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Hölder continuity for parabolic Anderson equation with non-Gaussian noise ☆

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ABSTRACT

Based on the existence, uniqueness and uniform boundedness of the mild solution of the parabolic Anderson equation driven by a non-Gaussian noise, we derive the spatial and temporal Hölder continuity of the mild solution.

1. Introduction

In this paper, we consider the following parabolic Anderson equation in Itô sense [6]:

$$\partial_t u(t,x) = \frac{1}{2}\Delta u(t,x) + u(t,x)V(x)\dot{W}(t), \quad (t,x) \in (0,\infty)\times\mathbb{R}^d \tag{1.1}$$

with initial data $u(0,x) = u_0(x)$ a bounded and measurable function. Here $V(\cdot) := \int_{\mathbb{R}^d} K(\cdot - y)\omega(dy)$ is a Gaussian potential with a deterministic function $K \geq 0$ and a centered Brownian sheet ω, and W is a one-dimensional Brownian motion which is independent of V.

Equation (1.1) is a type of stochastic heat equation (SHE). There has been a widespread interest in Hölder continuity for SHE, with several motivations for its study. To analyze hitting probabilities of the solution, one often requires a priori estimation about Hölder continuity (see e.g. [4] and references therein). Hölder exponents are also used to verify the optimality of convergent rate of numerical schemes in stochastic problems (see e.g. [1]).

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In the existing literatures, most authors concern the Hölder continuity for Gaussian noises driving SHE. Walsh [8] use Green's function technique to show that the solution of a SHE with Lipschitz coefficients driven by space–time white noise is spatially $(\frac{1}{2}-\epsilon)$- and temporally $(\frac{1}{4}-\epsilon)$-Hölder continuous. Here and what follows we denote ϵ by a generic positive and small constant. It is generalized by Sanz-Solé and Sarrà [7] to the case of spatially correlated noise. A different approach is established by Hu et al. [5]. They derive the Hölder continuity for a SHE with fractional noise in terms of Feynman–Kac formula.

Our present work concentrates on the spatial and temporal Hölder continuity of the mild solution u of (1.1) driven by the non-Gaussian noise $V\dot{W}$. To this end, we first present the global existence, uniqueness and uniform boundedness of u, which generalizes the local existence result in [6]. Then we prove the $(\min(\alpha,\beta)-\epsilon)$- and $(\frac{1}{2}\min(\alpha,\beta)-\epsilon)$-Hölder continuity for u in space and time respectively, provided that u_0 is α-Hölder continuous and V is β-Hölder continuous in mean square sense (see (3.1) in Section 3). In particular, the Hölder exponents of u coincide with that of space–time white noise driving SHE ([8]) when V is a Brownian sheet.

As (1.1) involves a non-Gaussian noise, the classical Green's function approaches ([7,8]) and Feynman–Kac representation methods ([5]) are not available. To overcome this difficulty, we transform the estimations for Green's function into the estimations for V and u. Then Gronwall's inequality combined with the uniform boundedness of u is used to yield the spatial and temporal regularity of u.

The rest of the paper is organized as follows. In Section 2, we prove the global existence and uniqueness of the mild solution of (1.1), and obtain the uniform boundedness of the mild solution. In Section 3, we derive the spatial and temporal Hölder continuity of the mild solution.

In the remainder of the article, C is a generic constant whose value may vary in different occurrences.

2. The existence and uniqueness of mild solution

Let $(W(t))_{t\geq 0}$ be a Brownian motion on a probability space $(\Omega_1,\mathcal{F}_1,\mathbb{P}_1)$ and $(\omega(x))_{x\in\mathbb{R}^d}$ be a Brownian sheet on another probability space $(\Omega_2,\mathcal{F}_2,\mathbb{P}_2)$. We denote the product space $(\Omega_1\otimes\Omega_2,\mathcal{F}_1\otimes\mathcal{F}_2,\mathbb{P}_1\otimes\mathbb{P}_2)$ by $(\Omega,\mathcal{F},\mathbb{P})$. Fix a positive and finite constant T and define

$$\mathcal{H}_p := \{Y \text{ is predictable} : \sup_{t\in[0,T]}\sup_{x\in\mathbb{R}^d}\|Y(t,x)\|_{L_p(\Omega)} < \infty\}. \tag{2.1}$$

Then $\mathcal{H}_p$ equipped with the norm $\|Y\|_{\mathcal{H}_p} := \sup_{t\in[0,T]}\sup_{x\in\mathbb{R}^d}\|Y(t,x)\|_{L_p(\Omega)}$ is a Banach space.

In this section, we show the existence and uniqueness of mild solution of (1.1) in $\mathcal{H}_p$. Recall that a random field u is called a mild solution of (1.1) if for any $(t,x)\in[0,T]\times\mathbb{R}^d$,

$$u(t,x) = p_t * u_0(x) + \int_0^t\int_{\mathbb{R}^d} p_{t-s}(x-y)u(s,y)V(y)dydW(s) \quad a.s., \tag{2.2}$$

where $p_t(x) = (2\pi t)^{-\frac{d}{2}}e^{-\frac{|x|^2}{2t}}$ is the heat kernel of $\frac{1}{2}\Delta$ on $\mathbb{R}^d$.

Theorem 2.1. *Assume that $\int_{\mathbb{R}^d} K^2(x)dx < +\infty$. There exists a unique mild solution u of (1.1), and for all $p\geq 2$ and $T>0$,*

$$\sup_{t\in[0,T]}\sup_{x\in\mathbb{R}^d}\left(\mathbb{E}_\omega\otimes\mathbb{E}_W|u(t,x)|^p\right) < \infty. \tag{2.3}$$

Proof. For each $u\in\mathcal{H}_p$ with $p\geq 2$, define

$$P(u)(t,x) = p_t * u_0(x) + \int_0^t \int_{\mathbb{R}^d} p_{t-s}(x-y)V(y)u(s,y)dydW(s), \ (t,x) \in [0,T] \times \mathbb{R}^d.$$

We first show that $P(u) \in \mathcal{H}_p$. Since u_0 is bounded and measurable,

$$\|p_t * u_0\|_{\mathcal{H}_p} = \sup_{t\in[0,T]} \sup_{x\in\mathbb{R}^d} |p_t * u_0(x)| \leq \|u_0\|_{L^\infty}. \tag{2.4}$$

For general $p \in [2,\infty)$, applying Burkholder–Davis–Gundy inequality and Minkowski inequality, we derive that

$$\begin{aligned}
&\mathbb{E}_\omega \otimes \mathbb{E}_W \left| \int_0^t \int_{\mathbb{R}^d} p_{t-s}(x-y)V(y)u(s,y)dydW(s) \right|^p \\
&\leq C_p \mathbb{E}_\omega \otimes \mathbb{E}_W \left(\int_0^t \left| \int_{\mathbb{R}^d} p_{t-s}(x-y)V(y)u(s,y)dy \right|^2 ds \right)^{\frac{p}{2}} \\
&\leq C_p \left(\int_0^t \left(\mathbb{E}_\omega \otimes \mathbb{E}_W \left| \int_{\mathbb{R}^d} p_{t-s}(x-y)V(y)u(s,y)dy \right|^p \right)^{\frac{2}{p}} ds \right)^{\frac{p}{2}} \\
&\leq C_p \left(\int_0^t \left(\mathbb{E}_W \otimes \mathbb{E}_\omega \left| \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} p_{t-s}(x-y)K(y-z)u(s,y)dy\omega(dz) \right|^p \right)^{\frac{2}{p}} ds \right)^{\frac{p}{2}} \\
&\leq C_p^2 \left(\int_0^t \left(\mathbb{E}_W \otimes \mathbb{E}_\omega \left[\int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} p_{t-s}(x-y)K(y-z)u(s,y)dy \right|^2 dz \right]^{\frac{p}{2}} \right)^{\frac{2}{p}} ds \right)^{\frac{p}{2}}.
\end{aligned}$$

Here C_p is the usual constant appeared in Burkholder–Davis–Gundy inequality (Theorem 4.37 [2]). Due to Hölder inequality and the property $\int_{\mathbb{R}^d} p_{t-s}(x-y)dy = 1$ for any $x \in \mathbb{R}^d$, the above integrand to dz has the following estimate:

$$\left| \int_{\mathbb{R}^d} p_{t-s}(x-y)K(y-z)u(s,y)dy \right|^2 \leq \int_{\mathbb{R}^d} p_{t-s}(x-y)K^2(y-z)u^2(s,y)dy.$$

Thus, we conclude that

$$\begin{aligned}
&\mathbb{E}_\omega \otimes \mathbb{E}_W \left| \int_0^t \int_{\mathbb{R}^d} p_{t-s}(x-y)V(y)u(s,y)dydW(s) \right|^p \\
&\leq C_p^2 \left(\int_0^t \left(\mathbb{E}_\omega \otimes \mathbb{E}_W \left[\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} p_{t-s}(x-y)K^2(y-z)u^2(s,y)dydz \right]^{\frac{p}{2}} \right)^{\frac{2}{p}} ds \right)^{\frac{p}{2}}
\end{aligned}$$

$$
\begin{aligned}
&\le C_p^2 \left(\int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} p_{t-s}(x-y) K^2(y-z) \Big(\mathbb{E}_\omega \otimes \mathbb{E}_W |u(s,y)|^p \Big)^{\frac{2}{p}} dydzds \right)^{\frac{p}{2}} \\
&\le C_p^2 \left(\int_0^t \left(\sup_{y\in\mathbb{R}^d} \mathbb{E}_\omega \otimes \mathbb{E}_W |u(s,y)|^p \right)^{\frac{2}{p}} \left(\int_{\mathbb{R}^d} p_{t-s}(x-y) dy \right) ds \right)^{\frac{p}{2}} \|K\|_{L^2}^p \\
&\le C_p^2 \|K\|_{L^2}^p \left(\int_0^t \sup_{y\in\mathbb{R}^d} (\mathbb{E}_\omega \otimes \mathbb{E}_W |u(s,y)|^p)^{\frac{2}{p}} ds \right)^{\frac{p}{2}}.
\end{aligned} \tag{2.5}
$$

Combining (2.4) and (2.5), we obtain

$$
\begin{aligned}
&\|P(u)\|_{\mathcal{H}_p} \\
&\le \|p_t * u_0\|_{\mathcal{H}_p} + \sup_{t\in[0,T]} \sup_{x\in\mathbb{R}^d} \left(\mathbb{E}_\omega \otimes \mathbb{E}_W \left| \int_0^t \int_{\mathbb{R}^d} p_{t-s}(x-y) V(y) u(s,y) dy dW(s) \right|^p \right)^{\frac{1}{p}} \\
&\le \|u_0\|_{L^\infty} + (C_p)^{\frac{2}{p}} T^{\frac{1}{2}} \|K\|_{L^2} \|u\|_{\mathcal{H}_p},
\end{aligned} \tag{2.6}
$$

which implies that the mapping P is well-defined for general $p \in [2, \infty)$.

Next we show that P is a contraction operator. To this end, we introduce a new norm $\|\cdot\|_{\lambda,*}$ defined by

$$
\|u\|_{\lambda,*} = \sup_{t\in[0,T]} \left(\exp(-\lambda t) \sup_{x\in\mathbb{R}^d} (\mathbb{E}_\omega \otimes \mathbb{E}_W |u(t,x)|^p)^{\frac{1}{p}} \right), \quad \lambda > 0,\ u \in \mathcal{H}_p,
$$

which is equivalent to the $\mathcal{H}_p$-norm. Assume that $u_1, u_2 \in \mathcal{H}_p$. As P is a linear operator, similar to the proof of (2.6), we obtain

$$
\begin{aligned}
&\sup_{x\in\mathbb{R}^d} (\mathbb{E}_\omega \otimes \mathbb{E}_W |P(u_1)(t,x) - P(u_2)(t,x)|^p)^{\frac{2}{p}} \\
&\le (C_p)^{\frac{4}{p}} \|K\|_{L^2}^2 \int_0^t \sup_{y\in\mathbb{R}^d} (\mathbb{E}_\omega \otimes \mathbb{E}_W |u_1(s,y) - u_2(s,y)|^p)^{\frac{2}{p}} ds \\
&\le (C_p)^{\frac{4}{p}} \|K\|_{L^2}^2 \|u_1 - u_2\|_{\lambda,*}^2 \left[\int_0^t \exp(2\lambda s) ds \right] \\
&= (C_p)^{\frac{4}{p}} \|K\|_{L^2}^2 \|u_1 - u_2\|_{\lambda,*}^2 \frac{\exp(2\lambda t) - 1}{2\lambda},
\end{aligned}
$$

from which we have

$$
\|P(u_1) - P(u_2)\|_{\lambda,*}^2 \le (C_p)^{\frac{4}{p}} \|K\|_{L^2}^2 \frac{1 - \exp(-2\lambda T)}{2\lambda} \|u_1 - u_2\|_{\lambda,*}^2.
$$

Take $\lambda > \frac{1}{2}(C_p)^{\frac{4}{p}} \|K\|_{L^2}^2$. Then the mapping P is a contraction with respect to $\|\cdot\|_{\lambda,*}$-norm. By the contraction mapping principle, there exists a unique fixed point $u \in \mathcal{H}_p$ which is the unique mild solution of (1.1). Moreover, by Gronwall's inequality, (2.4) and (2.5) yield that there exists a positive constant C such that

$$\sup_{t\in[0,T]}\sup_{x\in\mathbb{R}^d}(\mathbb{E}_\omega\otimes\mathbb{E}_W|u(t,x)|^p)\le C\|u_0\|_{L^\infty}^p\exp(CT),$$

which shows the uniform boundedness of the p-th moment for u. □

3. The spatial and temporal Hölder continuity

Assume that the Gaussian field V is β-Hölder continuous with $\beta\in(0,1)$ in mean square sense:

$$\mathbb{E}_\omega|V(x)-V(y)|^2=\int_{\mathbb{R}^d}|K(z-x)-K(z-y)|^2dz\le C\,|x-y|^{2\beta}\,. \tag{3.1}$$

Theorem 3.1. *Suppose that $\int_{\mathbb{R}^d}K^2(x)dx<+\infty$, u_0 is α-Hölder continuous and V satisfies (3.1). For $p\ge 2$, there exists a positive constant $C=C(K,T,p)$ such that*

$$(\mathbb{E}_\omega\otimes\mathbb{E}_W|u(t_1,x_1)-u(t_2,x_2)|^p)^{\frac1p}\le C(|x_1-x_2|^{\min(\alpha,\beta)}+|t_1-t_2|^{\frac12\min(\alpha,\beta)})$$

for every $t_1,t_2\in[0,T],x_1,x_2\in\mathbb{R}^d$. As a consequence, u is spatially $(\min(\alpha,\beta)-\epsilon)$- and temporally $(\frac12\min(\alpha,\beta)-\epsilon)$-Hölder continuous.

Proof. It is well-known that (see e.g. Theorem 2.1 [7]) for any $t_1,t_2\in[0,T]$ and $x_1,x_2\in\mathbb{R}^d$,

$$|p_{t_1}*u_0(x_1)-p_{t_2}*u_0(x_2)|\le C(|t_2-t_1|^{\frac{\alpha}{2}}+|x_2-x_1|^{\alpha}). \tag{3.2}$$

We only need to consider

$$\tilde P(u)(t,x):=\int_0^t\int_{\mathbb{R}^d}p_{t-s}(x-y)V(y)u(s,y)dydW(s).$$

Similar to the proof of Theorem 2.1, applying Burkholder–Davis–Gundy inequality and Minkowski inequality, we obtain

$$\begin{aligned}
&(\mathbb{E}_\omega\otimes\mathbb{E}_W|\tilde P(u)(t,x_1)-\tilde P(u)(t,x_2)|^p)^{\frac1p}\\
&=\left(\mathbb{E}_\omega\otimes\mathbb{E}_W\left|\int_0^t\int_{\mathbb{R}^d}(p_{t-s}(x_2-y)-p_{t-s}(x_1-y))u(s,y)\int_{\mathbb{R}^d}K(y-z)\omega(dz)dydW(s)\right|^p\right)^{\frac1p}\\
&\le C\left(\int_0^t\int_{\mathbb{R}^d}\left(\mathbb{E}_\omega\otimes\mathbb{E}_W\left|\int_{\mathbb{R}^d}(p_{t-s}(x_2-y)-p_{t-s}(x_1-y))u(s,y)K(y-z)dy\right|^p\right)^{\frac2p}dzds\right)^{\frac12}\\
&=C\left(\int_0^t\int_{\mathbb{R}^d}\left(\mathbb{E}_\omega\otimes\mathbb{E}_W\left|\int_{\mathbb{R}^d}p_{t-s}(v)[u(s,x_1-v)K(x_1-v-z)-u(s,x_2-v)K(x_2-v-z)]dv\right|^p\right)^{\frac2p}dzds\right)^{\frac12}\\
&\le C\bigg(\int_0^t\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}p_{t-s}(v)(\mathbb{E}_\omega\otimes\mathbb{E}_W|u(s,x_1-v)|^p)^{\frac2p}|K(x_1-v-z)-K(x_2-v-z)|^2dvdzds\\
&\quad+\int_0^t\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}p_{t-s}(v)(\mathbb{E}_\omega\otimes\mathbb{E}_W|u(s,x_1-v)-u(s,x_2-v)|^p)^{\frac2p}|K(x_2-v-z)|^2dvdzds\bigg)^{\frac12}.
\end{aligned}$$

By the assumptions (3.1) and $\int_{\mathbb{R}^d} K^2(x)dx < +\infty$, we have

$$(\mathbb{E}_\omega \otimes \mathbb{E}_W |\tilde{P}(u)(t,x_1) - \tilde{P}(u)(t,x_2)|^p)^{\frac{1}{p}}$$
$$\le C \left(|x_1 - x_2|^\beta + \int_0^t \int_{\mathbb{R}^d} p_{t-s}(v)(\mathbb{E}_\omega \otimes \mathbb{E}_W |u(s,x_1-v) - u(s,x_2-v)|^p)^{\frac{1}{p}} dvds \right).$$

The above inequality and (3.2) yield

$$(\mathbb{E}_\omega \otimes \mathbb{E}_W |u(t,x_1) - u(t,x_2)|^p)^{\frac{1}{p}}$$
$$\le (\mathbb{E}_\omega \otimes \mathbb{E}_W |p_t * u_0(x_1) - p_t * u_0(x_2)|^p)^{\frac{1}{p}} + (\mathbb{E}_\omega \otimes \mathbb{E}_W |\tilde{P}(u)(t,x_1) - \tilde{P}(u)(t,x_2)|^p)^{\frac{1}{p}}$$
$$\le C \left(|x_1 - x_2|^{\min(\alpha,\beta)} + \int_0^t \int_{\mathbb{R}^d} p_{t-s}(v)(\mathbb{E}_\omega \otimes \mathbb{E}_W |u(s,x_1-v) - u(s,x_2-v)|^p)^{\frac{1}{p}} dvds \right).$$

Substituting x_1 by $x_1 - m$ and x_2 by $x_2 - m$, we obtain

$$(\mathbb{E}_\omega \otimes \mathbb{E}_W |u(s,x_1-m) - u(s,x_2-m)|^p)^{\frac{1}{p}}$$
$$\le C\,|x_2 - x_1|^{\min(\alpha,\beta)} + C \int_0^t \int_{\mathbb{R}^d} p_{t-s}(v)(\sup_{v\in\mathbb{R}^d} \mathbb{E}_\omega \otimes \mathbb{E}_W |u(s,x_1-m-v) - u(s,x_2-m-v)|^p)^{\frac{1}{p}} dvds$$
$$\le C|x_2 - x_1|^{\min(\alpha,\beta)} + C \int_0^t (\sup_{v\in\mathbb{R}^d} \mathbb{E}_\omega \otimes \mathbb{E}_W |u(s,x_1-m-v) - u(s,x_2-m-v)|^p)^{\frac{1}{p}} ds.$$

Taking supremum for m on both sides of above inequality, we get the spatial regularity by Gronwall's inequality,

$$(\mathbb{E}_\omega \otimes \mathbb{E}_W |u(t,x_1) - u(t,x_2)|^p)^{\frac{1}{p}} \le (\sup_m \mathbb{E}_\omega \otimes \mathbb{E}_W |u(s,x_1-m) - u(s,x_2-m)|^p)^{\frac{1}{p}}$$
$$\le C|x_2 - x_1|^{\min(\alpha,\beta)} \exp(CT).$$

Now we prove the temporal regularity of u, which can be transformed to the spatial regularity. Due to (3.2),

$$(\mathbb{E}_\omega \otimes \mathbb{E}_W |u(t_1,x) - u(t_2,x)|^p)^{\frac{1}{p}}$$
$$\le (\mathbb{E}_\omega \otimes \mathbb{E}_W |p_{t_1} * u_0(x) - p_{t_2} * u_0(x)|^p)^{\frac{1}{p}} + (\mathbb{E}_\omega \otimes \mathbb{E}_W |\tilde{P}(u)(t_1,x) - \tilde{P}(u)(t_2,x)|^p)^{\frac{1}{p}}$$
$$\le C|t_2 - t_1|^{\frac{\alpha}{2}} + \left(\mathbb{E}_\omega \otimes \mathbb{E}_W \left| \int_{t_1}^{t_2} \int_{\mathbb{R}^d} p_{t_2-s}(x-y)u(s,y)V(y)dydW(s) \right|^p \right)^{\frac{1}{p}}$$
$$+ \left(\mathbb{E}_\omega \otimes \mathbb{E}_W \left| \int_0^{t_1} \int_{\mathbb{R}^d} (p_{t_2-s}(x-y) - p_{t_1-s}(x-y))u(s,y)V(y)dydW(s) \right|^p \right)^{\frac{1}{p}}$$
$$=: C|t_2 - t_1|^{\frac{\alpha}{2}} + II + III.$$

Term II can be estimated by the same procedure as the proof of Theorem 2.1, so we have $II \le C|t_2 - t_1|^{\frac{1}{2}}$. Then it remains to estimate term III. According to Burkholder–Davis–Gundy inequality and Fubini's theorem,

$$III \le C\left(\int_0^{t_1}\int_{\mathbb{R}^d}\left(\mathbb{E}_\omega \otimes \mathbb{E}_W\left|\int_{\mathbb{R}^d}\Big(p_{t_2-s}(x-y) - p_{t_1-s}(x-y)\Big)u(s,y)K(y-z)dy\right|^p\right)^{\frac{2}{p}}dzds\right)^{\frac{1}{2}}$$

$$\le C\Bigg(\int_0^{t_1}\int_{\mathbb{R}^d}\Bigg(\mathbb{E}_\omega \otimes \mathbb{E}_W\Bigg|\int_{(\mathbb{R}^d)^{\otimes 2}} p_{t_1-s}(x-y)p_{t_2-t_1}(y-w)\Big(u(s,w)K(w-z)$$

$$- u(s,y)K(y-z)\Big)dydw\Bigg|^p\Bigg)^{\frac{2}{p}}dzds\Bigg)^{\frac{1}{2}},$$

where in the last step we used the fact

$$\int_{\mathbb{R}^d} p_{t_2-s}(x-y)u(s,y)K(y-z)dy$$

$$= \int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d} p_{t_1-s}(x-y)p_{t_2-t_1}(y-w)dy\right)u(s,w)K(w-z)dw.$$

Denoting

$$f_1 := (\mathbb{E}_\omega \otimes \mathbb{E}_W|u(s,w)K(w-z) - u(s,y)K(y-z)|^p)^{\frac{2}{p}},$$

by Minkowski inequality and Hölder inequality, we have

$$III \le C\left(\int_0^{t_1}\int_{\mathbb{R}^d}\int_{(\mathbb{R}^d)^{\otimes 2}} p_{t_1-s}(x-y)p_{t_2-t_1}(y-w)f_1(x)dydwdzds\right)^{\frac{1}{2}}$$

$$\le C\Bigg(\int_0^{t_1}\int_{(\mathbb{R}^d)^{\otimes 3}} p_{t_1-s}(x-y)p_{t_2-t_1}(y-w)(\mathbb{E}_\omega \otimes \mathbb{E}_W|u(s,w)|^p)^{\frac{2}{p}}|K(w-z) - K(y-z)|^2dydwdzds$$

$$+ \int_0^{t_1}\int_{(\mathbb{R}^d)^{\otimes 3}} p_{t_1-s}(x-y)p_{t_2-t_1}(y-w)(\mathbb{E}_\omega \otimes \mathbb{E}_W|u(s,w) - u(s,y)|^p)^{\frac{2}{p}}K^2(y-z)dydwdzds\Bigg)^{\frac{1}{2}}.$$

Let $y - w := (t_2 - t_1)^{\frac{1}{2}}z$. According to the spatial regularity of $u(t,x)$, we obtain

$$III \le C\left(\int_0^{t_1}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} p_{t_1-s}(x-y)p_{t_2-t_1}(y-w)(|y-w|^{2\alpha} + |y-w|^{2\beta})dwdyds\right)^{\frac{1}{2}}$$

$$\le C\left(\int_0^{t_1}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} p_{t_1-s}(x-y)p_{t_2-t_1}(z)(|z|^{2\alpha}|t_2-t_1|^{\alpha} + |z|^{2\beta}|t_2-t_1|^{\beta})dzdyds\right)^{\frac{1}{2}}$$

$$\le C\,|t_2 - t_1|^{\frac{1}{2}\min(\alpha,\beta)}.$$

It follows that

$$\left(\mathbb{E}_\omega \otimes \mathbb{E}_W (u(t_1,x)-u(t_2,x))^p\right)^{\frac{1}{p}} \le C\,|t_2-t_1|^{\frac{1}{2}\min(\alpha,\beta)} + C\,|t_2-t_1|^{\frac{1}{2}}.$$

With a simple triangle inequality, the theorem is finished. By Kolmogorov's continuity criterion theorem, u is spatially $(\min(\alpha,\beta)-\epsilon)$- and temporally $(\frac{1}{2}\min(\alpha,\beta)-\epsilon)$-Hölder continuous. □

Remark 3.1. In the proof of Theorem 3.1, the estimates of the Green's function $p_t(x-y)$ cannot be applied to the term

$$\int_0^t \int_{\mathbb{R}^d} \left(\mathbb{E}_\omega \otimes \mathbb{E}_W \left| \int_{\mathbb{R}^d} (p_{t-s}(x_2-y)-p_{t-s}(x_1-y))u(s,y)K(y-z)dy \right|^p \right)^{\frac{2}{p}} dzds$$

or

$$\int_0^{t_1} \int_{\mathbb{R}^d} \left(\mathbb{E}_\omega \otimes \mathbb{E}_W \left| \int_{\mathbb{R}^d} (p_{t_2-s}(x-y)-p_{t_1-s}(x-y))u(s,y)K(y-z)dy \right|^p \right)^{\frac{2}{p}} dzds.$$

This is the main difference in the proofs between the cases of Gaussian noise and non-Gaussian noise.

Remark 3.2. The sharpness of the solutions' Hölder exponents for SPDEs is very important. To our knowledge, only [3] considers this problem for stochastic wave equation with additive Gaussian noise. The sharpness of the Hölder exponents for our problem remains open.

We finish this section by giving some examples about K such that Gaussian potential V is β-Hölder continuous in mean square sense, i.e., (3.1) holds.

Example 3.1. 1) For any fixed $x=(x_1,\cdots,x_d)^T \in \mathbb{R}^d$, we define $K(x-y) := I_{[0,x_1]}(y_1)\cdots I_{[0,x_d]}(y_d)$ for $y=(y_1,\cdots,y_d)^T \in \mathbb{R}^d$, where $I_{[0,x_i]} = -I_{[x_i,0]}$ if $x_i<0$. Then $V(x)$ is the d-dimensional Brownian sheet $\omega(x)$, which is $\frac{1}{2}$-Hölder continuous in mean square sense, i.e., $\beta=\frac{1}{2}$. In this case, the solution u is spatially $(\frac{1}{2}-\epsilon)$- and temporally $(\frac{1}{4}-\epsilon)$-Hölder continuous provided that $\alpha=\frac{1}{2}$, which is similar to the case of space–time white noise driving SHE [8].

2) $K(x)=|x|^{q_1}\exp(-|x|^{q_2})$, $x\in\mathbb{R}^d$ for $0\vee(1-\frac{d}{2})<q_1<1$, $q_2>0$. Then $V(x)$ is q_1-Hölder continuous in mean square sense, i.e., $\beta=q_1$.

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