On the uniqueness of heat flow of harmonic maps and hydrodynamic flow of nematic liquid crystals

Fanghua Lin* Changyou Wang†‡

Dedicated to Professor Roger Temam on the occasion of his 70th birthday

Abstract

For any n-dimensional compact Riemannian manifold (M, g) without boundary and another compact Riemannian manifold (N, h), we establish the uniqueness of the heat flow of harmonic maps from M to N in the class $C([0, T), W^{1,n})$. For the hydrodynamic flow (u, d) of nematic liquid crystals in dimensions $n = 2$ or 3, we show the uniqueness holds for the class of weak solutions provided either (i) for $n = 2$, $u \in L_t^\infty L_x^2 \cap L_t^2 H_x^1$, $\nabla P \in L_t^{\frac{4}{3}} L_x^{\frac{4}{3}}$, and $\nabla d \in L_t^\infty L_x^2 \cap L_t^2 H_x^2$; or (ii) for $n = 3$, $u \in L_t^\infty L_x^2 \cap L_t^2 H_x^1 \cap C([0, T), L^n)$, $P \in L_t^{\frac{n}{2}} L_x^{\frac{n}{2}}$, and $\nabla d \in L_t^2 L_x^2 \cap C([0, T), L^n)$. This answers affirmatively the uniqueness question posed by Lin-Lin-Wang. The proofs are very elementary.

1 Introduction and statement of results

For geometric nonlinear evolution equations or systems with critical nonlinearities, it is well-known that the short time smooth solutions may develop finite time singularities. The natural classes of solutions to such systems usually involve weak solutions in various larger function spaces. Although the existence of such weak solutions may be established, the uniqueness and regularity often remain to be very challenging.

*Courant Institute of Mathematical Sciences, New York University, New York, NY 10012
†Department of Mathematics, University of Kentucky, Lexington, KY 40506
‡Both authors are partially supported by NSF

Here we mention two examples. The first one is the celebrated work made by Leray [14] in 1934 on the existence of so-called Leray-Hopf type weak solutions to the Naiver-Stokes equation. Both uniqueness and regularity for the Leray-Hopf type weak solutions to NSE in dimension three still remain largely open. The second example is the heat flow of harmonic maps. It is well-known that in dimensions two or higher, the heat flow of harmonic maps can indeed develop singularities in finite time, see for example the works by Chang-Ding-Ye [3] for dimension two and Chen-Ding [2] in dimensions at least three. On the other hand, weak solutions that allow possible singularities to the heat flow of harmonic maps have been established by Struwe [21] and Chang [1] in dimension two and by Chen-Struwe [6] and Chen-Lin [5] in dimensions higher. While Freire [9] proved that Struwe's solution is unique in the class of weak solutions whose energies are monotonically decreasing in dimension two, whether Chen-Struwe's solution is unique in certain classes in higher dimensions is unknown.

These two examples motivate us to investigate the uniqueness issue of weak solutions to both the heat flow of harmonic maps and the equation of liquid crystal flows in certain critical L^p spaces. The later equation is a simplified version of the Ericksen-Leslie system modeling the hydrodynamics of liquid crystal materials developed by Ericksen [8] and Leslie [13] in 1960's. It is a macroscopic continuum description of the time evolution of the material under the influence of both the flow filed and the macroscopic description of the microscopic orientation configurations of rod-like liquid crystals. Mathematically, it is a strongly coupled system between the Navier-Stokes equation and the transported heat flow of harmonic maps into sphere.

Now let's describe the problems and our results. First, we describe the heat flow of harmonic maps. Let (M, g) be a n-dimensional compact or complete Riemannian manifold without boundary, $(N, h) \subset \mathbb{R}^k$ be a compact Riemannian manifold without boundary, isometrically embedded into the Euclidean space $\mathbb{R}^k$. Consider the heat flow of harmonic maps $u : M \times \mathbb{R}_+ \to N$:

$$u_t - \Delta u = A(u)(\nabla u, \nabla u) \tag{1.1}$$

$$u|_{t=0} = u_0 \tag{1.2}$$

where $A(\cdot)(\cdot,\cdot)$ is the second fundamental form of N, and $u_0 : M \to N$ is a given map. For $1 \le p < +\infty$, recall the Sobolev space $W^{1,p}(M,N)$ is defined by

$$W^{1,p}(M,N) = \left\{ v \in W^{1,p}(M,\mathbb{R}^k) : \ v(x) \in N \text{ a.e. } x \in M \right\}.$$

For $0 < T \le \infty$, $H^1(M \times [0,T], N)$ is defined by

$$H^1(M\times[0,T],N) = \left\{ v \in H^1(M \times [0,T], \mathbb{R}^k) : \ v(x,t) \in N \text{ a.e. } (x,t) \in M \times [0,T] \right\}.$$

For $u_0 \in W^{1,2}(M,N)$ and $0 < T \le +\infty$, recall that a map $u \in H^1(M \times [0,T], N)$ is a weak solution of (1.1) and (1.2) if u satisfies (1.1) in the sense of distributions and (1.2) in the sense of trace.

Our first result is the following uniqueness theorem.

Theorem 1.1 *For $n \ge 2$, $0 < T \le \infty$, and $u_0 \in W^{1,n}(M,N)$, suppose that $u, v \in H^1(M \times [0,T], N) \cap C([0,T), W^{1,n}(M,N))$ are two weak solutions to (1.1) on $M \times (0,T)$ such that $u|_{t=0} = v|_{t=0} = u_0$ on M. Then $u \equiv v$ on $M \times [0,T)$.*

Remark 1.2 We would like to point out that when considering the heat flow of harmonic maps on manifolds M with boundaries, Theorem 1.1 remains to be true under the initial condition and the boundary condition: $u = u_0$ on $\partial M \times (0,T)$, provided that $u_0 \in C^2(\partial M, N)$. The interested readers can check that slight modifications of the proof presented in §2 will achieve this.

Next we start to describe the liquid crystal flows in dimensions two and three. For $n = 2$ or 3, let $\Omega \subset \mathbb{R}^n$ be either a bounded smooth domain or $\mathbb{R}^n$. First, let's briefly recall that the equation of hydrodynamic flow of nematic liquid crystals on Ω. The interested readers can refer to [8], [13], [15], and [17] for the detailed background.

For $0 < T \le +\infty$, let $u : \Omega \times [0,T) \to \mathbb{R}^n$ be the fluid velocity field, and $d : \Omega \times [0,T) \to S^2$ be the director field of the nematic liquid crystals. Then the initial value problem for the equation of hydrodynamic flow of liquid crystals is given by

$$\begin{aligned} u_t + u\cdot\nabla u - \Delta u + \nabla P &= -\nabla\cdot(\nabla d\otimes\nabla d) && (1.3)\\ \nabla\cdot u &= 0 && (1.4)\\ d_t + u\cdot\nabla d - \Delta d &= |\nabla d|^2 d. && (1.5)\\ (u,d)|_{t=0} &= (u_0,d_0) && (1.6)\end{aligned}$$

where $P:\Omega\times[0,T)\to\mathbb{R}$ is the pressure function, $\nabla d\otimes\nabla d = \left(\langle\frac{\partial d}{\partial x_i},\frac{\partial d}{\partial x_j}\rangle\right)_{1\le i,j\le n}$ is the stress tensor induced by the director field d, $\nabla\cdot$ denotes the divergence operator, $u_0\in L^2(\Omega,\mathbb{R}^n)$, with $\nabla\cdot u_0=0$, is the initial velocity field, and $d_0:\Omega\to S^2$, with $\nabla d_0\in L^2(\Omega,\mathbb{R}^{3n})$, is the initial director field.

When Ω is a bounded smooth domain of $\mathbb{R}^n$, we will consider the system (1.3), (1.4), (1.5), and (1.6) along with the boundary condition:

$$(u,d) = (0,d_0) \text{ on } \partial\Omega\times(0,T). \tag{1.7}$$

For $n=2$, we will establish the uniqueness for the class of Leray-Hopf type weak solutions to the equation of hydrodynamic flow of nematic liquid crystals. More precisely, we have

Theorem 1.3 *For $0<T\le+\infty$, $u_0\in L^2(\Omega,\mathbb{R}^2)$ with $\nabla\cdot u_0=0$, and $d_0:\Omega\to S^2$ with $\nabla d_0\in L^2(\Omega,\mathbb{R}^6)$, suppose that for $i=1,2$, $u_i\in L^\infty_t L^2_x\cap L^2_t H^1_x(\Omega\times[0,T],\mathbb{R}^2)$, $\nabla P_i\in L^{\frac43}_t L^{\frac43}_x$, and $d_i\in L^\infty_t\dot H^1_x\cap L^2_t\dot H^2_x(\Omega\times[0,T],S^2)$ are a pair of weak solutions [1] to (1.3), (1.4), (1.5) under either*
(i) when $\Omega=\mathbb{R}^2$, the same initial condition:

$$(u_i,d_i)\big|_{t=0} = (u_0,d_0),\ \ i=1,2, \tag{1.8}$$

or

(ii) when $\Omega\subset\mathbb{R}^2$ is a bounded domain, the same initial and boundary conditions:

$$(u_i,d_i)=(u_0,d_0) \text{ on } \Omega\times\{0\},\ (u_i,d_i)=(0,d_0) \text{ on } \partial\Omega\times(0,T),\ i=1,2, \tag{1.9}$$

with $d_0\in C^{2,\beta}(\partial\Omega,S^2)$ for some $\beta\in(0,1)$.
Then $(u_1,d_1)\equiv(u_2,d_2)$ in $\Omega\times[0,T)$.

[1] The reader can refer to [19] Definition 1.1 for the exact definition of weak solutions.

We would like to point out that theorem 1.3 answers affirmatively that the global weak solutions obtained by Lin-Lin-Wang [19] (Remark 1.5 (i)) is unique in the same class of weak solutions.

For simplicity, when $n = 3$, we only consider the uniqueness of the Cauchy problem of the hydrodynamic flow of nematic liquid crystals in the entire space, i.e. $\Omega = \mathbb{R}^n$.

Theorem 1.4 *For $n = 3$ and $0 < T \leq +\infty$, $u_0 \in L^n(\mathbb{R}^n, \mathbb{R}^n)$ with $\nabla \cdot u_0 = 0$, and $d_0 : \mathbb{R}^n \to S^2$ with $\nabla d_0 \in L^n(\mathbb{R}^n, \mathbb{R}^{3n})$, suppose $u_i \in \left(L_t^\infty L_x^2 \cap L_t^2 L_x^2(\mathbb{R}^n \times [0,T])\right) \cap C([0,T), L^n(\mathbb{R}^n))$, $P_i \in L_t^{\frac{n}{2}} L_x^{\frac{n}{2}}(\mathbb{R}^n \times [0,T])$, and $d_i \in L_t^2 \dot{H}_x^1 \cap C([0,T), \dot{W}^{1,n}(\mathbb{R}^n, S^2))^2$, $i = 1, 2$, are a pair of weak solutions to (1.3), (1.4), (1.5) under the same initial condition:*

$$(u_i, d_i)\big|_{t=0} = (u_0, d_0)\,, \ i = 1, 2. \tag{1.10}$$

Then $(u_1, d_i) \equiv (u_2, d_2)$ on $\mathbb{R}^n \times [0, T)$.

Remark 1.5 (i) When we consider the equation of hydrodynamic flow of liquid crystals (1.3), (1.4), (1.5) and (1.6) on smooth bounded domains $\Omega \subset \mathbb{R}^3$, the uniqueness theorem 1.4 remains to be true under the boundary condition (1.7), provided that $d_0 \in C^{2,\beta}(\partial\Omega, S^2)$ for some $\beta \in (0.1)$. The interested readers can check that this follows from an ϵ_0-boundary regularity estimate similar to lemma 3.2, which can be proved by suitable modifications of the interior ϵ_0-regular lemma 3.2.
(ii) It is also true that both Theorem 1.4 and (i) remain to hold for $n \geq 4$.

The rest of this paper is organized as follows. In §2, we first establish a small energy regularity for (1.1) and then prove Theorem 1.1. In §3, we first establish a uniqueness result under the extra assumption on the blow up rate of $(\|u(t)\|_{L^\infty} + \|\nabla d(t)\|_{L^\infty})$, and then verify that this assumption holds for the class of weak solutions dealt in both Theorem 1.3 and 1.4.

[2]Here $\dot{H}^1$ and $\dot{W}^{1,n}$ denotes the homogeneous Sobolev space on $\mathbb{R}^n$.

2 Proof of Theorem 1.1

For the simplicity of presentation, we assume that $(M, g) = (\mathbb{R}^n, dx^2)$ is the n-dimensional euclidean space equipped with the standard metric.

For $x \in \mathbb{R}^n$, $t > 0$, and $R > 0$, let $B_R(x)$ be the ball in $\mathbb{R}^n$ with center x and radius R and denote $B_R = B_R(0)$; and let

$$P_R(x,t) = B_R(x) \times [t - R^2, t]$$

be the parabolic ball in $\mathbb{R}^{n+1}$ with center (x,t) and radius R and denote $P_R = P_R(0,0)$.

The proof of theorem 1.1 relies on the following two lemmas. The first is an ϵ_0-regularity estimate.

Lemma 2.1 *There is $\epsilon_0 > 0$ such that if $u \in H^1(P_1, N) \cap L^\infty([-1,0], W^{1,n}(B_1, N))$ is a weak solution to (1.1) satisfying*

$$\|\nabla u\|_{L^\infty([-1,0],L^n(B_1))} \leq \epsilon_0, \tag{2.11}$$

then $u \in C^\infty(P_{\frac{1}{2}}, N)$ and

$$\|\nabla u\|_{C^m(P_{\frac{1}{2}})} \leq C(m, \epsilon_0) \|\nabla u\|_{L^2(P_1)}, \ \forall m \geq 0. \tag{2.12}$$

Proof. The reader can refer to Wang [22] for the proof in the critical dimension $n = 2$. Here we present a proof, which is valid for $n \geq 3$.

For any $(x,t) \in P_{\frac{1}{2}}$ and $0 < r < \frac{1}{2}$, it follows from (2.11) that

$$\|\nabla u\|_{L^\infty([t-r^2,t],L^n(B_r(x)))} \leq \epsilon_0.$$

Let $v : P_r(x,t) \to \mathbb{R}^k$ solve

$$\begin{cases} v_t - \Delta v = 0 & \text{in } P_r(x,t) \\ v = u & \text{on } \partial_p P_r(x,t) \end{cases} \tag{2.13}$$

where $\partial_p P_r(x,t) = \big(\partial B_r(x) \times [t-r^2, t]\big) \cup \big(B_r(x) \times \{t - r^2\}\big)$ denotes the parabolic boundary of $P_r(x,t)$.

Multiplying both (1.1) and (2.13) by $u-v$, subtracting the resulting equations, and integrating over $P_r(x,t)$, we obtain

$$\begin{aligned}
&\int_{P_r(x,t)} |\nabla(u-v)|^2 \\
\lesssim\ &\int_{P_r(x,t)} |\nabla u|^2 |u-v| \\
\lesssim\ &\int_{t-r^2}^{t} \|\nabla u\|_{L^2(B_r(x))} \|\nabla u\|_{L^n(B_r(x))} \|u-v\|_{L^{2^*}(B_r(x))} \quad \left(2^* = \frac{2n}{n-2}\right) \\
\lesssim\ &\|\nabla u\|_{L^\infty([t-r^2,t],L^n(B_r(x)))} \int_{t-r^2}^{t} \|\nabla u\|_{L^2(B_r(x))} \|\nabla(u-v)\|_{L^2(B_r(x))} \\
\leq\ &C\epsilon_0 \|\nabla u\|_{L^2(P_r(x,t))} \|\nabla(u-v)\|_{L^2(P_r(x,t))},
\end{aligned}$$

where we have used the Sobolev embedding inequality. Hence we have

$$\int_{P_r(x,t)} |\nabla(u-v)|^2 \leq C\epsilon_0^2 \int_{P_r(x,t)} |\nabla u|^2. \tag{2.14}$$

On the other hand, by the standard theory on the heat equation, we have that for any $\theta \in (0,1)$,

$$(\theta r)^{-n} \int_{P_{\theta r}(x,t)} |\nabla v|^2 \lesssim \theta^2 r^{-n} \int_{P_r(x,t)} |\nabla u|^2. \tag{2.15}$$

Combining (2.14) with (2.15) yields

$$(\theta r)^{-n} \int_{P_{\theta r}(x,t)} |\nabla u|^2 \leq C\left(\theta^2 + \epsilon_0^2 \theta^{-n}\right) r^{-n} \int_{P_r(x,t)} |\nabla u|^2 \tag{2.16}$$

for any $(x,t) \in P_{\frac{1}{2}}$, $0 < r \leq \frac{1}{2}$, and $\theta \in (0,1)$.

For any $\alpha \in (0,1)$, first choose $\theta_0 \in (0,1)$ such that $2C\theta_0^2 \leq \theta_0^{2\alpha}$ and then choose ϵ_0 such that $2C\epsilon_0^2 \leq \theta_0^{n+2\alpha}$, we obtain

$$(\theta_0 r)^{-n} \int_{P_{\theta_0 r}(x,t)} |\nabla u|^2 \leq \theta_0^{2\alpha} r^{-n} \int_{P_r(x,t)} |\nabla u|^2, \ \forall (x,t) \in P_{\frac{1}{2}},\ 0 < r \leq \frac{1}{2}. \tag{2.17}$$

By iterating (2.17), we conclude that for any $\alpha \in (0,1)$, it holds [3]

$$r^{-n} \int_{P_r(x,t)} |\nabla u|^2 \leq C r^{2\alpha} \int_{P_1} |\nabla u|^2,\ \forall (x,t) \in P_{\frac{1}{2}},\ 0 < r \leq \frac{1}{2}. \tag{2.19}$$

[3] We would like to point out that (2.19) would imply the Hölder continuity of u, provided that u satisfies the following local energy inequality:

$$r^{2-n} \int_{P_r(x,t)} |u_t|^2 \lesssim (2r)^{-n} \int_{P_{2r}(x,t)} |\nabla u|^2. \tag{2.18}$$

However, (2.18) doesn't seem to hold automatically for the class of weak solutions of (1.1), $u \in L^\infty([-1,0], W^{1,n}(B_1))$ for $n=3$.

To conclude (2.12) from (2.19) without the local energy inequality (2.18), we employ the estimate of parabolic Riesz potentials in the parabolic Morrey spaces that was established by Huang-Wang [11] recently. For the convenience of readers, we outline the main steps.

First recall the parabolic Morrey spaces on $\mathbb{R}^{n+1}$. For $1 \le p < +\infty$, $0 \le \lambda \le n+2$, and an open set $U \subset \mathbb{R}^{n+1}$, the Morrey space $M^{p,\lambda}(U)$ is defined by

$$M^{p,\lambda}(U) = \left\{ f \in L^p_{\text{loc}}(U) : \ \|f\|^p_{M^{p,\lambda}(U)} \equiv \sup_{z \in U, r>0} r^{\lambda-(n+2)} \int_{P_r(z)\cap U} |f|^p < +\infty \right\}.$$

It is clear that (2.19) implies that for any $\alpha \in (0,1)$, $\nabla u \in M^{2,2-2\alpha}(P_{\frac{1}{2}})$ and

$$\|\nabla u\|_{M^{2,2-2\alpha}(P_{\frac{1}{2}})} \le C \|\nabla u\|_{L^2(P_1)}. \tag{2.20}$$

Now we have

Claim. $\nabla u \in L^q(P_{\frac{1}{2}})$ for any $1 < q < +\infty$ and

$$\|\nabla u\|_{L^q(P_{\frac{1}{2}})} \le C(q) \left[1 + \|\nabla u\|_{L^2(P_1)}\right]. \tag{2.21}$$

To show this claim, let $\eta \in C_0^\infty(P_1)$ be such that $0 \le \eta \le 1$, $\eta \equiv 1$ on $P_{\frac{1}{2}}$, and

$$|\eta_t| + |\nabla \eta| + |\nabla^2 \eta| \le 64.$$

Set $v(z) = \eta(z)u(z)$. Then

$$v_t - \Delta v = F, \ F \equiv [\eta A(u)(\nabla u, \nabla u) - u(\eta_t - \Delta\eta) - 2\nabla u \nabla \eta].$$

Then we have

$$v(z) = \int_{\mathbb{R}^{n+1}} G(z-w)F(w)\, dw$$

where G is the fundamental solution of the heat equation on $\mathbb{R}^n$. By [11] lemma 3.2, we have

$$\begin{aligned} |\nabla v(z)| &= \left| \int_{\mathbb{R}^{n+1}} \nabla G(z-w)F(w) \right| \\ &\le C \int_{\mathbb{R}^{n+1}} \frac{|F(w)|}{\delta(z,w)^{n+1}}\, dw = C I_1(|F|)(z), \end{aligned}$$

where $\delta(z,w)=\max\{|x-y|,\sqrt{|t-s|}\}$ is the parabolic distance between $z=(x,t)$ and $w=(y,s)$, and I_1 is the parabolic Riesz potential of order one [4]. Since $F\equiv 0$ outside P_1, it is not hard to see from (2.20) that $F\in M^{1,2-2\alpha}(\mathbb{R}^{n+1})$ and

$$\|F\|_{M^{1,2-2\alpha}(\mathbb{R}^{n+1})}\leq C\left[1+\|\nabla u\|_{L^2(P_1)}\right].$$

Hence, by the estimate of Riesz potential in Morrey spaces (see [11] Theorem 3.1), we have that $\nabla v\in L^{\frac{2-2\alpha}{1-2\alpha},*}(\mathbb{R}^{n+1})$ [5] and

$$\|\nabla v\|_{L^{\frac{2-2\alpha}{1-2\alpha},*}(\mathbb{R}^{n+1})}\leq C\|F\|_{M^{1,2-2\alpha}(\mathbb{R}^{n+1})}\leq C\left[1+\|\nabla u\|_{L^2(P_1)}\right].$$

Since $\lim_{\alpha\uparrow\frac{1}{2}}\frac{2-2\alpha}{1-2\alpha}=+\infty$, we conclude that $\nabla u\in L^q(P_{\frac{1}{2}})$ for any $1<q<+\infty$ and (2.21) holds.

It is readily seen that claim 2 implies $u\in C^\infty(P_{\frac{1}{2}})$ and (2.12) holds. This completes the proof. $\square$

By suitable translations and dilation of lemma 2.1, we can obtain the blow-up rate of $\|\nabla u(t)\|_{L^\infty(\mathbb{R}^n)}$ as t tends to zero. More precisely, we have

Lemma 2.2 *For $T>0$ and $u_0\in W^{1,n}(\mathbb{R}^n,N)$, suppose $u\in H^1(\mathbb{R}^n\times[0,T],N)\cap C([0,T),W^{1,n}(\mathbb{R}^n,N))$ is a weak solution to (1.1) and (1.2), then there exists $0<t_0\leq T$ depending on u_0,n such that $u\in C^\infty(\mathbb{R}^n\times(0,t_0],N)$ and*

$$\sup_{0<t\leq t_0}\sqrt{t}\,\|\nabla u(t)\|_{L^\infty(\mathbb{R}^n)}<+\infty, \tag{2.22}$$

and

$$\lim_{t\downarrow 0^+}\sqrt{t}\,\|\nabla u(t)\|_{L^\infty(\mathbb{R}^n)}=0. \tag{2.23}$$

Proof. Since $u\in C([0,T),W^{1,n}(\mathbb{R}^n))$ and $u(0)=u_0\in W^{1,n}(\mathbb{R}^n)$, there exist $r_0=r_0(u_0)>0$ and $0<t_0=t_0(u_0)\leq\min\{r_0^2,T\}$ such that

$$\sup_{x\in\mathbb{R}^n,0\leq t\leq t_0}\|\nabla u(t)\|_{L^n(B_{r_0}(x)}\leq\epsilon_0, \tag{2.24}$$

[4] The parabolic Riesz potential of order $0\leq\beta<n+2$ is defined by

$$I_\beta(f)(z)=\int_{\mathbb{R}^{n+1}}\frac{f(w)}{\delta(z,w)^{n+2-\beta}}\,dw.$$

[5] Here $L^{p,*}(\mathbb{R}^{n+1})$ denotes the weak L^p-space for $p\geq 1$.

where $\epsilon_0 > 0$ is given by lemma 2.1. In particular, we have that for any $x \in \mathbb{R}^n$ and $0 < \tau \le \sqrt{t_0}$,

$$\|\nabla u\|_{L^\infty([0,\tau^2],L^n(B_\tau(x)))} \le \epsilon_0.$$

Define $v(y,s) = u(x+\tau y, \tau^2 + \tau^2 s)$ for $(y,s) \in P_1$. Then v solves (1.1) on P_1 and

$$\|\nabla v\|_{L^\infty([-1,0],L^n(B_1))} \le \epsilon_0.$$

Applying lemma 2.1, we conclude that $v \in C^\infty(P_{\frac{1}{2}})$ and

$$\|\nabla v\|_{L^\infty(P_{\frac{1}{2}})} \le C \|\nabla v\|_{L^2(P_1)}.$$

Back to the original scales, this implies $u \in C^\infty(P_{\frac{\tau}{2}}(x,\tau^2))$ and

$$\tau \|\nabla u\|_{L^\infty(P_{\frac{\tau}{2}}(x,\tau^2))} \le C(\tau^{-n} \int_{P_\tau(x,\tau^2)} |\nabla u|^2)^{\frac{1}{2}} \le C \|\nabla u\|_{L^\infty([0,\tau^2],L^n(B_\tau(x)))}. \quad (2.25)$$

Taking supremum over all $x \in \mathbb{R}^n$ and $0 < \tau \le \sqrt{t_0}$ yields (2.22). To see (2.23), observe that for any $0 < \epsilon \le \epsilon_0$, there exist $\tau_\epsilon > 0$ and $r_\epsilon > 0$ such that

$$\sup_{0 \le t \le \tau_\epsilon} \left(\int_{\mathbb{R}^n} |\nabla u(t) - \nabla u_0|^n \right)^{\frac{1}{n}} \le \frac{\epsilon}{2},$$

and

$$\sup_{x \in \mathbb{R}^n, 0 < r \le r_\epsilon} \left(\int_{B_r(x)} |\nabla u_0|^n \right)^{\frac{1}{n}} \le \frac{\epsilon}{2}.$$

Hence there exists $t_\epsilon > 0$ such that

$$\sup_{x \in \mathbb{R}^n, 0 \le \tau \le t_\epsilon} \|\nabla u\|_{L^\infty([0,\tau^2],L^n(B_\tau(x)))} \le \epsilon.$$

Hence (2.25) yields

$$\sup_{0 < \tau \le t_\epsilon} \tau \left\|\nabla u(\tau^2)\right\|_{L^\infty(\mathbb{R}^n)} \le C\epsilon.$$

This clearly implies (2.23). The proof is now complete. □

Proof of Theorem 1.1. First, by interpolation inequalities, (2.22) and (2.23) imply that for any $n < p \le +\infty$,

$$\sup_{0 < t \le t_0} \sqrt{t}^{1-\frac{n}{p}} \|\nabla u(t)\|_{L^p(\mathbb{R}^n)} < +\infty, \ \lim_{t \downarrow 0} \sqrt{t}^{1-\frac{n}{p}} \|\nabla u(t)\|_{L^p(\mathbb{R}^n)} = 0. \quad (2.26)$$

Set $w = u - v$. Then $w \in C([0,t_0], W^{1,n}(\mathbb{R}^n))$ solves

$$\begin{aligned} w_t - \Delta w &= A(u)(\nabla u, \nabla u) - A(v)(\nabla v, \nabla v) \text{ in } \mathbb{R}^n \times (0, t_0) \\ w\big|_{t=0} &= 0. \end{aligned}$$

Direct calculations imply

$$|A(u)(\nabla u, \nabla u) - A(v)(\nabla v, \nabla v)| \leq C\left[(|\nabla u| + |\nabla v|)|\nabla w| + |\nabla v|^2 |w|\right]. \tag{2.27}$$

By the Duhamel's formula, we have

$$\begin{aligned} |w(t)| &= \left|\int_0^t e^{-(t-s)\Delta}(A(u)(\nabla u, \nabla u) - A(v)(\nabla v, \nabla v))(s)\, ds\right| \\ &\lesssim \int_0^t e^{-(t-s)\Delta}\left[(|\nabla u| + |\nabla v|)|\nabla w| + |\nabla v|^2|w|\right](s)\, ds, \end{aligned} \tag{2.28}$$

and

$$\begin{aligned} |\nabla w(t)| &= \left|\int_0^t \nabla e^{-(t-s)\Delta}(A(u)(\nabla u, \nabla u) - A(v)(\nabla v, \nabla v))(s)\, ds\right| \\ &\lesssim \int_0^t \left|\nabla e^{-(t-s)\Delta}\left[(|\nabla u| + |\nabla v|)|\nabla w| + |\nabla v|^2|w|\right](s)\right| ds. \end{aligned} \tag{2.29}$$

To proceed with the proof, we need three claims.

Claim 1. For any $0 < \delta < 1$, there exists $C = C(\delta) > 0$ such that for $0 < t \leq t_0$,

$$t^{-\frac{\delta}{2}} \|w(t)\|_{L^{\frac{n}{\delta}}(\mathbb{R}^n)} \leq C\left(\sup_{0\leq s\leq t}(\|\nabla u(s)\|_{L^n(\mathbb{R}^n)} + \|\nabla v(s)\|_{L^n(\mathbb{R}^n)})\right)^2. \tag{2.30}$$

To see it, applying the standard estimate of the heat kernel[6] to (2.28) yields

$$\begin{aligned} \|w(t)\|_{L^{\frac{n}{\delta}}(\mathbb{R}^n)} &\lesssim \int_0^t (t-s)^{-\frac{2-\delta}{2}}\left(\|\nabla u(s)\|_{L^n(\mathbb{R}^n)} + \|\nabla v(s)\|_{L^n(\mathbb{R}^n)}\right)^2 ds \\ &\lesssim \left(\int_0^t (t-s)^{-\frac{2-\delta}{2}}\, ds\right)\left(\sup_{0\leq s\leq t}(\|\nabla u(s)\|_{L^n(\mathbb{R}^n)} + \|\nabla v(s)\|_{L^n(\mathbb{R}^n)}\right)^2 \\ &= Ct^{\frac{\delta}{2}}\left(\sup_{0\leq s\leq t}(\|\nabla u(s)\|_{L^n(\mathbb{R}^n)} + \|\nabla v(s)\|_{L^n(\mathbb{R}^n)})\right)^2. \end{aligned}$$

[6]For $1 \leq p \leq q \leq +\infty$,

$$\left\|e^{-t\Delta} f\right\|_{L^q(\mathbb{R}^n)} \lesssim t^{-(\frac{1}{p}-\frac{1}{q})\frac{n}{2}} \|f\|_{L^p(\mathbb{R}^n)}, \quad \left\|\nabla e^{-t\Delta} f\right\|_{L^q(\mathbb{R}^n)} \lesssim t^{-(1+\frac{1}{p}-\frac{1}{q})\frac{n}{2}} \|f\|_{L^p(\mathbb{R}^n)}.$$

For $0 < t \le t_0$, set

$$A(t) = \sup_{0<s\le t} \sqrt{s}\left(\|\nabla u(s)\|_{L^\infty(\mathbb{R}^n)} + \|\nabla v(s)\|_{L^\infty(\mathbb{R}^n)}\right),$$

$$B_\delta(t) = \sup_{0<s\le t} \sqrt{s}^{1-\frac{\delta}{2}}\left(\|\nabla u(s)\|_{L^{\frac{n}{\delta}}(\mathbb{R}^n)} + \|\nabla v(s)\|_{L^{\frac{n}{\delta}}(\mathbb{R}^n)}\right).$$

Then we have

Claim 2. There exists $C = C(\delta) > 0$ such that for any $0 < t \le t_0$,

$$\begin{aligned} t^{-\frac{\delta}{2}} \|w(t)\|_{L^{\frac{n}{\delta}}(\mathbb{R}^n)} &\le C[A^2(t)(\sup_{0<s\le t} s^{-\frac{\delta}{2}}\|w(s)\|_{L^{\frac{n}{\delta}}(\mathbb{R}^n)}) \\ &\quad + B_\delta(t)(\sup_{0\le s\le t} \|\nabla w(s)\|_{L^n(\mathbb{R}^n)})]. \end{aligned} \tag{2.31}$$

This is a refinement of claim 1. By (2.28) and (2.27), we have

$$\begin{aligned} \|w(t)\|_{L^{\frac{n}{\delta}}(\mathbb{R}^n)} &\lesssim \int_0^t (t-s)^{-\frac{1}{2}}(\|\nabla u(s)\|_{L^{\frac{n}{\delta}}(\mathbb{R}^n)} + \|\nabla v(s)\|_{L^{\frac{n}{\delta}}(\mathbb{R}^n)})\|\nabla w(s)\|_{L^n(\mathbb{R}^n)}\, ds \\ &+ \int_0^t \|\nabla v(s)\|^2_{L^\infty(\mathbb{R}^n)}\|w(s)\|_{L^{\frac{n}{\delta}}(\mathbb{R}^n)}\, ds \\ &= I + II. \end{aligned}$$

I can be estimated by

$$\begin{aligned} I &\lesssim \left(\int_0^t (t-s)^{-\frac{1}{2}} s^{-\frac{1-\delta}{2}}\, ds\right) \cdot \sup_{0<s\le t} \sqrt{s}^{\frac{1-\delta}{2}}\left(\|\nabla u(s)\|_{L^{\frac{n}{\delta}}(\mathbb{R}^n)} + \|\nabla v(s)\|_{L^{\frac{n}{\delta}}(\mathbb{R}^n)}\right) \\ &\quad \cdot \sup_{0\le s\le t} \|\nabla w(s)\|_{L^n(\mathbb{R}^n)} \\ &\lesssim t^{\frac{\delta}{2}} B_\delta(t) \cdot \sup_{0\le s\le t} \|\nabla w(s)\|_{L^n(\mathbb{R}^n)}, \end{aligned}$$

since

$$\int_0^t (t-s)^{-\frac{1}{2}} s^{-\frac{1-\delta}{2}}\, ds = t^{\frac{\delta}{2}} \int_0^1 (1-s)^{-\frac{1}{2}} s^{-\frac{1-\delta}{2}}\, ds = Ct^{\frac{\delta}{2}}.$$

II can be estimated by

$$\begin{aligned} II &\lesssim \left(\int_0^t s^{-1+\frac{\delta}{2}}\, ds\right) \cdot \left[\sup_{0<s\le t} \sqrt{s}\|\nabla v(s)\|_{L^\infty(\mathbb{R}^n)}\right]^2 \cdot \left[\sup_{0<s\le t} s^{-\frac{\delta}{2}}\|\nabla w(s)\|_{L^{\frac{n}{\delta}}(\mathbb{R}^n)}\right] \\ &\lesssim t^{\frac{\delta}{2}} A^2(t) \cdot \sup_{0<s\le t} s^{-\frac{\delta}{2}} \|w(s)\|_{L^{\frac{n}{\delta}}(\mathbb{R}^n)}. \end{aligned}$$

Putting these two estimates together yields (2.31). Finally, we need

Claim 3. There exists $C = C(\delta) > 0$ such that for any $0 < t \le t_0$,

$$\begin{aligned} \|\nabla w(t)\|_{L^n(\mathbb{R}^n)} &\le C[A(t)\left(\sup_{0\le s\le t} \|\nabla v(s)\|_{L^n(\mathbb{R}^n)}\right)\left(\sup_{0<s\le t} s^{-\frac{\delta}{2}}\|w(s)\|_{L^{\frac{n}{\delta}}(\mathbb{R}^n)}\right) \\ &\quad + B_\delta(t) \sup_{0\le s\le t} \|\nabla w(s)\|_{L^n(\mathbb{R}^n)}]. \end{aligned} \tag{2.32}$$

To show (2.32), observe that (2.29) and the standard estimate on the heat kernel imply

$$\begin{aligned}
&\|\nabla w(t)\|_{L^n(\mathbb{R}^n)}\\
\lesssim\ & \int_0^t (t-s)^{-\frac{1+\delta}{2}} \left(\|\nabla u(s)\|_{L^{\frac{n}{\delta}}(\mathbb{R}^n)} + \|\nabla v(s)\|_{L^{\frac{n}{\delta}}(\mathbb{R}^n)}\right) \|\nabla w(s)\|_{L^n(\mathbb{R}^n)}\, ds\\
+\ & \int_0^t (t-s)^{-\frac{1+\delta}{2}} \left\||\nabla v(s)|^2\right\|_{L^n(\mathbb{R}^n)} \|w(s)\|_{L^{\frac{n}{\delta}}(\mathbb{R}^n)}\, ds\\
=\ & III + IV.
\end{aligned}$$

III can be estimated by

$$\begin{aligned}
III &\lesssim \left(\int_0^t (t-s)^{-\frac{1+\delta}{2}} s^{-\frac{1-\delta}{2}}\, ds\right) B_\delta(t) \left(\sup_{0\le s\le t} \|\nabla w(s)\|_{L^n(\mathbb{R}^n)}\right)\\
&\lesssim B_\delta(t) \left(\sup_{0\le s\le t} \|\nabla w(s)\|_{L^n(\mathbb{R}^n)}\right),
\end{aligned}$$

since

$$\int_0^t (t-s)^{-\frac{1+\delta}{2}} s^{-\frac{1-\delta}{2}}\, ds = \int_0^1 (1-s)^{-\frac{1+\delta}{2}} s^{-\frac{1-\delta}{2}}\, ds < +\infty.$$

IV can be estimated by

$$\begin{aligned}
IV &\lesssim \left(\int_0^t (t-s)^{-\frac{1+\delta}{2}} s^{\frac{\delta-1}{2}}\, ds\right)\cdot\\
&\quad A(t) \left(\sup_{0\le s\le t} \|\nabla v(s)\|_{L^n(\mathbb{R}^n)}\right) \left(\sup_{0< s\le t} s^{-\frac{\delta}{2}} \|w(s)\|_{L^{\frac{n}{\delta}}(\mathbb{R}^n)}\right)\\
&\lesssim A(t) \left(\sup_{0\le s\le t} \|\nabla v(s)\|_{L^n(\mathbb{R}^n)}\right) \left(\sup_{0< s\le t} s^{-\frac{\delta}{2}} \|w(s)\|_{L^{\frac{n}{\delta}}(\mathbb{R}^n)}\right),
\end{aligned}$$

since

$$\int_0^t (t-s)^{-\frac{1+\delta}{2}} s^{\frac{\delta-1}{2}}\, ds = \int_0^1 (1-s)^{-\frac{1+\delta}{2}} s^{\frac{\delta-1}{2}}\, ds < +\infty.$$

Putting these two estimates together yields (2.32).

Now define the function $\Phi : (0, t_0] \to \mathbb{R}_+$ by

$$\Phi(t) = \left[\sup_{0\le s\le t} \|\nabla w(s)\|_{L^n(\mathbb{R}^n)} + \sup_{0< s\le t} s^{-\frac{\delta}{2}} \|w(s)\|_{L^{\frac{n}{\delta}}(\mathbb{R}^n)}\right],\ 0 < t \le t_0.$$

Without loss of generality, we may assume that

$$\Phi(t) = \left[\|\nabla w(t)\|_{L^n(\mathbb{R}^n)} + t^{-\frac{\delta}{2}} \|w(t)\|_{L^{\frac{n}{\delta}}(\mathbb{R}^n)}\right].$$

Then (2.31) and (2.32) imply

$$\Phi(t) \leq C\left[\left(1+\sup_{0\leq s\leq t}\|\nabla v(s)\|_{L^n(\mathbb{R}^n)}\right)A(t)+B_\delta(t)\right]\Phi(t).$$

It follows from (2.26) that there exists sufficiently small $0 < t_1 \leq t_0$ such that

$$C\left[\left(1+\sup_{0\leq s\leq t_1}\|\nabla v(s)\|_{L^n(\mathbb{R}^n)}\right)A(t_1)+B_\delta(t_1)\right]\leq \frac{1}{2}.$$

Hence

$$\Phi(t_1) \leq \frac{1}{2}\Phi(t_1).$$

This implies $\Phi(t_1) = 0$. Thus $u \equiv v$ on $\mathbb{R}^n \times [0, t_1)$. Repeating the above argument at $t = t_1$, we can conclude $u \equiv v$ on $\mathbb{R}^n \times [0, T)$. This completes the proof. □

3 Proof of Theorem 1.3 and Theorem 1.4

In this section, we will present the proof of the uniqueness theorem for the hydrodynamic flow of liquid crystals. There are two steps to prove Theorem 1.3 and 1.4: (i) we establish the uniqueness under the extra assumption that

$$\sqrt{t}\left[\|u(t)\|_{L^\infty(\Omega)}+\|\nabla d(t)\|_{L^\infty(\Omega)}\right]\to 0,\ \text{as } t\downarrow 0,$$

and (ii) we verify that this assumption holds for the class of weak solutions we consider in Theorem 1.3 and 1.4.

Lemma 3.1 *For $n = 2$ or 3 and $0 < T < +\infty$, suppose that for $i = 1, 2$, $(u_i, d_i) : \Omega \times [0, T) \to \mathbb{R}^n \times S^2$ are a pair of weak solutions of (1.3), (1.4), (1.5), (1.6) (and (1.7) when $\Omega \subset \mathbb{R}^n$ is a bounded domain) with $u_i \in L^\infty_t L^2_x \cap L^2_t H^1_x(\Omega \times [0, T])$ and $\nabla d_i \in L^2(\Omega \times [0, T])$. There exists $\epsilon_0 > 0$ such that if for some $0 < t_0 \leq T$*

$$\max_{i=1,2}\sup_{0<t\leq t_0}\sqrt{t}\left[\|u_i(t)\|_{L^\infty(\Omega)}+\|\nabla d_i\|_{L^\infty(\Omega)}\right]\leq \epsilon_0, \tag{3.33}$$

then $u_1 \equiv u_2$ and $d_1 \equiv d_2$ on $\Omega \times [0, t_0]$[7].

[7] It is known that the weak solutions (u_i, d_i), $i = 1, 2$, are smooth in $\Omega \times (0, t_0]$, see for example [19].

Proof. For $1 < p < +\infty$, let $\mathbb{E}^p$ be the closure in $L^p(\Omega, \mathbb{R}^n)$ of all divergence-free vector fields with compact support in Ω. Let $\mathbb{P} : L^2(\Omega, \mathbb{R}^n) \to \mathbb{E}^2$ be the Leray projection operator. It is well-known that P can be extended to a bounded operator from $L^p(\Omega, \mathbb{R}^n)$ to $\mathbb{E}^p$ for all $1 < p < +\infty$. Let $\mathbb{A} = \mathbb{P}\Delta$ be the Stokes operator [8]

Let $w = u_1 - u_2$ and $d = d_1 - d_2$. Applying $\mathbb{P}$[9] to both sides of (1.3) for u_1 and u_2 and subtracting the resulting equations, it is not hard to see that (w, d) satisfies:

$$\begin{aligned} w_t - \mathbb{A}w &= -\mathbb{P}\nabla \cdot (w \otimes u_1 + u_2 \otimes w + \nabla d \otimes \nabla d_1 + \nabla d_2 \otimes \nabla d) && (3.34)\\ \nabla \cdot w &= 0 && (3.35)\\ d_t - \Delta d &= \big[(\nabla d_1 + \nabla d_2) \cdot \nabla d d_1 + |\nabla d_2|^2 d\big] - [w \cdot \nabla d_1 + u_2 \cdot \nabla d] && (3.36)\\ (w, d)|_{t=0} &= (0, 0), && (3.37) \end{aligned}$$

and

$$(w, d) = (0, 0) \text{ on } \partial\Omega \times (0, T) \tag{3.38}$$

when $\Omega \subset \mathbb{R}^n$ is a bounded domain.

For $0 < t \le t_0$, set

$$A_i(t) = \sqrt{t}\left[\|u_i(t)\|_{L^\infty(\Omega)} + \|\nabla d_i\|_{L^\infty(\Omega)}\right],\ i = 1, 2,$$

$$C(t) = \left[\|u_1(t)\|_{L^n(\Omega)} + \|u_2(t)\|_{L^n(\Omega)} + \|\nabla d_1(t)\|_{L^n(\Omega)} + \|\nabla d_2(t)\|_{L^n(\Omega)}\right],$$

and for fixed $0 < \delta < 1$,

$$D_\delta(t) = t^{\frac{1-\delta}{2}}\left(\|u_1(t)\|_{L^{\frac{n}{\delta}}(\Omega)} + \|u_2(t)\|_{L^{\frac{n}{\delta}}(\Omega)} + \|\nabla d_1(t)\|_{L^{\frac{n}{\delta}}(\Omega)} + \|\nabla d_2(t)\|_{L^{\frac{n}{\delta}}(\Omega)}\right).$$

Then we have, by interpolation inequalities, that

$$D_\delta(t) \le C(t)^\delta \left(A_1(t) + A_2(t)\right)^{1-\delta},\ \forall 0 < t \le t_0. \tag{3.39}$$

By the Duhamel formula, we have

$$w(t) = -\int_0^t e^{-(t-s)\mathbb{A}}\mathbb{P}\nabla \cdot (w \otimes u_1 + u_2 \otimes w + \nabla d \otimes \nabla d_1 + \nabla d_2 \otimes \nabla d)(s), \tag{3.40}$$

[8]Note that if $\Omega = \mathbb{R}^n$, then $\mathbb{A} = \Delta$ on $\mathbb{E}^p \cap W_0^{2,p}(\mathbb{R}^n, \mathbb{R}^n)$.

[9]This is possible, since the assumption (3.33) can imply that for $i = 1, 2$, $d_i \in W_q^{2,1}(\mathbb{R}^n \times [t_1, t_0])$ for any $1 < q < +\infty$ and hence we can choose the pressure P_i such that $\nabla P_i \in L^2(\mathbb{R}^n \times [t_1, t_0])$ for any $t_1 > 0$.

and

$$d(t) = \int_0^t e^{-(t-s)\Delta}\left[(\nabla d_1 + \nabla d_2)\cdot\nabla d d_1 + |\nabla d_2|^2 d - w\cdot\nabla d_1 - u_2\cdot\nabla d\right](s). \quad (3.41)$$

Similar to the proof of Theorem 1.1, we can estimate d as follows. We need to estimate

$$\sup_{0<t\le t_0} t^{-\frac{\delta}{2}}\|d(t)\|_{L^{\frac{n}{\delta}}(\Omega)}.$$

To proceed, we first claim

$$t^{-\frac{\delta}{2}}\|d(t)\|_{L^{\frac{n}{\delta}}(\Omega)} \lesssim \sup_{0<t\le t_0} C(t),\ \forall 0<t\le t_0. \quad (3.42)$$

In fact, since $|d|\le|d_1|+|d_2|=2$, (3.41) and the standard estimate on the heat kernel imply that for $0<t\le t_0$,

$$\begin{aligned}
&\|d(t)\|_{L^{\frac{n}{\delta}}(\Omega)}\\
\lesssim\ & \int_0^t (t-s)^{-\frac{2-\delta}{2}}\left[\sum_{i=1}^2(\|\nabla d_i(s)\|_{L^n(\Omega)}+\|u_i(s)\|_{L^n(\Omega)})\right]^2 ds\\
\lesssim\ & \left(\int_0^t (t-s)^{-\frac{2-\delta}{2}}\,ds\right)\left(\sup_{0\le s\le t} C(s)\right)\\
\lesssim\ & t^{\frac{\delta}{2}}\sup_{0\le s\le t} C(s).
\end{aligned}$$

This yields (3.42).

Next we want to refine the above estimate as follows. (3.41) and the standard estimate on the heat kernel imply that for $0<t\le t_0$,

$$\begin{aligned}
&\|d(t)\|_{L^{\frac{n}{\delta}}(\Omega)}\\
\lesssim\ & \int_0^t (t-s)^{-\frac{1}{2}}[\sum_{i=1}^2(\|\nabla d_i(s)\|_{L^{\frac{n}{\delta}}(\Omega)}+\|u_i(s)\|_{L^{\frac{n}{\delta}}(\Omega)})]\cdot(\|\nabla d(s)\|_{L^n(\Omega)}+\|w(s)\|_{L^n(\Omega)})\\
+\ & \int_0^t (t-s)^{-\frac{1}{2}}\|\nabla d_2(s)\|_{L^\infty(\Omega)}\|\nabla d_2(s)\|_{L^n(\Omega)}\|d(s)\|_{L^{\frac{n}{\delta}}(\Omega)}\\
\lesssim\ & \left(\int_0^t (t-s)^{-\frac{1}{2}} s^{\frac{\delta-1}{2}}\,ds\right)\left(\sup_{0<s\le t} D_\delta(s)\right)\left(\sup_{0\le s\le t}(\|\nabla d(s)\|_{L^n(\Omega)}+\|w(s)\|_{L^n(\Omega)})\right)\\
+\ & \left(\int_0^t (t-s)^{-\frac{1}{2}} s^{\frac{\delta-1}{2}}\,ds\right)\left(\sup_{0<s\le t} A_2(s)\right)\left(\sup_{0\le s\le t} C(s)\right)\left(\sup_{0<s\le t} s^{-\frac{\delta}{2}}\|d(s)\|_{L^{\frac{n}{\delta}}(\Omega)}\right)\\
\lesssim\ & t^{\frac{\delta}{2}}\left[\sup_{0<s\le t} D_\delta(s)+\left(\sup_{0<s\le t} A_2(s)\right)\left(\sup_{0<s\le t} C(s)\right)\right]\\
&\cdot\left[\sup_{0<s\le t}\left(\|\nabla d(s)\|_{L^n(\Omega)}+\|w(s)\|_{L^n(\Omega)}+s^{-\frac{\delta}{2}}\|d(s)\|_{L^{\frac{n}{\delta}}(\Omega)}\right)\right],
\end{aligned} \quad (3.43)$$

where we have used the inequality

$$\int_0^t (t-s)^{-\frac{1}{2}} s^{\frac{\delta-1}{2}}\, ds \lesssim t^{\frac{\delta}{2}}.$$

Applying ∇ of both sides of (3.41) and employing the standard L^p-estimate of $\nabla e^{-t\Delta}$, we have that for $0 < t \le t_0$,

$$\begin{aligned}
& \|\nabla d(t)\|_{L^n(\Omega)} \\
\lesssim\ & \int_0^t (t-s)^{-\frac{1+\delta}{2}} \left(\|\nabla d_1(s)\|_{L^{\frac{n}{\delta}}(\Omega)} + \|\nabla d_2(s)\|_{L^{\frac{n}{\delta}}(\Omega)}\right) \|\nabla d(s)\|_{L^(\Omega)}\, ds \\
+\ & \int_0^t (t-s)^{-\frac{1+\delta}{2}} \left(\|\nabla d_1(s)\|_{L^{\frac{n}{\delta}}(\Omega)} \|w(s)\|_{L^n(\Omega)} + \|u_2(s)\|_{L^{\frac{n}{\delta}}(\Omega)} \|\nabla d(s)\|_{L^n(\Omega)}\right) ds \\
+\ & \int_0^t (t-s)^{-\frac{1+\delta}{2}} \|\nabla d_2(s)\|_{L^\infty(\Omega)} \|\nabla d_2(s)\|_{L^n(\Omega)} \|d(s)\|_{L^{\frac{n}{\delta}}(\Omega)}\, ds \\
\lesssim\ & \left(\int_0^t (t-s)^{-\frac{1+\delta}{2}} s^{\frac{\delta-1}{2}}\, ds\right) \left(\sup_{0<s\le t} D_\delta(s)\right) \left(\sup_{0\le s\le t} \left(\|w(s)\|_{L^n(\Omega)} + \|\nabla d\|_{L^n(\Omega)}\right)\right) \\
+\ & \left(\int_0^t (t-s)^{-\frac{1+\delta}{2}} s^{\frac{\delta-1}{2}}\, ds\right) \left(\sup_{0<s\le t} A_2(s)\right) \left(\sup_{0<s\le t} C(s)\right) \left(\sup_{0<s\le t} s^{-\frac{\delta}{2}} \|d(s)\|_{L^{\frac{n}{\delta}}(\Omega)}\right) \\
\le\ & C \left(\sup_{0<s\le t} D_\delta(s)\right) \left(\sup_{0\le s\le t} \left(\|w(s)\|_{L^n(\Omega)} + \|\nabla d\|_{L^n(\Omega)}\right)\right) \\
+\ & C \left(\sup_{0<s\le t} A_2(s)\right) \left(\sup_{0<s\le t} C(s)\right) \left(\sup_{0<s\le t} s^{-\frac{\delta}{2}} \|d(s)\|_{L^{\frac{n}{\delta}}(\Omega)}\right),
\end{aligned} \tag{3.44}$$

where we have used

$$\int_0^t (t-s)^{-\frac{1+\delta}{2}} s^{\frac{\delta-1}{2}}\, ds = \int_0^1 (1-s)^{-\frac{1+\delta}{2}} s^{\frac{\delta-1}{2}}\, ds < +\infty.$$

Now we want to estimate $\|w(t)\|_{L^n(\Omega)}$. Before doing it, we need to recall the following $L^p - L^q$ estimate of $e^{-t\mathbb{A}}\mathbb{P}\nabla$:

$$\left\| e^{-t\mathbb{A}} \mathbb{P} \nabla f \right\|_{L^q(\Omega)} \lesssim t^{-\frac{1+(\frac{n}{p}-\frac{n}{q})}{2}} \|f\|_{L^p(\Omega)}\ \forall 1 < p \le q < +\infty. \tag{3.45}$$

The reader can find the proof of (3.45) by Kato[12] when $\Omega = \mathbb{R}^n$, and by Giga [10] when $\Omega \subset \mathbb{R}^n$ is a bounded domain.

Applying (3.45) with $p = \frac{n}{1+\delta}$ and $q = n$ to (3.40), we have that for $0 < t \le t_0$,

$$\begin{aligned}
\|w(t)\|_{L^n(\Omega)} &\lesssim \int_0^t (t-s)^{-\frac{1+\delta}{2}} (\|u_1(s)\|_{L^{\frac{n}{\delta}}(\Omega)} + \|u_2(s)\|_{L^{\frac{n}{\delta}}(\Omega)}) \|w(s)\|_{L^n(\Omega)}\, ds \\
&+ \int_0^t (t-s)^{-\frac{1+\delta}{2}} (\|\nabla d_1(s)\|_{L^{\frac{n}{\delta}}(\Omega)} + \|\nabla d_2(s)\|_{L^{\frac{n}{\delta}}(\Omega)}) \|\nabla d(s)\|_{L^n(\Omega)}\, ds \\
&\lesssim \left(\int_0^t (t-s)^{-\frac{1+\delta}{2}} s^{\frac{\delta-1}{2}}\, ds \right) \left(\sup_{0<s\le t} D_\delta(s) \right) \\
&\quad \cdot \left(\sup_{0\le s\le t} (\|w(s)\|_{L^n(\Omega)} + \|\nabla d\|_{L^n(\Omega)}) \right) \\
&\le C \left(\sup_{0<s\le t} D_\delta(s) \right) \left[\sup_{0\le s\le t} \left(\|w(s)\|_{L^n(\Omega)} + \|\nabla d\|_{L^n(\Omega)} \right) \right]. \qquad (3.46)
\end{aligned}$$

Finally, set the function $\Phi : (0, t_0] \to \mathbb{R}_+$ by

$$\Phi(t) = \sup_{0<s\le t} \left(\|\nabla d(s)\|_{L^n(\Omega)} + \|w(s)\|_{L^n(\Omega)} + s^{-\frac{\delta}{2}} \|d(s)\|_{L^{\frac{n}{\delta}}(\Omega)} \right).$$

Combining the inequalities (3.46)-(3.44)-(3.43) together, we obtain that for $0 < t \le t_0$,

$$\begin{aligned}
\Phi(t) &\le C \left[\sup_{0<s\le t} D_\delta(s) + \sup_{0<s\le t} (A_1(s) + A_2(s)) \sup_{0<s\le t} C(s) \right] \Phi(t) \\
&\le \frac{1}{2} \Phi(t) \qquad (3.47)
\end{aligned}$$

provided that $\epsilon_0 > 0$ is sufficiently small such that

$$C \left[\sup_{0<s\le t_0} D_\delta(s) + \sup_{0<s\le t_0} (A_1(s) + A_2(s)) \sup_{0<s\le t_0} C(s) \right] \le C(\delta)[\epsilon_0^{1-\delta} + \epsilon_0] \le \frac{1}{2}.$$

This implies $\Phi(t) \equiv 0$ for $0 < t \le t_0$. Hence $(u_1, d_1) \equiv (u_2, d_2)$ on $\Omega \times [0, t_0]$. □

Proof of Theorem 1.3:

First it follows [19] Theorem 1.2 that (i) $(u_i, d_i) \in C^\infty(\Omega \times (0, T])$ for $i = 1, 2$, and (ii) for $\Omega \subset \mathbb{R}^2$ a bounded domain, since $d_0 \in C^{2,\beta}(\partial\Omega, S^2)$ for some $\beta \in (0, 1)$, $(u_i, d_i) \in C^{2,1}_\beta(\overline{\Omega} \times (0, T])$[10]. Moreover, by a simple scaling argument, we have that for any $0 < t \le T$,

$$A_i(t) \equiv \sup_{0<s\le t} \sqrt{s} \left(\|u_i(s)\|_{L^\infty(\Omega)} + \|\nabla d_i(s)\|_{L^\infty(\Omega)} \right) < +\infty, \ i = 1, 2. \qquad (3.48)$$

[10]Here $C^{2,1}_\beta(\overline{\Omega} \times (0, T])$ denotes the spaces of C^1 functions f such that $\nabla_x^2 f, f_t \in C^\beta(\overline{\Omega} \times (0, T])$.

It remains to show that

$$\lim_{t\downarrow 0} A_i(t) = 0,\ i = 1, 2. \tag{3.49}$$

To see (3.49), recall that the weak solution (u_i, d_i), $i = 1, 2$, in Theorem 1.3 satisfies the following energy inequality (see [19]):

$$\int_\Omega (|u_i(t)|^2 + |\nabla d_i(t)|^2) + 2\int_0^t \int_\Omega (|\nabla u_i|^2 + |\Delta d_i + |\nabla d_i|^2 d_i|^2)$$
$$\leq \int_\Omega (|u_0|^2 + |\nabla d_0|^2). \tag{3.50}$$

In particular, for $i = 1, 2$, $E_i(t) = \int_\Omega (|u_i(t)|^2 + |\nabla d_i(t)|^2)$ is monotonically nonincreasing with respect to $t \geq 0$. Hence

$$\lim_{t\downarrow 0} E_i(t) \leq E(0) \equiv \int_\Omega (|u_0|^2 + |\nabla d_0|^2).$$

On the other hand, for $i = 1, 2$, since $(u_i(t), \nabla d_i(t))$ converges weakly to $(u_0, \nabla d_0)$ in $L^2(\Omega)$ as $t \downarrow 0$, the lower semicontinuity implies

$$\lim_{t\downarrow 0} E_i(t) \geq E(0).$$

Thus

$$\lim_{t\downarrow 0} E_i(t) = E(0),\ i = 1, 2$$

and hence $E_i(t) \in C([0, T])$ for $i = 1, 2$. Now we can use the argument similar to that of Theorem 1.1 to show that

$$\lim_{t\downarrow 0} \sup_{x\in\overline{\Omega}} \int_{B_t(x)\times[0,t^2]} \sum_{i=1}^2 \left(|u_i|^2 + |\nabla d_i|^2\right) = 0.$$

Applying [19] Theorem 1.2 again, this implies (3.49). It is clear that (3.49) and Lemma 3.1 imply that there exists $0 < t_0 < T$ such that $(u_1, d_1) = (u_2, d_2)$ on $\Omega \times [0, t_0]$. For $i = 1, 2$, since $(u_i, d_i) \in C^\infty(\Omega \times [t_0, T]) \cap C^{2,1}_\beta(\overline{\Omega} \times [t_0, T])$ solves (1.3), (1.4), (1.5), under either the same initial condition for $\Omega = \mathbb{R}^2$ or the same initial and boundary conditions for $\Omega \subset \mathbb{R}^2$ being a bounded domain, the uniqueness for classical solutions implies $(u_1, d_1) = (u_2, d_2)$ on $\Omega \times [t_0, T]$. This completes the proof. □

In order to prove Theorem 1.4 for $n = 3$, we need to establish an ϵ_0-regularity estimate similar to lemma 2.1 for the heat flow of harmonic maps. More precisely, we have

Lemma 3.2 *For $n = 3$, there exists $\epsilon_0 > 0$ such that if $u \in L^\infty([-1,0], L^n(B_1, \mathbb{R}^n))$, $P \in L^{\frac{n}{2}}(P_1)$, and $d \in L^\infty([-1,0], W^{1,n}(B_1, S^2))$ is a weak solution of (1.3), (1.4), and (1.5) that satisfies*

$$\left[\|u\|_{L^\infty([-1,0],L^n(B_1))} + \|\nabla d\|_{L^\infty([-1,0],L^n(B_1))}\right] \le \epsilon_0. \tag{3.51}$$

Then $(u, d) \in C^\infty(P_{\frac{1}{4}}, \mathbb{R}^n \times S^2)$ and

$$\left[\|u\|_{L^\infty(P_{\frac{1}{4}})} + \|\nabla d\|_{L^\infty(P_{\frac{1}{4}})}\right] \le C(\epsilon_0). \tag{3.52}$$

Proof. It is divided into several steps. First, we have

Claim 1. $\nabla d \in L^q(P_{\frac{1}{2}})$ for any $1 < q < +\infty$ and

$$\|\nabla d\|_{L^q(P_{\frac{1}{2}})} \le C(q)\left[1 + \|\nabla d\|_{L^2(P_1)}\right]. \tag{3.53}$$

The proof of claim 1 is similar to that of lemma 2.1, which is sketched here. For any $z = (x,t) \in P_{\frac{1}{2}}$ and $0 < r < \frac{1}{2}$, (3.51) implies

$$\left[\|u\|_{L^\infty([t-r^2,t],L^n(B_r(x)))} + \|\nabla d\|_{L^\infty([t-r^2,t],L^n(B_r(x)))}\right] \le \epsilon_0. \tag{3.54}$$

Let $v : P_r(x,t) \to \mathbb{R}^k$ solve

$$\begin{cases} v_t - \Delta v = 0 & \text{in } P_r(x,t) \\ v = d & \text{on } \partial_p P_r(x,t) \end{cases} \tag{3.55}$$

Multiplying (1.5) and (3.55) by $d - v$ and integrating the resulting equations and then subtracting each other, we obtain

$$\begin{aligned}
&\int_{P_r(x,t)} |\nabla(d-v)|^2 \\
\lesssim\ & \int_{P_r(x,t)} \left[|u||\nabla d| + |\nabla d|^2\right] |d - v| \\
\lesssim\ & \int_{t-r^2}^t \left[\|u\|_{L^n(B_r(x))} + \|\nabla d\|_{L^n(B_r(x))}\right] \|\nabla d\|_{L^2(B_r(x))} \|d - v\|_{L^{\frac{2n}{n-2}}(B_r(x))} \\
\lesssim\ & \left[\|u\|_{L^\infty([t-r^2,t],L^n(B_r(x)))} + \|\nabla d\|_{L^\infty([t-r^2,t],L^n(B_r(x)))}\right] \\
& \cdot \int_{t-r^2}^t \|\nabla d\|_{L^2(B_r(x))} \|\nabla(d-v)\|_{L^2(B_r(x))} \\
\le\ & C\epsilon_0 \|\nabla d\|_{L^2(P_r(x,t))} \|\nabla(d-v)\|_{L^2(P_r(x,t))}.
\end{aligned}$$

Hence we have

$$\int_{P_r(x,t)} |\nabla(d-v)|^2 \le C\epsilon_0^2 \int_{P_r(x,t)} |\nabla d|^2. \tag{3.56}$$

For v, we have that for any $\theta \in (0,1)$,

$$(\theta r)^{-n} \int_{P_{\theta r}(x,t)} |\nabla v|^2 \le C\theta^2 r^{-n} \int_{P_r(x,t)} |\nabla d|^2. \tag{3.57}$$

Combining (3.56) with (3.57) yields

$$(\theta r)^{-n} \int_{P_{\theta r}(x,t)} |\nabla d|^2 \le C\left(\theta^2 + \epsilon_0^2\theta^{-n}\right) r^{-n} \int_{P_r(x,t)} |\nabla d|^2 \tag{3.58}$$

for any $(x,t) \in P_{\frac{1}{2}}$, $0 < r \le \frac{1}{2}$, and $\theta \in (0,1)$. Similar to lemma 2.1, choosing sufficiently small $\theta = \theta_0$ first and sufficiently small ϵ_0 second and finally iterating the resulting inequality, (3.58) yields that for any $\alpha \in (0,1)$, there exists $C = C(\epsilon_0, \alpha) > 0$ such that

$$r^{-n} \int_{P_r(x,t)} |\nabla d|^2 \le C r^{2\alpha} \int_{P_1} |\nabla d|^2, \ \forall (x,t) \in P_{\frac{1}{2}}, \ 0 < r \le \frac{1}{2}, \tag{3.59}$$

or equivalently,

$$\|\nabla d\|_{M^{2,2-2\alpha}(P_{\frac{1}{2}})} \le C \|\nabla d\|_{L^2(P_1)}. \tag{3.60}$$

Now we perform the Riesz potential estimate in Morrey spaces by the same way as in lemma 2.1. More precisely, let $\eta \in C_0^\infty(P_1)$ be a cut-off function of $P_{\frac{1}{2}}$ and set $w = \eta d$. Then w satisfies

$$w_t - \Delta w = H, \ H \equiv \eta(|\nabla d|^2 d - u \cdot \nabla d) + (\eta_t - \Delta\eta)d - 2\nabla\eta \cdot \nabla d. \tag{3.61}$$

Since $H \equiv 0$ outside P_1, and $u \cdot \nabla d \in M^{1,2-\alpha}(P_{\frac{1}{2}})$ satisfies

$$\|u \cdot \nabla d\|_{M^{1,2-\alpha}(P_{\frac{1}{2}})} \le C \|u\|_{L^\infty([-1,0],L^n(B_1))} \|\nabla d\|_{L^2(P_1)},$$

it is easy to see from (3.60) that $H \in M^{1,2-\alpha}(\mathbb{R}^{n+1})$ and

$$\|H\|_{M^{1,2-\alpha}(\mathbb{R}^{n+1})} \le C\left[1 + \|\nabla d\|_{L^2(P_1)}\right].$$

Similar to lemma 2.1, we have

$$|\nabla w(z)| \le C \int_{\mathbb{R}^{n+1}} \frac{|H(w)|}{\delta(z,w)^{n+1}}\, dw = CI_1(|H|)(z).$$

Hence $\nabla w \in L^{\frac{2-\alpha}{1-\alpha},*}(\mathbb{R}^{n+1})$ and

$$\|\nabla w\|_{L^{\frac{2-\alpha}{1-\alpha},*}(\mathbb{R}^{n+1})} \le C\,\|H\|_{M^{1,2-\alpha}(\mathbb{R}^{n+1})} \le C\left[1+\|\nabla d\|_{L^2(P_1)}\right].$$

Since $\lim_{\alpha\uparrow 1^-}\frac{2-\alpha}{1-\alpha}=+\infty$, we can see that $\nabla d\in L^q(P_{\frac12})$ for any $1<q<+\infty$, and (3.53) holds.

Next we want to modify the standard argument on the small energy regularity on nonhomogeneous Naiver-Stokes equations, see for example Caffarelli-Kohn-Nirenberg [4], Lin [16], Seregin [20], Escauriaza-Seregin-Sverak [7], and Lin-Liu [18], prove that

Claim 2. $u\in L^\infty(P_{\frac{5}{16}})$.

First observe that since (3.51) and (3.53) imply that for any $1<q<3$,

$$d_t-\Delta d = (|\nabla d|^2 d - u\cdot\nabla d)\in L^q(P_{\frac12}). \tag{3.62}$$

Hence by the L^q-estimate on the heat equation we have that $d\in W_q^{2,1}(P_{\frac38})$ for any $1<q<3$, and

$$\left\|\nabla^2 d\right\|_{L^q(P_{\frac38})} \lesssim \left\||\nabla d|^2 d - u\cdot\nabla d\right\|_{L^q(P_{\frac12})} \le C(q)\epsilon_0\left[1+\|\nabla d\|_{L^2(P_1)}\right].$$

Since

$$u_t + u\cdot\nabla u - \Delta u + \nabla P = f,\ \text{in } P_{\frac38}, \tag{3.63}$$

where $f\equiv -\nabla\cdot(\nabla d\otimes\nabla d)\in L^q(P_{\frac38})$ for any $1<q<3$ and

$$\|f\|_{L^q(P_{\frac38})} \le C(q)\epsilon_0\left[1+\|\nabla d\|_{L^2(P_1)}\right]. \tag{3.64}$$

Since $u\in L_t^\infty L_x^2\cap L_t^2 H_x^1(P_{\frac38})$ and $P\in L^{\frac n2}(P_{\frac38})$, it is not hard to verify that u is a suitable weak solution to (3.63), i.e. u satisfies

$$\begin{aligned}&\int_{B_{\frac38}}|u(t)|^2\phi(t)+2\int_{B_{\frac38}\times[0,t]}|\nabla u|^2\phi\\ \le\ &\int_{B_{\frac38}\times[0,t]}\left[|u|^2(\phi_t+\Delta\phi)+(|u|^2+2P)u\cdot\nabla\phi+2f\cdot u\phi\right]\end{aligned} \tag{3.65}$$

for a.e. $t\in\left[-(\frac38)^2,0\right]$ and for any nonnegative function $\phi\in C_0^\infty(P_{\frac38})$.

Since P satisfies that for a.e $t \in \left[-(\frac{3}{8})^2, 0\right]$

$$\Delta P = \nabla \cdot (f - u \cdot \nabla u) \text{ in } B_{\frac{3}{8}},$$

the same argument as [20] page 1022, Lemma 3.1, with the help of (3.51) and (3.64), implies that there exists $\theta_0 \in (0,1)$ such that

$$\left(\frac{1}{\theta_0^2} \int_{P_{\theta_0}(z)} |P|^{\frac{n}{2}}\right)^{\frac{2}{n}} \leq C\epsilon_0, \ \forall z \in P_{\frac{1}{4}}. \tag{3.66}$$

Since u is a suitable weak solution of (3.63) that satisfies the smallness conditions (3.51), (3.64) for all $1 < q < 3$, and (3.66), it is well-known (see for example [4] [16] [20]) that $u \in C^\alpha(P_{\frac{5}{16}})$ for some $\alpha \in (0,1)$, and

$$\|u\|_{L^\infty(P_{\frac{5}{16}})} \leq C(\epsilon_0). \tag{3.67}$$

Now substituting (3.67) into (3.62), we conclude that $d_t - \Delta d \in L^q(P_{\frac{5}{16}})$ for all $1 < q < +\infty$. Hence $d \in W_q^{2,1}(P_{\frac{1}{4}})$ for any $1 < q < +\infty$. This and the Sobolev embedding theorem imply that $\nabla d \in L^\infty(P_{\frac{1}{4}})$ and

$$\|\nabla d\|_{L^\infty(P_{\frac{1}{4}})} \leq C(\epsilon_0). \tag{3.68}$$

It is clear that (3.52) follows from (3.67) and (3.68). The proof is now complete. □

Proof of Theorem 1.4:

With the help of lemma 3.2, it can be done similar to that of Theorem 1.3. First, since $(u_i, \nabla d_i) \in C([0,T), L^n(\mathbb{R}^n))$ for $i = 1, 2$, it follows that

$$\lim_{t \downarrow 0^+} \sup_{x \in \mathbb{R}^n} \int_{P_t(x,t^2)} \sum_{i=1}^{2} (|u_i|^n + |\nabla d_i|^n) = 0. \tag{3.69}$$

By translation and scaling, (3.69) and lemma 3.2 then imply

$$\lim_{t \downarrow 0^+} \left(\|u_i\|_{L^\infty(\mathbb{R}^n)} + \|\nabla d_i\|_{L^\infty(\mathbb{R}^n)}\right) = 0, \ i = 1, 2.$$

Hence lemma 3.1 implies that there exists $0 < t_0 < T$ such that $u_1 \equiv u_2$ and $d_1 \equiv d_2$ on $\mathbb{R}^n \times [0, t_0]$. Repeating the same argument at $t = t_0$ can eventually lead to $(u_1, d_1) \equiv (u_2, d_2)$ on $\mathbb{R}^n \times [0, T)$. This completes the proof. □

References

[1] K. Chang, *Heat flow and boundary value problem for harmonic maps*, Annales de l'institut Henri Poincaré Analyse non linairé, 6 no. 5 (1989), p. 363-395.

[2] Y. Chen, W. Ding, *Blow-up and global existence for heat flows of harmonic maps*, Invent. Math. 99, no. 3 (1990): 567-578.

[3] K. Chang, W. Ding, R. Ye, *Finite-time blow-up of the heat flow of harmonic maps from surfaces*, J. Differential Geom. 36 (1992), 507-515.

[4] L. Caffarelli, R. Kohn, L. Nirenberg, *Partial regularity of suitable weak solutions of the Navier-Stokes equations*, Comm. Pure Appl. Math. 35 (1982), 771-831.

[5] Y. Chen, F. Lin, *Evolution of harmonic maps with Dirichlet boundary conditions*, Comm. Anal. Geom. 1(3-4) (1993), 327-346.

[6] Y. Chen, M. Struwe, *Existence and partial regularity results for the heat flow for harmonic maps*, Math. Z. 201, no. 1 (1989): 83-103.

[7] L. Escauriaza, G. Serëgin, V. Sverak, $L_{3,\infty}$*-solutions of Navier-Stokes equations and backward uniqueness.* (Russian) Uspekhi Mat. Nauk 58 (2003), no. 2(350), 3-44; translation in Russian Math. Surveys 58 (2003), no. 2, 211-250.

[8] J. L. Ericksen, *Hydrostatic theory of liquid crystal*, Arch. Rational Mech. Anal. 9 (1962), 371-378.

[9] A. Freire, *Uniqueness for the harmonic map flow from surfaces to general targets*, Comment Math. Helvetici, Vol. 70, No. 1 (1995), 310-338.

[10] Y. Giga, *Solutions for Semilinear Parabolic Equations in* L^p *and Regularity of Weak Solutions of the Naiver-Stokes System.* J. Diff. Eqns. 61 (1986), 186-212.

[11] T. Huang, C. Y. Wang, *Notes on the regularity of harmonic map systems.* Proc. Amer. Math. Soc. 138 (2010), 2015-2023.

[12] T. Kato, *Strong* L^p *solutions of the Navier-Stokes equations in* $\mathbb{R}^m$*, with applications to weak solutions*, Math. Z. 187 (1984), 471-480.

[13] F. M. Leslie, *Some constitutive equations for liquid crystals*, Arch. Rational Mech. Anal. 28, 1968, 265-283.

[14] J. Leray, *Sur le mouvement d'un liquide visqueux emplissant l'espace*, Acta Math. 63 (1934), 193-248.

[15] F. H. Lin, *Nonlinear theory of defects in nematic liquid crystal: phase transition and flow phenomena*, Comm. Pure Appl. Math. 42, 1989, 789-814.

[16] F. H. Lin, *A new proof of the Caffarelli-Kohn-Nirenberg Theorem*, Comm. Pure. Appl. Math. LI (1998), 0241-0257.

[17] F. H. Lin, C. Liu, *Nonparabolic Dissipative Systems Modeling the Flow of Liquid Crystals*, Comm. Pure. Appl. Math., Vol. XLVIII, 501-537 (1995).

[18] F. H. Lin, C. Liu, *Partial regularities of the nonlinear dissipative systems modeling the flow of liquid crystals*, Dis. Cont. Dyn. Sys., 2(1996) 1-23.

[19] F. H. Lin, J. Y. Lin, C. Y. Wang, *Liquid Crystal Flows in Two Dimensions.* Arch. Rational Mech. Anal., Vol. 197. No. 1 (2010), 297-336.

[20] G. Seregin, *On the number of singular points of weak solutions to the Naiver-Stokes equations.* Comm. Pure. Appl. Math., Vol. LIV (2001) 1019-1028.

[21] M. Struwe, *On the evolution of harmonic mappings of Riemannian surfaces.* Comment. Math. Helvetici 60 (1985), 558-581.

[22] C. Y. Wang, *A remark on harmonic map flows from surfaces.* Differential Integral Equations 12 (1999), no. 2, 161-166.