial Differential Equations **45**, no. 1-2, 165-191 (2012).

[13] F. Hélein, *Régularité des applications faiblement harmoniques entre une surface et une variété riemannienne.* C. R. Acad. Sci. Paris Sér. I Math. **312**, 591-596 (1991).

[14] T. Huang, C. Y. Wang, *On the uniqueness of heat flow of harmonic maps.* Preprint, arXiV:1208.1470.

[15] T. Huang, C. Y. Wang, *Notes on the regularity of harmonic map systems.* Proc. Amer. Math. Soc., **138** (6), 2015-2023 (2010).

[16] M. C. Hong, C. Y. Wang, *Regularity and relaxed problems of minimizing biharmonic maps into spheres.* Calc. Var. Partial Differential Equations **23**, no. 4, 425-450 (2005).

[17] M. C. Hong, H. Yin, *Partial regularity of a minimizer of the relaxed energy for biharmonic maps.* J. Funct. Anal. **262**, no. 2, 681-718 (2012).

[18] F. John, L. Nirenberg, *On functions of bounded mean oscillation.* Comm. Pure. Appl. Math., **14**, 415-426 (1961).

[19] Y. Ku, *Interior and boundary regularity of intrinsic biharmonic maps to spheres.* Pacific J. Math. **234**, 43-67 (2008).

[20] H. Koch, T. Lamm, *Geometric flows with rough initial data.* Asian J. Math., **16** (2), 209-235 (2012).

[21] T. Lamm, *Biharmonischer Wärmefluss.* Diplomarbeit Universität Freiburg, 2001.

[22] T. Lamm, *Heat flow for extrinsic biharmonic maps with small initial energy.* Ann. Global Anal. Geom., **26** (4), 369-384 (2004).

[23] T. Lamm, T. Rivière, *Conservation laws for fourth order systems in four dimensions.* Comm. Partial Differ.Equ. **33**, 245-262 (2008).

[24] T. Lamm, C. Wang, *Boundary regularity for polyharmonic maps in the critical dimension.* Adv. Calc. Var. **2** (1), 1-16 (2009).

[25] F. H. Lin, *Gradient estimates and blow-up analysis for stationary harmonic maps.* Ann. Math. **149**, 785-829 (1999).

[26] R. Moser, *Unique solvability of the Dirichlet problem for weakly harmonic maps.* Manuscripta Math., **105**, no. 3, 379-399 (2001).

[27] R. Moser, *Remarks on the regularity of biharmonic maps in four dimensions.* Comm. Pure Appl. Math. **59**, no. 3, 317-329 (2006).

[28] R. Moser, *Regularity of minimizing extrinsic polyharmonic maps in the critical dimension.* Manuscripta Math. **131**, no. 3-4, 475-485 (2010).

[29] R. Moser, *Weak solutions of a biharmonic map heat flow.* Adv. Calc. Var. **2**, no. 1, 73-92 (2009).

[30] P. Strzelecki, *On biharmonic maps and their generalizations.* Calc. Var. Partial Differ. Equ. **18** (4), 401-432 (2003).

[31] T. Rivière, *Conservation laws for conformally invariant variational problems.* Invent. Math. **168** (1), 1-22 (2007).

[32] M. Rupflin, *Uniqueness for the heat flow for extrinsic polyharmonic maps in the critical dimension.* Comm. Partial Differential Equations **36**, no. 7, 1118-1144 (2011).

[33] C. Scheven, *Dimension reduction for the singular set of biharmonic maps.* Adv. Calc. Var. **1** no. 1, 53-91 (2008).

[34] C. Scheven, *An optimal partial regularity result for minimizers of an intrinsically defined second-order functional.* Ann. Inst. H. Poincaré Anal. Non Linéaire **26**, no. 5, 1585-1605 (2009).

[35] R. Schoen, *Analytic aspets of harmonic map problem.* Seminar on nonlinear partial differential euqations (Berkeley, Calif., 1983), 321-358, Math. Sci. Res. Inst. Publ., 2, Springer, New York-Berlin, 1984.

[36] L. Simon, *Asymptotics for a class of nonlinear evolution equations, with applications to geometric problems.* Ann. Math., (2) **118**, no. 3, 525-571 (1983).

[37] M. Struwe, *Uniqueness of harmonic maps with small energy.* Manuscripta Math., **96**, no. 4, 463-486 (1998).

[38] M. Struwe, *Partial regularity for biharmonic maps, revisited.* Calc. Var. Partial Differ. Equ. **33**(2), 249-262 (2008).

[39] R. Schoen, K. Uhlenbeck, *A regularity theory for harmonic maps.* J. Diff. Geom. **17**, no.2, 307-335 (1982).

[40] E. M. Stein, Singular integrals and differentiability properties of functions. Princeton Univ. Press, 1970.

[41] C. Y. Wang, *Biharmonic maps from* $\mathbb{R}^4$ *into a Riemannian manifold.* Math. Z., **247** (1), 65-87 (2004).

[42] C. Y. Wang, *Remarks on biharmonic maps into spheres.* Calc. Var. & PDE, **21**, 221-242 (2004).

[43] C. Y. Wang, *Stationary biharmonic maps from* $\mathbb{R}^m$ *into a Riemannian manifold.* Comm. Pure Appl. Math., **57** (4), 419-444 (2004).

[44] C. Y. Wang, *Heat flow of biharmonic maps in dimensions four and its application.* Pure Appl. Math., Q. **3** (2), part 1, 595-613 (2007).

[45] C. Y. Wang, *Well-posedness for the heat flow of biharmonic maps with rough initial data.* J. Geom. Anal., **22**, 223-243 (2012).