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NUMERICAL ANALYSIS ON ERGODIC LIMIT OF APPROXIMATIONS FOR STOCHASTIC NLS EQUATION VIA MULTI-SYMPLECTIC SCHEME*

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Abstract. We consider a finite dimensional approximation of the stochastic nonlinear Schrödinger equation driven by multiplicative noise, which is derived by applying a symplectic method to the original equation in spatial direction. Both the unique ergodicity and the charge conservation law for this finite dimensional approximation are obtained on the unit sphere. To simulate the ergodic limit over long time for the finite dimensional approximation, we discretize it further in the temporal direction to obtain a fully discrete scheme, which inherits not only the stochastic multi-symplecticity and charge conservation law of the original equation but also the unique ergodicity of the finite dimensional approximation. The temporal average of the fully discrete numerical solution is proved to converge to the ergodic limit with order 1 with respect to the time step for a fixed spatial step. Numerical experiments verify our theoretical results on charge conservation, ergodicity, and weak convergence.

Key words. stochastic Schrödinger equation, multiplicative noise, unique ergodicity, multi-symplectic scheme, weak error

AMS subject classifications. 37M25, 60H35, 65C30, 65P10

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1. Introduction.

For the stochastic nonlinear Schrödinger (NLS) equation with a multiplicative noise in Stratonovich sense,

$$
\begin{cases} du = \mathbf{i}\big(\Delta u + \lambda |u|^2 u\big)dt + \mathbf{i}u \circ dW, \\ u(t,0) = u(t,1) = 0, \ t \geq 0, \\ u(0,x) = u_0(x), \ x \in [0,1] \end{cases} \tag{1}
$$

with $\lambda = \pm 1$, we consider the case that W is a real valued Q-Wiener process on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ with paths in $H_0^1 := H_0^1(0,1)$ with Dirichlet boundary condition. The Karhunen–Loève expansion of W is as follows:

$$
W(t,x,\omega) = \sum_{k=0}^{\infty} \beta_k(t,\omega) Q^{\frac{1}{2}} e_k(x), \quad t \geq 0, \quad x \in [0,1], \quad \omega \in \Omega,
$$

where $(e_k = \sqrt{2}\sin(k\pi x))_{k\geq 1}$ is an eigenbasis of the Dirichlet Laplacian Δ in $L^2 := L^2(0,1)$ and $(\beta_k)_{k\geq 1}$ is a sequence of independent real valued Brownian motions

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associated with the filtration $(\mathcal{F}_t)_{t\geq 0}$. In addition, the covariance operator Q is assumed to commute with the Laplacian and satisfies

$$Qe_k = \eta_k e_k, \quad \eta_k > 0, \quad \forall k \in \mathbb{N}, \quad \eta := \sum_{k=1}^{\infty} \eta_k < \infty.$$

We refer to [9] for additional assumptions on the well-posedness of (1). It is shown that (1) is a Hamiltonian system with stochastic multi-symplectic structure and charge conservation law (see [6, 9, 11] and references therein). Structure-preserving numerical schemes have remarkable superiority over conventional schemes to numerically solve Hamiltonian systems over long time. As another kind of long-time behavior, the ergodicity for this kind of conservative system is an important and difficult problem which is still open. Motivated by [10], we study the ergodicity for a finite dimensional approximation (FDA) of the original equation instead.

In this paper, we investigate the ergodicity for a symplectic FDA of (1) and approximate its ergodic limit via a multi-symplectic and ergodic scheme. As we show that the FDA is charge conserved, without loss of generality, we consider the ergodicity in the finite dimensional unit sphere $\mathcal{S}$. There have been some papers considering the additive noise case with dissipative assumptions, and also some papers requiring a uniformly elliptic assumption on the whole space to ensure unique ergodicity (see, e.g., [3, 12, 13, 15, 16]). A conservative FDA with linear multiplicative noise has an uncertain nondegeneracy, which relies heavily on the solution. To overcome this difficulty, we construct an invariant control set $\mathcal{M}_0 \subset \mathcal{S}$, in which the FDA is shown to be nondegenerate. Together with the Krylov–Bogoliubov theorem and the Hörmander condition, we prove that the solution U possesses a unique invariant measure μ_h (i.e., U is uniquely ergodic) with

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \mathbb{E}f(U(t))dt = \int_{\mathcal{M}_0} f d\mu_h = \int_{\mathcal{S}} f d\mu_h.$$

For many physical applications, the approximation of the invariant measure is of fundamental importance, especially when the invariant measure is unknown (see, e.g., [1, 3, 4, 5, 7, 13, 14, 15, 16]). Some papers construct numerical schemes which also possess unique invariant measures, and then show the approximate error between invariant measures. For example, [7, 15] work with dissipative systems driven by additive noise, and [16] considers elliptic stochastic differential equations (SDEs) with bounded coefficients and dissipative type condition. There is also some work concentrating on the approximation of the invariant measure, i.e., the approximation of the ergodic limit $\int_{\mathcal{S}} f d\mu_h$, in which case the numerical schemes may not be uniquely ergodic. For instance, [3] approximates the invariant measure of stochastic partial differential equations with an additive noise based on Kolmogorov equation. Reference [13] gives error estimates for time-averaging estimators of numerical schemes based on the associated Poisson equation and the assumption of local weak convergence order. Authors in [14] calculate the ergodic limit for Langevin equations with dissipations via quasi-symplectic integrators. There have been few results on constructing conservative and uniquely ergodic schemes to calculate the ergodic limit for conservative systems, to our knowledge. We focus on the approximation of the ergodic limit via a multi-symplectic scheme, which is also shown to be uniquely ergodic. For a fixed spacial dimension, the local weak error of this fully discrete scheme (FDS) in temporal direction is of order 2, which yields order 1 for the approximate error of the ergodic limit based on the associated Poisson equation (see also [4, 13]) and a priori estimates of the numerical solutions. That is,

$$\left|\mathbb{E}\left[\frac{1}{N}\sum_{n=0}^{N-1} f(U^n) - \int_{\mathcal{S}} f d\mu_h\right]\right| \le C_h\left(\frac{1}{T}+\tau\right).$$

The paper is organized as follows. In section 2, we apply a symplectic semidiscrete scheme to the original equation to get the FDA, and show the unique ergodicity as well as the charge conservation law for the FDA. In section 3, we present a multi-symplectic and ergodic FDS to approximate the ergodic limit, and show the approximate error based on a priori estimates and local weak error. In section 4, the discrete charge evolution compared with those of the Euler–Maruyama scheme and implicit Euler scheme, the ergodic limit, and global weak convergence order are tested numerically. Section 5 is the appendix containing proofs of some a priori estimates.

2. Unique ergodicity. In this section, we first apply the central finite difference scheme to (1) in the spatial direction to obtain a FDA, which is also a Hamiltonian system. To investigate the ergodicity of this conservative system, we then construct an invariant control set $\mathcal{M}_0 \subset \mathcal{S}$ with respect to a control function introduced in section 2.2. The FDA is proved to be ergodic in $\mathcal{M}_0$ based on the Krylov–Bogoliubov theorem and the Hörmander condition.

2.1. Finite dimensional approximation (FDA). Based on the central finite difference scheme and the notation $u_j := u_j(t)$, $j = 1, \dots, M$, we consider the following spatial semidiscretization:

$$du_j = \mathbf{i}\left[\frac{u_{j+1}-2u_j+u_{j-1}}{h^2} + \lambda|u_j|^2 u_j\right]dt + \mathbf{i}u_j\sum_{k=1}^{K}\sqrt{\eta_k}e_k(x_j)\circ d\beta_k(t)$$

with a truncated noise $\sum_{k=1}^{K}\sqrt{\eta_k}e_k(x)\beta_k(t)$, $K \in \mathbb{N}$, a given uniform step size $h = \frac{1}{M+1}$ for some $M \le K$, and $x_j = jh$, $j = 1, \dots, M$. The condition $M \le K$ here ensures the existence of the solution for the control function. Denoting vectors $U := U(t) = (u_1, \dots, u_M)^T \in \mathbb{C}^M$, $\beta(t) = (\beta_1(t), \dots, \beta_K(t))^T \in \mathbb{R}^K$, and matrices $F(U) = \operatorname{diag}\{|u_1|^2, \dots, |u_M|^2\}$, $E_k = \operatorname{diag}\{e_k(x_1), \dots, e_k(x_M)\}$, $\Lambda = \operatorname{diag}\{\sqrt{\eta_1}, \dots, \sqrt{\eta_K}\}$, $Z(U) = \operatorname{diag}\{u_1, \dots, u_M\}E_{MK}\Lambda$,

$$A = \begin{pmatrix} -2 & 1 & & \\ 1 & -2 & 1 & \\ & \ddots & \ddots & \ddots \\ & & 1 & -2 \end{pmatrix} \in \mathbb{R}^{M\times M}, \quad E_{MK} = \begin{pmatrix} e_1(x_1) & \cdots & e_K(x_1) \\ \vdots & & \vdots \\ e_1(x_M) & \cdots & e_K(x_M) \end{pmatrix}_{M\times K},$$

then the FDA is in the following form:

$$\begin{cases} dU = \mathbf{i}\left[\frac{1}{h^2}AU + \lambda F(U)U\right]dt + \mathbf{i}Z(U)\circ d\beta(t), \\ U(0) = c_*\left(u_0(x_1), \dots, u_0(x_M)\right)^T, \end{cases} \tag{2}$$

where c_* is a normalized constant. The noise term in (2) has an equivalent Itô form

$$\mathbf{i}Z(U)\circ d\beta(t) = \mathbf{i}\sum_{k=1}^{K}\sqrt{\eta_k}E_kU\circ d\beta_k(t) = -\frac{1}{2}\sum_{k=1}^{K}\eta_k E_k^2 U dt + \mathbf{i}\sum_{k=1}^{K}\sqrt{\eta_k}E_kU d\beta_k(t)$$

$$=: -\hat{E}U dt + \mathbf{i}\sum_{k=1}^{K}\sqrt{\eta_k}E_k U d\beta_k(t) \tag{3}$$

with $\hat{E} = \frac{1}{2}\sum_{k=1}^{K}\eta_k E_k^2$. In the following, $\|\cdot\|$ denotes the 2-norm for both matrices and vectors, which satisfies $\|BV\| \le \|B\|\|V\|$ for any matrices $B \in \mathbb{C}^{m\times n}$ and vectors $V \in \mathbb{C}^n$, $m, n \in \mathbb{N}$. It is then easy to show that $\|A\| \le 4$, which is independent of the dimension M.

PROPOSITION 2.1. *The FDA* (2) *possesses the charge conservation law, i.e.,*

$$\|U(t)\|^2 = \|U(0)\|^2, \quad \forall\, t \ge 0, \quad \mathbb{P}\text{-a.s.}$$

where $\|U(t)\| = (\|P(t)\|^2 + \|Q(t)\|^2)^{\frac{1}{2}} = \left(\sum_{m=1}^{M}(|p_m(t)|^2 + |q_m(t)|^2)\right)^{\frac{1}{2}}$*, and* $P(t) = (p_1(t), \dots, p_M(t))^T$ *and* $Q(t) = (q_1(t), \dots, q_M(t))^T$ *are the real and imaginary parts of* $U(t)$ *respectively.*

Proof. Noticing that matrices A and $F(U)$ are symmetric and the linear function $Z(U)$ satisfies

$$\overline{U}^T Z(U) = (\overline{u_1}, \dots, \overline{u_M})\begin{pmatrix} u_1 & & \\ & \ddots & \\ & & u_M \end{pmatrix} E_{MK}\begin{pmatrix} \sqrt{\eta_1} & & \\ & \ddots & \\ & & \sqrt{\eta_K} \end{pmatrix}$$

$$= (|u_1|^2, \dots, |u_M|^2)E_{MK}\begin{pmatrix} \sqrt{\eta_1} & & \\ & \ddots & \\ & & \sqrt{\eta_K} \end{pmatrix} \in \mathbb{R}^K, \tag{4}$$

where $\overline{U}$ denotes the conjugate of U, we multiply (2) by $\overline{U}^T$, take the real part, and then get the charge conservation law for U. ∎

In the following, without pointing out, all equations hold in the sense $\mathbb{P}$-a.s.

Remark 2.2. Equation. (1) can be rewritten into an infinite dimensional Hamiltonian system (see [11]). It is easy to verify that the central finite difference scheme (2) applied to (1) is equivalent to the symplectic Euler scheme applied to the infinite dimensional Hamiltonian form of (1), which implies the symplecticity of (2).

2.2. Unique ergodicity. As the charge of (2) is conserved shown in Proposition 2.1, without loss of generality, we assume that $U(0) \in \mathcal{S}$ and investigate the unique ergodicity of (2) on $\mathcal{S}$. As the nondegeneracy for (2) relies on the solution U as a result of the multiplicative noise, the standard procedure to show the irreducibility and strong Feller property on the whole $\mathcal{S}$ do not apply. So we need to construct an invariant control set.

DEFINITION 2.3. *(see, e.g.,* [2]*) A subset* $\mathcal{M} \ne \emptyset$ *of* $\mathcal{S}$ *is called an invariant control set for the control system*

$$d\phi = \mathbf{i}\left[\frac{1}{h^2}A\phi + \lambda F(\phi)\phi\right]dt + \mathbf{i}Z(\phi)d\Psi(t) \tag{5}$$

of (2) *with a differentiable deterministic function* Ψ*, if* $\overline{\mathcal{O}^+(x)} = \overline{\mathcal{M}}$*,* $\forall x \in \mathcal{M}$*, and* $\mathcal{M}$ *is maximal with respect to inclusion, where* $\mathcal{O}^+(x)$ *denotes the set of points reachable from* x *(i.e., connected with* x*) in any finite time and* $\overline{\mathcal{M}}$ *denotes the closure of* $\mathcal{M}$*.*

We state one of our main results in the following theorem.

THEOREM 2.4. *The FDA* (2) *possesses a unique invariant probability measure* μ_h *on an invariant control set* $\mathcal{M}_0$*, which implies the unique ergodicity of* (2)*. Moreover,*

$$\operatorname{supp}(\mu_h) = \mathcal{S} \text{ and } \ \mu_h(\mathcal{S}) = \mu_h(\mathcal{M}_0) = 1.$$

Proof.
Step 1. *Existence of invariant measures.*

From Proposition 2.1, we find $\pi_t(U(0), \mathcal{S}) = 1, \forall\, t \geq 0$, where $\pi_t(U(0), \cdot)$ denotes the transition probability (probability kernel) of $U(t)$. As the finite dimensional unit sphere $\mathcal{S}$ is tight, the family of measures $\pi_t(U(0), \cdot)$ is tight, which implies the existence of invariant measures by the Krylov–Bogoliubov theorem [8].

Step 2. *Invariant control set.*

Denoting $U = P + \mathbf{i}Q$ with P and Q being the real and imaginary parts of U respectively, we first consider the following subset of $\mathcal{S}$:

$$\mathcal{S}_1 = \{U = P + \mathbf{i}Q \in \mathcal{S} : \ P > 0\}.$$

For any $t > 0, \ y, z \in \mathcal{S}_1$, there exists a differentiable function ϕ satisfying $\phi(s) = (\phi_1(s), \ldots, \phi_M(s))^T \in \mathcal{S}_1, \ s \in [0, t], \ \phi(0) = y$, and $\phi(t) = z$ by polynomial interpolation argument. As $\operatorname{rank}(Z(\phi(s))) = M$ for $\phi(s) \in \mathcal{S}_1$ and $M \leq K$, the linear equations

$$Z(\phi(s))X = -\mathbf{i}\phi'(s) - \left[\frac{1}{h^2} A\phi(s) + \lambda F(\phi(s))\phi(s)\right]$$

possess a solution $X \in \mathbb{C}^M$. As, in addition, $Z(\phi(s)) = \operatorname{diag}\{\phi_1(s), \ldots, \phi_M(s)\} E_{MK}\Lambda$, where $\operatorname{diag}\{\phi_1(s), \ldots, \phi_M(s)\}$ is invertible for $\phi(s) \in \mathcal{S}_1$, the solution X depends continuously on s and is denoted by $X(s)$. Thus, there exists a differentiable function $\Psi(\cdot) := \int_0^\cdot X(s)ds$ which, together with ϕ defined above, satisfies the control function (5) with initial data $\Psi(0) = 0$. That is, for any $y, z \in \mathcal{S}_1$, y and z are connected, denoted by $y \leftrightarrow z$. The above argument also holds for the following subsets:

$$\begin{aligned}
\mathcal{S}_2 &= \{U = P + \mathbf{i}Q \in \mathcal{S} : \ P < 0\},\\
\mathcal{S}_3 &= \{U = P + \mathbf{i}Q \in \mathcal{S} : \ Q > 0\},\\
\mathcal{S}_4 &= \{U = P + \mathbf{i}Q \in \mathcal{S} : \ Q < 0\}.
\end{aligned}$$

For any $y \in \mathcal{S}_i, \ z \in \mathcal{S}_j$ with $i \neq j$ and $i, j \in \{1, 2, 3, 4\}$, there must exist $\mathcal{S}_l$, r_i, and r_j, satisfying $r_i \in \mathcal{S}_i \cap \mathcal{S}_l \neq \emptyset$ and $r_j \in \mathcal{S}_j \cap \mathcal{S}_l \neq \emptyset$ for some $l \in \{1, 2, 3, 4\}$, such that $y \leftrightarrow r_i \leftrightarrow r_j \leftrightarrow z$. Thus,

$$\mathcal{M}_0 := \mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3 \cup \mathcal{S}_4 = \{U = P + \mathbf{i}Q \in \mathcal{S} : P \neq 0 \text{ or } Q \neq 0\},$$

with $\overline{\mathcal{M}_0} = \mathcal{S}$, is an invariant control set for (5).

Step 3. *Uniqueness of the invariant measure.*

We rewrite (2) with P and Q according to its equivalent form in the Itô sense and obtain

$$\begin{aligned}
d\begin{pmatrix} P \\ Q \end{pmatrix} &= \begin{pmatrix} -\hat{E} & -\frac{1}{h^2}A - \lambda F(P,Q) \\ \frac{1}{h^2}A + \lambda F(P,Q) & -\hat{E} \end{pmatrix} \begin{pmatrix} P \\ Q \end{pmatrix} dt \\
&\quad + \sum_{k=1}^{K} \sqrt{\eta_k} \begin{pmatrix} 0 & -E_k \\ E_k & 0 \end{pmatrix} \begin{pmatrix} P \\ Q \end{pmatrix} d\beta_k(t)
\end{aligned}$$

$$=: X_0(P,Q)dt + \sum_{k=1}^{K} X_k(P,Q)d\beta_k(t). \tag{6}$$

To derive the uniqueness of the invariant measure, we consider the Lie algebra generated by the diffusions of (6),

$$L(X_0, X_1, \dots, X_K) = span\Big\{X_l, [X_i, X_j], [X_l, [X_i, X_j]], \dots, 0 \le l, i, j \le K\Big\}.$$

Choosing $p_* = 0$ and $q_* = \frac{-1}{\sqrt{M}}(1, \dots, 1)^T$ such that $z_* := p_* + \mathbf{i}q_* \in \mathcal{S}_4 \subset \mathcal{M}_0$, we derive that the vectors

$$X_k(p_*, q_*) = \sqrt{\frac{\eta_k}{M}} \begin{pmatrix} e_k(x_1) \\ \vdots \\ e_k(x_M) \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ [X_0, X_k](p_*, q_*) = \sqrt{\frac{\eta_k}{M}} \begin{pmatrix} -\hat{E} \begin{pmatrix} e_k(x_1) \\ \vdots \\ e_k(x_M) \end{pmatrix} \\ (\frac{1}{h^2}A + \frac{1}{M}I) \begin{pmatrix} e_k(x_1) \\ \vdots \\ e_k(x_M) \end{pmatrix} \end{pmatrix}$$

are independent of each other for $k = 1, \dots, M$, which hence implies the following Hörmander condition:

$$\dim L(X_0, X_1, \dots, X_K)(z_*) = 2M.$$

Then there is at most one invariant measure with $\text{supp}(\mu_h) = \mathcal{S}$ according to [2]. Actually, according to the above procedure, we obtain that the Hörmander condition holds uniformly for any $z \in \mathcal{M}_0$.

Combining the three steps above, we conclude that there exists a unique invariant measure μ_h on $\mathcal{M}_0$ for the FDA, with $\mu_h(\mathcal{S}) = \mu_h(\mathcal{M}_0) = 1$. ∎

From the theorem above, we can find out that for some other nonlinearities, e.g., $iF(x, |u|)u$ with F being some potential function, such that the equation still possesses the charge conservation law, we can still get the ergodicity of the finite dimensional approximation of the original equation through the procedure above. The procedure could also applied to higher dimensional Schrödinger equations with proper well-posed assumptions, but it may be more technical to verify the Hörmander condition.

Remark 2.5. According to the ergodicity of (2), we have

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \mathbb{E}f(U(t))dt = \int_{\mathcal{S}} f d\mu_h, \quad \forall\ f \in B_b(\mathcal{S}), \quad \text{in}\ \ L^2(\mathcal{S}, \mu_h),$$

where $B_b(\mathcal{S})$ denotes the set of bounded and measurable functions and $\int_{\mathcal{S}} f d\mu_h$ is known as the ergodic limit with respect to the invariant measure μ_h.

For more details, we refer to [8] and references therein.

3. Approximation of ergodic limit. A fully discrete scheme (FDS) with the discrete multi-symplectic structure and the discrete charge conservation law is constructed in this section, which could also inherit the unique ergodicity of the FDA. In addition, we prove that the time average of the FDS can approximate the ergodic limit $\int_{\mathcal{S}} f d\mu_h$ with order 1 with respect to the time step.

3.1. Fully discrete scheme (FDS). We apply the midpoint scheme to (2), and obtain the following FDS:

$$\begin{cases} U^{n+1}-U^n=\mathbf{i}\frac{\tau}{h^2}AU^{n+\frac{1}{2}}+\mathbf{i}\lambda\tau F(U^{n+\frac{1}{2}})U^{n+\frac{1}{2}}+\mathbf{i}Z(U^{n+\frac{1}{2}})\delta_{n+1}\beta, \\ U^0=U(0)\in\mathcal{S}, \end{cases} \tag{7}$$

where τ denotes the uniform time step, $t_n=n\tau$, $U^n=(u_1^n,\ldots,u_M^n)\in\mathbb{C}^M$, $U^{n+\frac{1}{2}}=\frac{U^{n+1}+U^n}{2}$, and $\delta_{n+1}\beta=\beta(t_{n+1})-\beta(t_n)$. For the FDS (7), which is implicit in both deterministic and stochastic terms, its well-posedness is stated in the following proposition.

PROPOSITION 3.1. *For any initial value $U^0=U(0)\in\mathcal{S}$, there exists a unique solution $(U^n)_{n\in\mathbb{N}}$ of* (7), *and it possesses the discrete charge conservation law, i.e.,*

$$\|U^{n+1}\|^2=\|U^n\|^2=1,\quad \forall\ n\in\mathbb{N}.$$

Proof. We multiply both sides of (7) by $\overline{U^{n+\frac{1}{2}}}$, take the real part, and obtain the existence of the numerical solution by the Brouwer fixed-point theorem as well as the discrete charge conservation law.

For the uniqueness, we assume that $X=(X_1,\ldots,X_M)^T$ and $Y=(Y_1,\ldots,Y_M)^T$ are two solutions of (7) with $U^n=z=(z_1,\ldots,z_M)^T\in\mathcal{S}$. It follows that $X,Y\in\mathcal{S}$ and

$$X-Y=\mathbf{i}\frac{\tau}{h^2}A\frac{X-Y}{2}+\frac{\mathbf{i}\lambda\tau}{8}H(X,Y,z)+\mathbf{i}Z\left(\frac{X-Y}{2}\right)\delta_{n+1}\beta, \tag{8}$$

where

$$H(X,Y,z)=\begin{pmatrix} |X_1+z_1|^2(X_1+z_1)-|Y_1+z_1|^2(Y_1+z_1) \\ \vdots \\ |X_M+z_M|^2(X_M+z_M)-|Y_M+z_M|^2(Y_M+z_M) \end{pmatrix}.$$

Based on the fact that $|a|^2a-|b|^2b=|a|^2(a-b)+|b|^2(a-b)+ab(\overline{a}-\overline{b})$ for any $a,b\in\mathbb{C}$, we have

$$\Im\left[(\overline{X}-\overline{Y})^TH(X,Y,z)\right]=\Im\left[\sum_{m=1}^M(X_m+z_m)(Y_m+z_m)(\overline{X_m}-\overline{Y_m})^2\right]$$

with $\Im[V]$ denoting the imaginary part of V. Multiplying (8) by $(\overline{X}-\overline{Y})^T$ and taking the real part, we get

$$\begin{aligned}\|X-Y\|^2&=-\frac{\lambda\tau}{8}\Im\left[(\overline{X}-\overline{Y})^TH(X,Y,z)\right]\\ &\le\frac{\tau}{8}\left(\max_{1\le m\le M}|X_m+z_m||Y_m+z_m|\right)\|X-Y\|^2\le\frac{\tau}{2}\|X-Y\|^2,\end{aligned}$$

where we have used the fact $X,Y,z\in\mathcal{S}$ and (4). For $\tau<1$, we get $X=Y$ and complete the proof. ∎

The proposition above shows that (7) possesses the discrete charge conservation law. Furthermore, (7) also inherits the unique ergodicity of the FDA and the stochastic multi-symplecticity of the original equation, which are stated in the following two theorems.

THEOREM 3.2. *The FDS* (7) *is also ergodic with a unique invariant measure* μ_h^τ *on the control set* $\mathcal{M}_0$, *such that* $\mu_h^\tau(\mathcal{S}) = \mu_h^\tau(\mathcal{M}_0) = 1$. *Also,*

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} f(U^n) = \int_{\mathcal{S}} f d\mu_h^\tau, \quad \forall\ f \in B_b(\mathcal{S}), \quad \textit{in}\ \ L^2(\mathcal{S}, \mu_h^\tau).$$

Proof. Based on the charge conservation law for $\{U^n\}_{n\geq 1}$, we obtain the existence of the invariant measure similar to the proof of Theorem 2.4.

To obtain the uniqueness of the invariant measure, we show that the Markov chain $\{U^{3n}\}_{n\geq 1}$ satisfies the minorization condition (see, e.g., [12]). First, Proposition 3.1 implies that for a given $U^n \in \mathcal{S}$, solution U^{n+1} can be defined through a continuous function $U^{n+1} = \kappa(U^n, \delta_{n+1}\beta)$. As $\delta_{n+1}\beta$ has a C^∞ density, we get a jointly continuous density for U^{n+1}. Second, similar to Theorem 2.4, for any given $y, z \in \mathcal{M}_0$, there must exist $i, j, k \in \{1, 2, 3, 4\}$ and $r_i, r_j \in \mathcal{M}_0$, such that $y \in \mathcal{S}_i, z \in \mathcal{S}_j, r_i \in \mathcal{S}_i \cap \mathcal{S}_k$, and $r_j \in \mathcal{S}_j \cap \mathcal{S}_k$. As $\frac{y+r_i}{2} \in \mathcal{S}_i$ and $Z(\frac{y+r_i}{2})$ is invertible, $\delta_{3n+1}\beta$ can be chosen to ensure that

$$r_i - y = \mathbf{i}\frac{\tau}{h^2}A\frac{y+r_i}{2} + \mathbf{i}\lambda\tau F(\frac{y+r_i}{2})\frac{y+r_i}{2} + \mathbf{i}Z(\frac{y+r_i}{2})\delta_{3n+1}\beta$$

holds, i.e., $r_i = \kappa(y, \delta_{3n+1}\beta)$. Similarly, based on the fact $\frac{r_i+r_j}{2} \in \mathcal{S}_k$ and $\frac{r_j+z}{2} \in \mathcal{S}_j$, we have $r_j = \kappa(r_i, \delta_{3n+2}\beta)$ and $z = \kappa(r_j, \delta_{3n+3}\beta)$. That is, for any given $y, z \in \mathcal{M}_0$, $\delta_{3n+1}\beta, \delta_{3n+2}\beta, \delta_{3n+3}\beta$ can be chosen to ensure that $U^{3n} = y$ and $U^{3(n+1)} = z$. Finally we obtain that, for any $\delta > 0$,

$$\mathbb{P}_3\left(y, B(z,\delta)\right) := \mathbb{P}\left(U^3 \in B(z,\delta)\middle| U^0 = y\right) > 0,$$

where $B(z, \delta)$ denotes the open ball of radius δ centered at z. ∎

The infinite dimensional system (1) has been shown to preserve the stochastic multi-symplectic conservation law locally (see, i.e., [11]):

$$d_t(dp \wedge dq) - \partial_x(dp \wedge dv + dq \wedge dw)dt = 0,$$

with p, q denoting the real and imaginary parts of solution u respectively and $v = p_x$, $w = q_x$ being the derivatives of p and q with respect to variable x. We now show that this ergodic FDS (7) not only possesses the discrete charge conservation law as shown in Proposition 3.1 but also preserves the discrete stochastic multi-symplectic structure.

THEOREM 3.3. *The implicit FDS* (7) *preserves the discrete multi-symplectic structure*

$$\begin{aligned}&\frac{1}{\tau}(dp_j^{n+1} \wedge dq_j^{n+1} - dp_j^n \wedge dq_j^n) - \frac{1}{h}(dp_j^{n+\frac12} \wedge dv_{j+1}^{n+\frac12} - dp_{j-1}^{n+\frac12} \wedge dv_j^{n+\frac12})\\ &\quad - \frac{1}{h}(dq_j^{n+\frac12} \wedge dw_{j+1}^{n+\frac12} - dq_{j-1}^{n+\frac12} \wedge dw_j^{n+\frac12}) = 0,\end{aligned}$$

where p_j^n, q_j^n *denote the real and imaginary parts of* u_j^n, $v_j = \frac{1}{h}(p_j^n - p_{j-1}^n)$, *and* $w_j = \frac{1}{h}(q_j^n - q_{j-1}^n)$.

Proof. Rewriting (7) with the real and imaginary parts of the components u_j^n of U^n, we get

$$
\begin{cases}
\frac{1}{\tau}(q_j^{n+1}-q_j^n)-\frac{1}{h}(v_{j+1}^{n+\frac12}-v_j^{n+\frac12})=\left((p_j^{n+\frac12})^2+(q_j^{n+\frac12})^2\right)p_j^{n+\frac12}+p_j^{n+\frac12}\zeta_j^K,\\
-\frac{1}{\tau}(p_j^{n+1}-p_j^n)-\frac{1}{h}(w_{j+1}^{n+\frac12}-w_j^{n+\frac12})=\left((p_j^{n+\frac12})^2+(q_j^{n+\frac12})^2\right)q_j^{n+\frac12}+q_j^{n+\frac12}\zeta_j^K,\\
\frac{1}{h}(p_j^{n+\frac12}-p_{j-1}^{n+\frac12})=v_j^{n+\frac12},\\
\frac{1}{h}(q_j^{n+\frac12}-q_{j-1}^{n+\frac12})=w_j^{n+\frac12},
\end{cases}
\tag{9}
$$

where $\zeta_j^K=\sum_{k=1}^K\sqrt{\eta_k}e_k(x_j)\circ d\beta_k(t)$. Denoting $z_j^{n+\frac12}=(p_j^{n+\frac12},q_j^{n+\frac12},v_j^{n+\frac12},w_j^{n+\frac12})^T$ and taking the differential in the phase space on both sides of (9), we obtain

$$
\begin{aligned}
&\frac{1}{\tau}d\begin{pmatrix} q_j^{n+1}-q_j^n\\ -(p_j^{n+1}-p_j^n)\\ 0\\ 0\end{pmatrix}+\frac{1}{h}d\begin{pmatrix} -(v_{j+1}^{n+\frac12}-v_j^{n+\frac12})\\ -(w_{j+1}^{n+\frac12}-w_j^{n+\frac12})\\ p_j^{n+\frac12}-p_{j-1}^{n+\frac12}\\ q_j^{n+\frac12}-q_{j-1}^{n+\frac12}\end{pmatrix}\\
&\quad=\nabla^2S_1(z_j^{n+\frac12})dz_j^{n+\frac12}+\nabla^2S_2(z_j^{n+\frac12})dz_j^{n+\frac12}\zeta_j^K,
\end{aligned}
\tag{10}
$$

where

$$
S_1(z_j^{n+\frac12})=\frac14\left((p_j^{n+\frac12})^2+(q_j^{n+\frac12})^2\right)^2+\frac12\left(v_j^{n+\frac12}\right)^2+\frac12\left(w_j^{n+\frac12}\right)^2
$$

and

$$
S_2(z_j^{n+\frac12})=\frac12\left(p_j^{n+\frac12}\right)^2+\frac12\left(q_j^{n+\frac12}\right)^2.
$$

Then the wedge product between $dz_j^{n+\frac12}$ and (10) concludes the proof based on the symmetry of ∇^2S_1 and ∇^2S_2. ∎

Before giving the approximate error of the ergodic limit, we give some essential a priori estimates about the stability of (7) and (2). In the following, C denotes a generic constant independent of T, N, τ, and h while C_h denotes a constant depending also on h, whose value may be different from line to line.

LEMMA 3.4. *For any initial value $U^0\in\mathcal{S}$ and $\gamma\geq1$, if $Q\in\mathcal{HS}(L^2,H^{\frac32-\frac1\gamma})$, then there exists a constant C such that the solution $(U^n)_{n\in\mathbb{N}}$ of (7) satisfies*

$$
\mathbb{E}\left\|U^{n+1}-U^n\right\|^{2\gamma}\leq C(\tau^{2\gamma}h^{-4\gamma}+\tau^\gamma),\quad\forall\, n\in\mathbb{N},
$$

where $\mathcal{HS}(L^{\gamma_1},H^{\gamma_2})$ denotes the space of Hilbert–Schmidt operators from L^{γ_1} to H^{γ_2}.

LEMMA 3.5. *For any initial value $U(0)\in\mathcal{S}$ and $\gamma\geq1$, there exists a constant C such that the solution $U(t)$ of (2) satisfies*

$$
\mathbb{E}\|U(t_{n+1})-U(t_n)\|^{2\gamma}\leq C(\tau^{2\gamma}h^{-4\gamma}+\tau^\gamma),\quad\forall\, n\in\mathbb{N}.
$$

The proofs of the lemmas above are given in the Appendix for the readers' convenience.

3.2. Approximation of ergodic limit. To approximate the ergodic limit of (2) and get the approximate error, we give an estimate of the local weak convergence between $U(\tau)$ and U^1, and the Poisson equation associated with (2) is also used (see [13]). Recall that the SDE (2) in the Stratonovich sense has an equivalent Itô form

$$\begin{aligned} dU &= \left[\mathbf{i}\frac{1}{h^2}AU + \mathbf{i}\lambda F(U)U - \hat{E}U\right]dt + \mathbf{i}Z(U)d\beta(t) \\ &=: b(U)dt + \sigma(U)d\beta(t) \end{aligned} \tag{11}$$

based on (3). For any fixed $f \in W^{4,\infty}(\mathcal{S})$, let $\hat{f} := \int_{\mathcal{S}} f d\mu_h$ and φ be the unique solution of the Poisson equation $\mathcal{L}\varphi = f - \hat{f}$, where

$$\mathcal{L} := b \cdot \nabla + \frac{1}{2}\sigma\sigma^T : \nabla^2$$

denotes the generator of (11). It is easy to find out that (11) satisfies the hypoelliptic setting (see, e.g., [13]) according to the Hörmander condition in Theorem 2.4. Thus, $\varphi \in W^{4,\infty}(\mathcal{S})$ according to Theorem 4.1 in [13]. Based on the well-posedness of the numerical solution $(U^n)_{n\in\mathbb{N}}$ and the implicit function theorem, (7) can be rewritten in the form

$$U^{n+1} = U^n + \tau\Phi(U^n, \tau, h, \delta_{n+1}\beta) \tag{12}$$

for some function Φ. Denoting by $D\varphi(u)\Phi_1$ and $D^k\varphi(u)(\Phi_1, \ldots, \Phi_k)$ the first and kth order weak derivatives evaluated in the directions Φ_j, $j = 1, \ldots, k$, with $D^k\varphi(u)(\Phi)^k$ for shorthand if all the directions are the same in the kth derivatives, then we have

$$\begin{aligned} \varphi(U^{n+1}) &= \varphi(U^n) + \tau\left[D\varphi(U^n)\Phi^n + \frac{1}{2}\tau D^2\varphi(U^n)(\Phi^n)^2\right] + \frac{1}{6}D^3\varphi(U^n)(\tau\Phi^n)^3 + R_n^\Phi \\ &=: \varphi(U^n) + \tau\mathcal{L}^\Phi\varphi(U^n) + \frac{1}{6}D^3\varphi(U^n)(\tau\Phi^n)^3 + R_n^\Phi, \end{aligned} \tag{13}$$

where $\Phi^n := \Phi(U^n, \tau, h, \delta_{n+1}\beta)$,

$$\mathcal{L}^\Phi\varphi(U^n) = D\varphi(U^n)\Phi^n + \frac{1}{2}\tau D^2\varphi(U^n)(\Phi^n)^2,$$

and

$$R_n^\Phi = \frac{1}{4!}D^4\varphi(\theta_n)(\tau\Phi^n)^4$$

for some $\theta_n \in [U^n, U^{n+1}] := [u_1^n, u_1^{n+1}] \times \ldots \times [u_M^n, u_M^{n+1}]$. Adding (13) together from $n = 0$ to $n = N-1$ for some fixed $N \in \mathbb{N}$, then dividing the result by $T = N\tau$, and noticing that $\mathcal{L}\varphi(U^n) = f(U^n) - \hat{f}$, we obtain

$$\begin{aligned} \frac{\varphi(U^N) - \varphi(U^0)}{N\tau} =& \frac{1}{N}\Bigg(\sum_{n=0}^{N-1}\left[\mathcal{L}^\Phi\varphi(U^n) - \mathcal{L}\varphi(U^n)\right] + \sum_{n=0}^{N-1}\mathcal{L}\varphi(U^n) \\ &+ \frac{1}{\tau}\sum_{n=0}^{N-1}\frac{1}{6}D^3\varphi(U^n)(\tau\Phi^n)^3 + \frac{1}{\tau}\sum_{n=0}^{N-1}R_n^\Phi\Bigg) \\ =& \frac{1}{N}\sum_{n=0}^{N-1}\left[\mathcal{L}^\Phi\varphi(U^n) - \mathcal{L}\varphi(U^n) + \frac{1}{6\tau}D^3\varphi(U^n)(\tau\Phi^n)^3\right] \end{aligned}$$

$$+\left(\frac{1}{N}\sum_{n=0}^{N-1} f(U^n)-\hat{f}\right)+\frac{1}{N\tau}\sum_{n=0}^{N-1} R_n^{\Phi},$$

which shows

$$\begin{aligned}\left|\mathbb{E}\left[\frac{1}{N}\sum_{n=0}^{N-1} f(U^n)-\hat{f}\right]\right| &\le \left|\frac{1}{N\tau}\mathbb{E}\left[\varphi(U^N)-\varphi(U^0)\right]\right|+\left|\frac{1}{N\tau}\sum_{n=0}^{N-1}\mathbb{E}R_n^{\Phi}\right| \\ &+\left|\frac{1}{N}\sum_{n=0}^{N-1}\mathbb{E}\left[\mathcal{L}^{\Phi}\varphi(U^n)-\mathcal{L}\varphi(U^n)+\frac{1}{6\tau}D^3\varphi(U^n)(\tau\Phi^n)^3\right]\right| =: I+II+III.\end{aligned} \tag{14}$$

The average $\frac{1}{N}\sum_{n=0}^{N-1} f(U^n)$ is regarded as an approximation of $\hat{f}$. We next begin to investigate the approximate error by estimating I, II, and III respectively.

According to the fact that $\varphi \in W^{4,\infty}(\mathcal{S})$ and Lemma 3.4, we have

$$I \le \frac{2\|\varphi\|_{0,\infty}}{N\tau} \le \frac{C}{T} \tag{15}$$

and

$$\begin{aligned} II \le&\frac{1}{N\tau}\sum_{n=0}^{N-1}\mathbb{E}\left[\|\tau\Phi^n\|^4\,\|D^4\varphi\|_{L^\infty}\right] \le \frac{C}{N\tau}\sum_{n=0}^{N-1}\mathbb{E}\left[\left\|U^{n+1}-U^n\right\|^4\right] \\ \le&\frac{C}{N\tau}\sum_{n=0}^{N-1}\left(\tau^4h^{-8}+\tau^2\right) \le C\left(\tau^3h^{-8}+\tau\right),\end{aligned} \tag{16}$$

where $\|\varphi\|_{\gamma,\infty} := \sup_{|\alpha|\le\gamma,u\in\mathcal{S}}|D^\alpha\varphi(u)|$, $\gamma\in\mathbb{N}$.

It then remains to estimate the term III. To this end, we need the estimate of the local weak convergence, which is stated in the following theorem. The proof of the following theorem is also given in the Appendix.

THEOREM 3.6. *For a fixed spatial approximation* (2), *and for any initial value* $U^0 \in \mathcal{S}$ *and* $\varphi \in W^{4,\infty}(\mathcal{S})$, *it holds under the condition* $Q \in \mathcal{HS}(L^2, H^{\frac{5}{4}})$ *and* $\tau = O(h^4)$ *that*

$$\left|\mathbb{E}\left[\varphi(U(\tau))-\varphi(U^1)\right]\right| \le C_h\tau^2$$

for some constant $C_h = C(\varphi,\eta,h)$.

Now we are in the position of showing the approximation error between the time average of FDS and the ergodic limit of FDA.

THEOREM 3.7. *Under the assumptions in Theorem* 3.6 *and for any* $f \in W^{4,\infty}(\mathcal{S})$, *there exists a positive constant* $C_h = C(f,\eta,h)$ *such that*

$$\left|\mathbb{E}\left[\frac{1}{N}\sum_{n=0}^{N-1} f(U^n)-\hat{f}\right]\right| \le C_h\left(\frac{1}{T}+\tau\right).$$

Proof. Based on (14)–(16), it suffices to estimate term III. For any $f \in W^{4,\infty}(\mathcal{S})$, we know from the statement above that the solution to the Poisson equation $\mathcal{L}\varphi = f-\hat{f}$ satisfies $\varphi \in W^{4,\infty}(\mathcal{S})$. Based on (13), Lemma 3.4, and the condition $\tau = O(h^4)$, we have

$$\begin{aligned}\varphi(U^1) &\overset{\mathbb{E}}{=} \varphi(U^0)+\tau\mathcal{L}^{\Phi}\varphi(U^0)+\frac{1}{6}D^3\varphi(U^0)(U^1-U^0)^3+O(\tau^2) \\ &\overset{\mathbb{E}}{=} \varphi(U^0)+\tau\mathcal{L}^{\Phi}\varphi(U^0)+O(\tau^2),\end{aligned} \tag{17}$$

where $\overset{\mathbb{E}}{=}$ means that the equation holds in expectation sense, and in the last step we have used the fact that

$$
\begin{aligned}
D^3\varphi(U^0)(U^1-U^0)^3 &= D^3\varphi(U^0)\left(\mathbf{i}\frac{\tau}{h^2}AU^{\frac{1}{2}} + \mathbf{i}\lambda\tau F(U^{\frac{1}{2}})U^{\frac{1}{2}} + \mathbf{i}Z(U^{\frac{1}{2}})\delta_1\beta\right)^3 \\
&\overset{\mathbb{E}}{=} D^3\varphi(U^0)\left(\mathbf{i}Z(U^{\frac{1}{2}})\delta_1\beta\right)^3 + O(\tau^2h^{-2}+\tau^2) \\
&\overset{\mathbb{E}}{=} D^3\varphi(U^0)\left(\frac{\mathbf{i}}{2}Z(U^1-U^0)\delta_1\beta + \mathbf{i}Z(U^0)\delta_1\beta\right)^3 + O(\tau^2h^{-2}+\tau^2) \\
&\overset{\mathbb{E}}{=} O(\tau^2h^{-2}+\tau^2) \qquad (18)
\end{aligned}
$$

based on the linearity of Z, Lemma 3.4, and that $\mathbb{E}\left(\mathbf{i}Z(U^0)\delta_1\beta\right)^3 = 0$. We can also get the following expression similar to (17) based on Taylor expansion and Lemma 3.5:

$$
\begin{aligned}
\varphi(U(\tau)) &\overset{\mathbb{E}}{=} \varphi(U^0) + \int_0^\tau \left(D\varphi(U^0)b(U(t)) + \frac{1}{2}D^2\varphi(U^0)\left(\sigma(U(t))\right)^2\right)dt \\
&\quad + \int_0^\tau D\varphi(U^0)\sigma(U(t))d\beta(t) + \frac{1}{6}D^3\varphi(U^0)(U(\tau)-U^0)^3 + O(\tau^2) \\
&\overset{\mathbb{E}}{=} \varphi(U^0) + \int_0^\tau \tilde{\mathcal{L}}_t\varphi(U^0)dt + O(\tau^2), \qquad (19)
\end{aligned}
$$

where

$$\tilde{\mathcal{L}}_t\varphi(U^0) := D\varphi(U^0)b(U(t)) + \frac{1}{2}D^2\varphi(U^0)\left(\sigma(U(t))\right)^2$$

and $\mathbb{E}\left[\int_0^\tau D\varphi(U^0)\sigma(U(t))d\beta(t)\right] = 0$. Thus, subtracting (17) from (19), we derive

$$\left|\mathbb{E}\left[\tau\mathcal{L}^\Phi\varphi(U^0) - \int_0^\tau \tilde{\mathcal{L}}_t\varphi(U^0)dt\right]\right| \le \left|\mathbb{E}\left[\varphi(U(\tau)) - \varphi(U^1)\right]\right| + C\tau^2. \qquad (20)$$

We notice that

$$
\begin{aligned}
\left|\int_0^\tau \mathbb{E}\left[\tilde{\mathcal{L}}_t\varphi(U^0) - \mathcal{L}\varphi(U^0)\right]dt\right| &\le \left|\int_0^\tau \mathbb{E}\left[D\varphi(U^0)\left(b(U(t)) - b(U^0)\right)\right]dt\right| \\
&\quad + \left|\frac{1}{2}\int_0^\tau \mathbb{E}\left[D^2\varphi(U^0)\left(\sigma(U(t))-\sigma(U^0), \sigma(U(t))+\sigma(U^0)\right)\right]dt\right|, \qquad (21)
\end{aligned}
$$

in which we have

$$
\begin{aligned}
\left|\mathbb{E}\left[D\varphi(U^0)\left(b(U(t))-b(U^0)\right)\right]\right| &= \Big|\mathbb{E}\Big[D^2\varphi(U^0)\Big(\mathbf{i}\frac{1}{h^2}A\left(U(t)-U^0\right) \\
&\quad + \mathbf{i}\lambda\Big(F(U(t))U(t) - F(U^0)U^0\Big) - \hat{E}(U(t)-U^0)\Big)\Big]\Big| \le C(th^{-2}+t)
\end{aligned}
$$

for the first term in (21). In the last step, we have used the fact that $g(V) := F(V)V$, $\forall\ V \in \mathcal{S}$, is a continuous differentiable function which satisfies $|D^kg(V)| \le C$ for $\|V\| \le 1$ and $k \in \mathbb{N}$, and then replaced $U(t)-U^0$ by the integral form of (2) to get the result. The second term in (21) can be estimated in the same way. Thus, we have

$$\left|\int_0^\tau \mathbb{E}\left[\tilde{\mathcal{L}}_t\varphi(U^0) - \mathcal{L}\varphi(U^0)\right]dt\right| \le C(\tau^2h^{-2}+\tau^2). \qquad (22)$$

We hence conclude based on (18), (20), (22) and Theorem 3.6 that

$$
\begin{aligned}
III &= \left| \frac{1}{N} \sum_{n=0}^{N-1} \mathbb{E}\left[\mathcal{L}^{\Phi}\varphi(U^n) - \mathcal{L}\varphi(U^n) + \frac{1}{6\tau} D^3\varphi(U^n)(U^{n+1}-U^n)^3 \right] \right| \\
&\le \frac{1}{\tau} \sup_{U^0\in\mathcal{S}} \left\{ \left| \mathbb{E}\left[\tau\mathcal{L}^{\Phi}\varphi(U^0) - \int_0^{\tau} \tilde{\mathcal{L}}_t\varphi(U^0)dt \right] \right| + \left| \int_0^{\tau} \mathbb{E}\left[\tilde{\mathcal{L}}_t\varphi(U^0) - \mathcal{L}\varphi(U^0) \right] dt \right| \right\} \\
&\quad + C(\tau h^{-2} + \tau) \le C_h\tau.
\end{aligned} \tag{23}
$$

Noticing that $\tau^3 h^{-8} = O(\tau)$ under the condition $\tau = O(h^4)$, from (15), (16), and (23), we finally obtain

$$
\left| \mathbb{E}\left[\frac{1}{N} \sum_{n=0}^{N-1} f(U^n) - \hat{f} \right] \right| \le C_h \left(\frac{1}{T} + \tau \right). \qquad \square
$$

Remark 3.8. Based on the theorem above and the ergodicity of (2), for a fixed h, we obtain

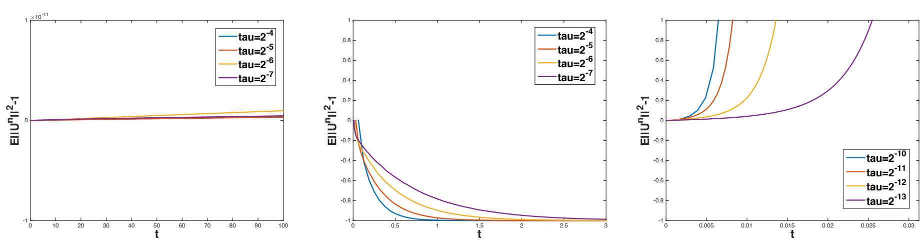
$$
\left| \mathbb{E}\left[\frac{1}{N} \sum_{n=0}^{N-1} f(U^n) - \frac{1}{T}\int_0^T f(U(t))dt \right] \right| \le C_h(B(T) + \tau),
$$

which implies that the global weak error is of order 1, i.e.,

$$
\left| \mathbb{E}\Big[f(U^n) - f(U(t)) \Big] \right| \le C_h(\tilde{B}(t) + \tau), \quad t \in [n\tau, (n+1)\tau],
$$

where $B(T) \to 0$ and $\tilde{B}(T) \to 0$ as $T \to \infty$. On the other hand, a time independent weak error in turn leads to the result stated in Theorem 3.7.

4. Numerical experiments. In this section, numerical experiments are given to test several properties of scheme (7) with $\lambda = 1$, i.e., the focusing case. In the following experiments, we simulate the noise $\delta_{n+1}\beta$ by $\sqrt{\tau}\xi_n$ with ξ_n being independent K-dimensional $N(0,1)$-random variables, and choose $\eta_k = k^{-4}$, $k = 1, \dots, K$. In addition, we approximate the expectation by taking averaged value over 500 paths, and the proposed scheme, which is implicit, is numerically solved utilizing the fixed point iteration. In the following, we will use the notation $\|U\|_\gamma^\gamma := \sum_{m=1}^M (|p_m|^\gamma + |q_m|^\gamma)$ for $U \in \mathbb{C}^M$ and $\gamma \in \mathbb{N}$ with $P = (p_1, \dots, p_M)^T$, and $Q = (q_1, \dots, q_M)^T$ being the real and imaginary parts of U. Notice that $\|\cdot\|_2 = \|\cdot\|$.



(a) Proposed scheme (b) IME scheme (c) EM scheme

FIG. 1. *Charge evolution* $\mathbb{E}\|U^n\|^2 - 1$ *for* (a) *the proposed scheme with* $T = 100$ *under steps* $\tau = 2^{-i}$ $(i = 4,5,6,7)$, (b) *IME scheme with* $T = 3$ *under steps* $\tau = 2^{-i}$ $(i = 4,5,6,7)$, *and* (c) *EM scheme with* $T = 2^{-5}$ *under steps* $\tau = 2^{-i}$ $(i = 10,11,12,13)$ $(h = 0.05,\ K = 30)$.

We omit the boundary nodes in the simulation; as a result, we may choose the normalized initial value $U^0 = c_*(U^0(1), \ldots, U^0(M))^T$ based on function $u_0(x)$ satisfying $U^0(m) = u_0(mh)$, $m = 1, \ldots, M$, in which $u_0(x)$ need not to satisfy the boundary condition in (1). Let $u_0(x) = 1$, and we get the normalized initial value U^0 satisfying $\|U^0\| = 1$, which is used in Figures 1, 3, and 4. We first simulate the discrete charge for the proposed scheme compared with the Euler–Maruyama (EM) scheme and implicit Euler (IE) scheme, respectively. Figure 1 shows that the proposed scheme possesses the discrete charge conservation law $\mathbb{E}\|U^n\|^2 = 1$, which coincides with Proposition 3.1, while both the EM scheme and the IE scheme do not. As the EM scheme is not stable—its solution will blow up in a short time—we choose the time step τ small enough for the EM scheme in the experiments.

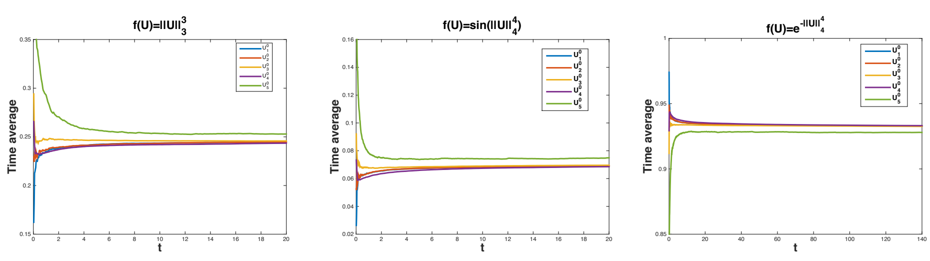
As the ergodic limit $\int_{\mathcal{S}} f d\mu_h$ is unknown, to verify the ergodicity of the numerical solution, we simulate the time averages $\frac{1}{N}\sum_{n=1}^{N} \mathbb{E}[f(U^n)]$ for the proposed scheme with the bounded function $f \in C_b(\mathcal{S})$ being (a) $f(U) = \|U\|_3^3$, (b) $f(U) = \sin(\|U\|_4^4)$, and (c) $f(U) = e^{-\|U\|_4^4}$ in Figure 2, starting from five different initial values U_l^0, $1 \leq l \leq 5$. It is known from Theorem 3.2 that, for almost every initial value $U^0 \in \mathcal{S}$, the time averages will converge to the same value, i.e., the ergodic limit. Thus, we choose five initial values

$$U_l^0 = c_*(U_l^0(1), \ldots, U_l^0(M))^T, \ l = 1, \ldots, 5$$

based on the following five functions:

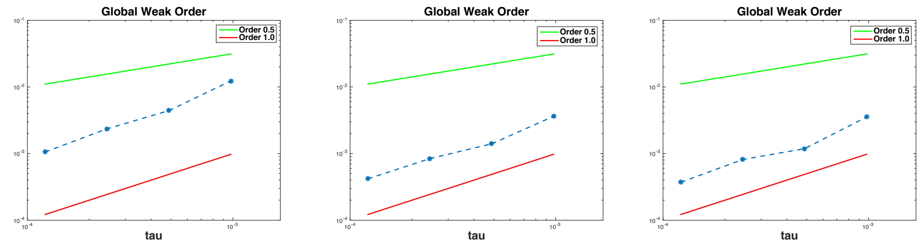
$$\begin{aligned}
u_{0,1}(x) &= \frac{1}{\sqrt{2}} + \frac{\mathbf{i}}{\sqrt{2}}, \quad u_{0,2}(x) = 1, \quad u_{0,3}(x) = 2x,\\
u_{0,4}(x) &= \left(1 - \sqrt{\frac{\pi}{2}(\exp\frac{1}{4} - 1)}\right)(1 - \exp(x(1-x))),\\
u_{0,5}(x) &= c_* \operatorname{sech}\left(\frac{x}{\sqrt{2}}\right)\exp\left(\mathbf{i}\frac{x}{2}\right)
\end{aligned}$$

with $U_l^0(m) = u_{0,l}(hm)$, $1 \leq m \leq M$, and c_* being normalized constants. The charge of all the initial functions equals 1, and $u_{0,4}(x)$ even satisfies the boundary condition in (1). Figure 2 shows that the proposed scheme starting from different initial values converges to the same value with error no more than $O(\tau)$ with $h = 0.05$ and $\tau = 2^{-6}$, which coincides with Theorem 3.7.



(a) $f(U) = \|U\|_3^3$, $T = 20$ (b) $f(U) = \sin(\|U\|_4^4)$, $T = 20$ (c) $f(U) = e^{-\|U\|_4^4}$, $T = 140$

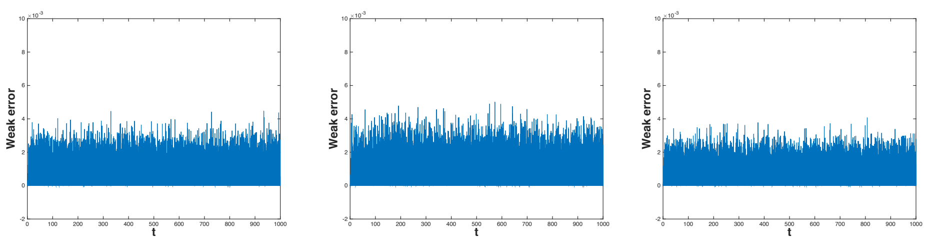
FIG. 2. *The time averages* $\frac{1}{N}\sum_{n=1}^{N} \mathbb{E}[f(U^n)]$ *for the proposed scheme with* (a) $f(U) = \|U\|_3^3$, (b) $f(U) = \sin(\|U\|_4^4)$, *and* (c) $f(U) = e^{-\|U\|_4^4}$ ($\tau = 2^{-6}$, $h = 0.05$, $K = 30$).



(a) $f(U) = \|U\|_3^3$ (b) $f(U) = \sin(\|U\|_4^4)$ (c) $f(U) = e^{-\|U\|_4^4}$

FIG. 3. *The weak convergence order of* $|\mathbb{E}[f(U^n) - f(U(T))]|$ *with* (a) $f(U) = \|U\|_3^3$, (b) $f(U) = \sin(\|U\|_4^4)$, *and* (c) $f(U) = e^{-\|U\|_4^4}$ ($\tau = 2^{-i}, 10 \le i \le 13$, $h = 0.05$, $T = 2^{-1}$, $K = 30$).

For a fixed h, Figures 3 and 4 show the weak convergence order in the temporal direction and the weak error over long time, respectively. Figure 3 shows that the proposed scheme is of order 1 in the weak sense for (a) $f(U) = \|U\|_3^3$, (b) $f(U) = \sin(\|U\|_4^4)$, and (c) $f(U) = e^{-\|U\|_4^4}$, which coincides with the statement in Remark 3.8. Furthermore, based on the ergodicity for both FDS and FDA, the weak error is supposed to be independent of time interval when time is large enough. To verify this property, we simulate the weak error over long time in Figure 4 for (a) $f(U) = \|U\|_3^3$, (b) $f(U) = \sin(\|U\|_4^4)$, and (c) $f(U) = e^{-\|U\|_4^4}$; it shows that the weak error for the proposed scheme would not increase before $T = 1000$ while the weak error for the EM scheme would increase with time.



(a) $f(U) = \|U\|_3^3$ (b) $f(U) = \sin(\|U\|_4^4)$ (c) $f(U) = e^{-\|U\|_4^4}$

FIG. 4. *The weak error* $|\mathbb{E}[f(U^n) - f(U(T))]|$ *for* (a) $f(U) = \|U\|_3^3$, (b) $f(U) = \sin(\|U\|_4^4)$, *and* (c) $f(U) = e^{-\|U\|_4^4}$ ($\tau = 2^{-12}$, $h = 0.05$, $T = 10^3$, $K = 30$).

5. Appendix.

5.1. Proof of Lemma 3.4. As proved in Proposition 3.1 that $\|U^n\| = 1$, $\forall\, n \in \mathbb{N}$, for the nonlinear term, we have

$$\mathbb{E}\left\|F(U^{n+\frac{1}{2}})U^{n+\frac{1}{2}}\right\|^{2\gamma} = \mathbb{E}\sum_{m=1}^{M}\left|u_m^{n+\frac{1}{2}}\right|^{6\gamma} \le \mathbb{E}\left(\sum_{m=1}^{M}\left|u_m^{n+\frac{1}{2}}\right|^2\right)^{3\gamma} \le \mathbb{E}\left\|U^{n+\frac{1}{2}}\right\|^{6\gamma} \le 1$$

by the convexity of $\mathcal{S}$, i.e., $\|U^{n+\frac{1}{2}}\| \le 1$, a.s. The noise term can be estimated as

$$\mathbb{E}\left\|Z(U^{n+\frac{1}{2}})\delta_{n+1}\beta\right\|^{2\gamma} = \mathbb{E}\left(\sum_{m=1}^{M}\left|\sum_{k=1}^{K}u_m^{n+\frac{1}{2}}e_k(x_m)\sqrt{\eta_k}\delta_{n+1}\beta_k\right|^2\right)^{\gamma}$$

$$\le \mathbb{E}\left(2\sum_{m=1}^{M}\left|u_m^{n+\frac{1}{2}}\right|^2\Big(\sum_{k=1}^{K}\sqrt{\eta_k}|\delta_{n+1}\beta_k|\Big)^2\right)^{\gamma} = \mathbb{E}\left(2\left\|U^{n+\frac{1}{2}}\right\|^2\Big(\sum_{k=1}^{K}\sqrt{\eta_k}|\delta_{n+1}\beta_k|\Big)^2\right)^{\gamma}$$

(24)

$$\le C\mathbb{E}\left(\sum_{k=1}^{K}\eta_k^{\frac{1}{4}}\eta_k^{\frac{1}{4}}|\delta_{n+1}\beta_k|\right)^{2\gamma} \le C\mathbb{E}\Big[\Big(\sum_{k=1}^{K}\eta_k^{\frac{\gamma}{2(2\gamma-1)}}\Big)^{2\gamma-1}\Big(\sum_{k=1}^{K}\eta_k^{\frac{\gamma}{2}}|\delta_{n+1}\beta_k|^{2\gamma}\Big)\Big] \le C\tau^{\gamma}$$

by $|e_k(x_m)| \le \sqrt{2}$ and Hölder's inequality. In the last step of (24) we notice that, as $Q \in \mathcal{HS}(L^2, H^{\frac{3}{2}-\frac{1}{\gamma}})$, $\sum_{k=1}^{\infty} k^{3-\frac{2}{\gamma}}\eta_k < \infty$, so $\eta_k = O(k^{-(4-\frac{2}{\gamma}+\epsilon)})$ for any $\epsilon > 0$. Thus,

$$\sum_{k=1}^{\infty}\eta_k^{\frac{\gamma}{2(2\gamma-1)}} \le C\sum_{k=1}^{\infty}k^{-(4-\frac{2}{\gamma}+\epsilon)\frac{\gamma}{2(2\gamma-1)}} = C\sum_{k=1}^{\infty}k^{-\left(1+\frac{\epsilon\gamma}{2(2\gamma-1)}\right)} < \infty.$$

In conclusion,

$$\begin{aligned}
&\mathbb{E}\left\|U^{n+1}-U^n\right\|^{2\gamma}\\
&\quad\le C\left(\mathbb{E}\left\|\frac{\tau}{h^2}AU^{n+\frac{1}{2}}\right\|^{2\gamma} + \mathbb{E}\left\|\lambda\tau F(U^{n+\frac{1}{2}})U^{n+\frac{1}{2}}\right\|^{2\gamma} + \mathbb{E}\left\|Z(U^{n+\frac{1}{2}})\delta_{n+1}\beta\right\|^{2\gamma}\right)\\
&\quad\le \frac{C\tau^{2\gamma}}{h^{4\gamma}}\mathbb{E}\left\|U^{n+\frac{1}{2}}\right\|^{2\gamma} + C\tau^{2\gamma} + C\tau^{\gamma} \le C\left(\tau^{2\gamma}h^{-4\gamma}+\tau^{\gamma}\right),
\end{aligned}$$

where we have used the fact that $\|A\| \le 4$.

5.2. Proof of Lemma 3.5. From (2) and (3), based on Hölder's inequality, we obtain

$$\begin{aligned}
&\mathbb{E}\|U(t_{n+1})-U(t_n)\|^{2\gamma}\\
&\quad= \mathbb{E}\left\|\int_{t_n}^{t_{n+1}}\left[\mathbf{i}\frac{1}{h^2}AU + \mathbf{i}\lambda F(U)U - \hat{E}U\right]dt + \int_{t_n}^{t_{n+1}}\mathbf{i}Z(U)d\beta(t)\right\|^{2\gamma}\\
&\quad\le C\Bigg(\int_{t_n}^{t_{n+1}}\mathbb{E}\left\|\mathbf{i}\frac{1}{h^2}AU + \mathbf{i}\lambda F(U)U - \hat{E}U\right\|^{2\gamma}dt\left(\int_{t_n}^{t_{n+1}}1^{\frac{2\gamma}{2\gamma-1}}dt\right)^{2\gamma-1}\\
&\qquad+ \mathbb{E}\left\|\int_{t_n}^{t_{n+1}}\mathbf{i}Z(U)d\beta(t)\right\|^{2\gamma}\Bigg)\\
&\quad\le C\tau^{2\gamma-1}\left\|\frac{1}{h^2}A\right\|^{2\gamma}\int_{t_n}^{t_{n+1}}\mathbb{E}\left\|U\right\|^{2\gamma}dt + C\tau^{2\gamma} + C\tau^{\gamma}\\
&\quad\le C(\tau^{2\gamma}h^{-4\gamma}+\tau^{\gamma}),
\end{aligned}$$

where we have used the boundedness of $F(U)U$ in $\mathcal{S}$ similar to that in Lemma 3.4. In the third step of the equation above, we also used

$$\begin{aligned}
\mathbb{E}\|\hat{E}U\|^{2\gamma} &\le C\mathbb{E}\left(\sum_{m=1}^{M}\left|\sum_{k=1}^{K}\eta_k e_k^2(x_m)u_m\right|^2\right)^{\gamma}\\
&\le C\mathbb{E}\left(\sum_{m=1}^{M}|u_m|^2\left(\sum_{k=1}^{K}\eta_k\right)^2\right)^{\gamma} \le C\eta^{2\gamma}\mathbb{E}\|U\|^{2\gamma} \le C
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{E}\left\|\int_{t_n}^{t_{n+1}}\mathbf{i}Z(U)d\beta(t)\right\|^{2\gamma} &\le C\left(\int_{t_n}^{t_{n+1}}\left(\mathbb{E}\|Z(U)\|_{\mathcal{HS}}^{2\gamma}\right)^{\frac{1}{\gamma}}dt\right)^{\gamma}\\
&\le C\left(\int_{t_n}^{t_{n+1}}\left(\mathbb{E}\left(\sum_{m=1}^{M}\sum_{k=1}^{K}|u_m e_k(x_m)\sqrt{\eta_k}|^2\right)^{\gamma}\right)^{\frac{1}{\gamma}}dt\right)^{\gamma}\\
&\le C\left(\int_{t_n}^{t_{n+1}}\left(\mathbb{E}\left(2\eta\|U\|^2\right)^{\gamma}\right)^{\frac{1}{\gamma}}dt\right)^{\gamma}\le C\tau^{\gamma}
\end{aligned}$$

according to the Burkholder–Davis–Gundy inequality and the fact that the Hilbert–Schmidt operater norm $\|Z(U)\|_{\mathcal{HS}}=\|Z(U)\|_F$, with $\|\cdot\|_F$ denoting the Frobenius norm.

5.3. Proof of Theorem 3.6. Based on Taylor expansion and Lemmas 3.4 and 3.5, we obtain

$$\begin{aligned}
&\mathbb{E}\left[\varphi(U(\tau))-\varphi(U^1)\right]=\mathbb{E}\left[D\varphi(U^1)\big(U(\tau)-U^1\big)+O\big(\|U(\tau)-U^1\|^2\big)\right]\\
&\quad=\mathbb{E}\left[D\varphi(U^0)\big(U(\tau)-U^1\big)\right]+\mathbb{E}\left[D^2\varphi(U^0)(U^1-U^0,U(\tau)-U^1)\right]\\
&\qquad+O\Big(\mathbb{E}\left[\|U^1-U^0\|^2\|U(\tau)-U^1\|\right]+\mathbb{E}\|U(\tau)-U^1\|^2\Big)\\
&\quad=:\mathcal{A}+\mathcal{B}+\mathcal{C}.
\end{aligned}$$

We give the mild solution and discrete mild solution of (2) and (7) respectively:

$$\begin{aligned}
U(\tau)=e^{\mathbf{i}\frac{1}{h^2}A\tau}U^0&+\int_0^{\tau}e^{\mathbf{i}\frac{1}{h^2}A(\tau-s)}\Big(\mathbf{i}\lambda F(U(s))U(s)-\hat{E}U(s)\Big)ds\\
&+\int_0^{\tau}e^{\mathbf{i}\frac{1}{h^2}A(\tau-s)}\mathbf{i}Z(U(s))d\beta(s),
\end{aligned}$$

$$\begin{aligned}
U^1=&\left(I-\frac{\mathbf{i}\tau}{2h^2}A\right)^{-1}\left(I+\frac{\mathbf{i}\tau}{2h^2}A\right)U^0+\left(I-\frac{\mathbf{i}\tau}{2h^2}A\right)^{-1}\mathbf{i}\lambda\tau F\left(U^{\frac{1}{2}}\right)U^{\frac{1}{2}}\\
&+\left(I-\frac{\mathbf{i}\tau}{2h^2}A\right)^{-1}\mathbf{i}Z\left(U^{\frac{1}{2}}\right)\delta_1\beta.
\end{aligned}$$

Estimation of $\mathcal{A}$. Considering the difference between the above equations, we have

$$\begin{aligned}
U(\tau)-U^1=&\left(e^{\mathbf{i}\frac{1}{h^2}A\tau}-\left(I-\frac{\mathbf{i}\tau}{2h^2}A\right)^{-1}\left(I+\frac{\mathbf{i}\tau}{2h^2}A\right)\right)U^0\\
&+\mathbf{i}\int_0^{\tau}\left[e^{\mathbf{i}\frac{1}{h^2}A(\tau-s)}-\left(I-\frac{\mathbf{i}\tau}{2h^2}A\right)^{-1}\right]\lambda F(U(s))U(s)ds\\
&+\mathbf{i}\int_0^{\tau}\left(I-\frac{\mathbf{i}\tau}{2h^2}A\right)^{-1}\lambda\left[F(U(s))U(s)-F\left(U^{\frac{1}{2}}\right)U^{\frac{1}{2}}\right]ds\\
&+\mathbf{i}\int_0^{\tau}\left[e^{\mathbf{i}\frac{1}{h^2}A(\tau-s)}-\left(I-\frac{\mathbf{i}\tau}{2h^2}A\right)^{-1}\right]Z(U(s))d\beta(s)
\end{aligned}$$

$$
\begin{aligned}
&+\mathbf{i}\int_0^\tau \left(I-\frac{\mathbf{i}\tau}{2h^2}A\right)^{-1} Z(U(s)-U^0)d\beta(s)\\
&-\left[\frac{\mathbf{i}}{2}\left(I-\frac{\mathbf{i}\tau}{2h^2}A\right)^{-1} Z(U^1-U^0)\delta_1\beta+\int_0^\tau e^{\mathbf{i}\frac{1}{h^2}A(\tau-s)}\hat{E}U(s)ds\right],\\
=:{}& \mathbf{a}+\mathbf{b}+\mathbf{c}+\mathbf{d}+\mathbf{e}+\mathbf{f},
\end{aligned}
$$

which, together with the fact that $\mathbb{E}[D\varphi(U^0)\mathbf{d}]=\mathbb{E}[D\varphi(U^0)\mathbf{e}]=0$, yields that

$$
\begin{aligned}
\mathcal{A}&=\mathbb{E}\left[D\varphi(U^0)\mathbf{a}\right]+\mathbb{E}\left[D\varphi(U^0)\mathbf{b}\right]+\mathbb{E}\left[D\varphi(U^0)\mathbf{c}\right]+\mathbb{E}\left[D\varphi(U^0)\mathbf{f}\right]\\
&=: A_1+A_2+A_3+A_4.
\end{aligned}
$$

Based on the estimates $e^x-(1-\frac{x}{2})^{-1}(1+\frac{x}{2})=O(x^3)$ for $\|x\|<1$, and

$$
\left\|e^{\mathbf{i}\frac{1}{h^2}A(\tau-s)}-(I-\frac{\mathbf{i}\tau}{2h^2}A)^{-1}\right\|\le C\left(\frac{\tau}{h^2}\|A\|\right)\le C\tau h^{-2},\quad \forall\, s\in[0,\tau], \tag{25}
$$

we have

$$
|A_1|\le C\|\varphi\|_{1,\infty}\|\tau h^{-2}A\|^3\mathbb{E}\|U^0\|\le C\tau^3h^{-6}\le C\tau^2h^{-2} \tag{26}
$$

under the condition $\tau=O(h^4)$, and

$$
|A_2|\le C\|\varphi\|_{1,\infty}\int_0^\tau \|\tau h^{-2}A\|\|F(U(s))U(s)\|ds\le C\tau^2h^{-2}. \tag{27}
$$

Term A_3 can be estimated based on Lemmas 3.4 and 3.5:

$$
\begin{aligned}
|A_3|=&\left|\mathbb{E}\left[D\varphi(U^0)\int_0^\tau\left(I-\frac{\mathbf{i}\tau}{2h^2}A\right)^{-1}\left[\left(F(U(s))U(s)-F(U^0)U^0\right)\right.\right.\right.\\
&\left.\left.\left.-\left(F\left(U^{\frac12}\right)U^{\frac12}-F(U^0)U^0\right)\right]ds\right]\right|,
\end{aligned}
$$

in which we have known from the proof of Theorem 3.7 that

$$
\begin{aligned}
&F(U(s))U(s)-F(U^0)U^0=g(U(s))-g(U^0)\\
&=Dg(U^0)(U(s)-U^0)+\frac12 D^2g(\theta(s))(U(s)-U^0)^2\\
&=Dg(U^0)\left(\int_0^s\frac{\mathbf{i}}{h^2}AU(r)+\mathbf{i}\lambda F(U(r))U(r)-\hat{E}U(r)dr+\int_0^s Z(U(r))d\beta(r)\right)\\
&\quad+\frac12 D^2g(\theta(s))(U(s)-U^0)^2
\end{aligned}
$$

for some $\theta(s)\in[U^0,U(s)]$ and $s\in[0,\tau]$, and the same for the term $F\left(U^{\frac12}\right)U^{\frac12}-F(U^0)U^0$. Based on the fact that $\mathbb{E}\left[Dg(U^0)\int_0^s Z(U(r))d\beta(r)\right]=0$, we hence get

$$
|A_3|\le C(\tau^2h^{-2}+\tau^2) \tag{28}
$$

similar to the proof of Lemma 3.5. We rewrite

$$
Z(U^1-U^0)\delta_1\beta=\begin{pmatrix}u_1^1-u_1^0 & & \\ & \ddots & \\ & & u_M^1-u_M^0\end{pmatrix}E_{MK}\Lambda\delta_1\beta
$$

$$
\begin{aligned}
&= \begin{pmatrix} \sum_{k=1}^K e_k(x_1)\sqrt{\eta_k}\delta_1\beta_k & & \\ & \ddots & \\ & & \sum_{k=1}^K e_k(x_M)\sqrt{\eta_k}\delta_1\beta_k \end{pmatrix} (U^1 - U^0) \\
&=: G(U^1 - U^0),
\end{aligned}
$$

where G satisfies that $\mathbb{E}[GU^0] = 0$. Utilizing that $\mathbb{E}[GF(U^0)U^0] = 0$, we can rewrite term A_4 as

$$
\begin{aligned}
A_4 &= -\mathbb{E}\Bigg[D\varphi(U^0)\left(\frac{\mathbf{i}}{2}\left(I - \frac{\mathbf{i}\tau}{2h^2}A\right)^{-1} G(U^1 - U^0) + \int_0^\tau e^{\mathbf{i}\frac{1}{h^2}A(\tau - s)}\hat{E}U(s)ds\right)\Bigg] \\
&= -\frac{\mathbf{i}}{2}\mathbb{E}\Bigg[D\varphi(U^0)\left(I - \frac{\mathbf{i}\tau}{2h^2}A\right)^{-1} G\left(\mathbf{i}\frac{\tau}{h^2}AU^{\frac{1}{2}} + \mathbf{i}\lambda\tau F(U^{\frac{1}{2}})U^{\frac{1}{2}} + \mathbf{i}GU^{\frac{1}{2}}\right)\Bigg] \\
&\quad - \mathbb{E}\Bigg[D\varphi(U^0)\int_0^\tau e^{\mathbf{i}\frac{1}{h^2}A(\tau - s)}\hat{E}U(s)ds\Bigg] \\
&= \frac{\tau}{4h^2}\mathbb{E}\left[D\varphi(U^0)\left(I - \frac{\mathbf{i}\tau}{2h^2}A\right)^{-1} GA(U^1 - U^0)\right] \\
&\quad + \frac{1}{2}\lambda\tau\mathbb{E}\left[D\varphi(U^0)\left(I - \frac{\mathbf{i}\tau}{2h^2}A\right)^{-1} G\left(F(U^{\frac{1}{2}})U^{\frac{1}{2}} - F(U^0)U^0\right)\right] \\
&\quad + \frac{1}{4}\mathbb{E}\left[D\varphi(U^0)\left(I - \frac{\mathbf{i}\tau}{2h^2}A\right)^{-1} G^2(U^1 - U^0)\right] \\
&\quad + \mathbb{E}\left[D\varphi(U^0)\left(\left(I - \frac{\mathbf{i}\tau}{2h^2}A\right)^{-1}\frac{1}{2}G^2U^0 - \int_0^\tau e^{\mathbf{i}\frac{1}{h^2}A(\tau - s)}\hat{E}U(s)ds\right)\right] \\
&=: A_{4,1} + A_{4,2} + A_{4,3} + A_{4,4},
\end{aligned}
$$

in which, based on $\mathbb{E}[G^3U^0] = 0$, $A_{4,3}$ can be expressed as

$$
\frac{1}{4}\mathbb{E}\left[D\varphi(U^0)\left(I - \frac{\mathbf{i}\tau}{2h^2}A\right)^{-1} G^2\left(\mathbf{i}\frac{\tau}{h^2}AU^{\frac{1}{2}} + \mathbf{i}\tau\lambda F(U^{\frac{1}{2}})U^{\frac{1}{2}} + \frac{\mathbf{i}}{2}G(U^1 - U^0)\right)\right].
$$

For any $U \in \mathbb{C}^M$, we have

$$
\mathbb{E}\|GU\| = \mathbb{E}\|Z(U)\delta_1\beta\| \le C\mathbb{E}\left(\|U\|^2\Big(\sum_{k=1}^K \sqrt{\eta_k}|\delta_1\beta_k|\Big)^2\right)^{\frac{1}{2}} \le C\tau^{\frac{1}{2}}\left(\mathbb{E}\|U\|^2\right)^{\frac{1}{2}}.
$$

Hence $\mathbb{E}\|G^3(U^1 - U^0)\| \le C\tau^{\frac{1}{2}}(\mathbb{E}\|G^2(U^1 - U^0)\|^2)^{\frac{1}{2}}$ can be further estimated based on (24) with $\gamma = 4$ under the condition $Q \in \mathcal{HS}(L^2, H^{\frac{5}{4}})$, which together with Lemma 3.4 and $\|U^{\frac{1}{2}}\| \le 1$ yields

$$
|A_{4,1} + A_{4,2} + A_{4,3}| \le C(\tau^{\frac{5}{2}}h^{-4} + \tau^2h^{-2} + \tau^2) \le C(\tau^2h^{-2} + \tau^2). \tag{29}
$$

For the term $A_{4,4}$, we have

$$
\frac{1}{2}G^2U^0 \overset{\mathbb{E}}{=} \frac{1}{2}\begin{pmatrix} \sum_{k=1}^K e_k^2(x_1)\eta_k(\delta_1\beta_k)^2u_1^0 \\ \vdots \\ \sum_{k=1}^K e_k^2(x_M)\eta_k(\delta_1\beta_k)^2u_M^0 \end{pmatrix}, \quad \hat{E}U(s) = \frac{1}{2}\begin{pmatrix} \sum_{k=1}^K e_k^2(x_1)\eta_k u_1(s) \\ \vdots \\ \sum_{k=1}^K e_k^2(x_M)\eta_k u_M(s) \end{pmatrix}.
$$

Thus, we obtain

$$
\begin{aligned}
A_{4,4} =& \frac{1}{2}\mathbb{E}\left[D\varphi(U^0)\left(I-\frac{\mathbf{i}\tau}{2h^2}A\right)^{-1}\begin{pmatrix}\sum_{k=1}^K e_k^2(x_1)\eta_k(\delta_1\beta_k)^2u_1^0\\ \vdots \\ \sum_{k=1}^K e_k^2(x_M)\eta_k(\delta_1\beta_k)^2u_M^0\end{pmatrix}\right]\\
&-\frac{1}{2}\mathbb{E}\left[D\varphi(U^0)\int_0^\tau e^{\mathbf{i}\frac{1}{h^2}A(\tau-s)}\begin{pmatrix}\sum_{k=1}^K e_k^2(x_1)\eta_k u_1(s)\\ \vdots \\ \sum_{k=1}^K e_k^2(x_M)\eta_k u_M(s)\end{pmatrix}ds\right]\\
=& \frac{1}{2}\mathbb{E}\left[D\varphi(U^0)\left(I-\frac{\mathbf{i}\tau}{2h^2}A\right)^{-1}\begin{pmatrix}\sum_{k=1}^K e_k^2(x_1)\eta_k((\delta_1\beta_k)^2-\tau)u_1^0\\ \vdots \\ \sum_{k=1}^K e_k^2(x_M)\eta_k((\delta_1\beta_k)^2-\tau)u_M^0\end{pmatrix}\right]\\
&+\frac{1}{2}\mathbb{E}\left[D\varphi(U^0)\int_0^\tau\left(\left(I-\frac{\mathbf{i}\tau}{2h^2}A\right)^{-1}-e^{\mathbf{i}\frac{1}{h^2}A(\tau-s)}\right)\begin{pmatrix}\sum_{k=1}^K e_k^2(x_1)\eta_k u_1^0\\ \vdots \\ \sum_{k=1}^K e_k^2(x_M)\eta_k u_M^0\end{pmatrix}ds\right]\\
&-\frac{1}{2}\mathbb{E}\left[D\varphi(U^0)\int_0^\tau e^{\mathbf{i}\frac{1}{h^2}A(\tau-s)}\begin{pmatrix}\sum_{k=1}^K e_k^2(x_1)\eta_k\left(u_1(s)-u_1^0\right)\\ \vdots \\ \sum_{k=1}^K e_k^2(x_M)\eta_k\left(u_M(s)-u_M^0\right)\end{pmatrix}ds\right],
\end{aligned}\tag{30}
$$

where in the last step we have used the fact that

$$
\begin{pmatrix}\sum_{k=1}^K e_k^2(x_1)\eta_k\tau u_1^0\\ \vdots \\ \sum_{k=1}^K e_k^2(x_M)\eta_k\tau u_M^0\end{pmatrix}=\int_0^\tau\begin{pmatrix}\sum_{k=1}^K e_k^2(x_1)\eta_k u_1^0\\ \vdots \\ \sum_{k=1}^K e_k^2(x_M)\eta_k u_M^0\end{pmatrix}ds.
$$

Noticing that the first term in (30) vanishes as $\mathbb{E}(\delta_1\beta_k)^2=\tau$ and replacing $U(s)-U^0$ by the integral type of (2), then further calculation shows that

$$
|A_{4,4}|\le C(\tau^2h^{-2}+\tau^2)\tag{31}
$$

based on (25) and the technique used in (28). We then conclude from (26)–(31) that

$$
|\mathcal{A}|\le C\big(\tau^2h^{-2}+\tau^2\big)\le C_h\tau^2.\tag{32}
$$

Estimation of $\mathcal{C}$. Estimations of A_1 and A_2 show that

$$
\mathbb{E}\|\mathbf{a}+\mathbf{b}\|^2\le C\big(\tau^6h^{-12}+\tau^4h^{-4}\big)\le C\tau^3.\tag{33}
$$

Based on Hölder's inequality, Itô isometry, and Lemmas 3.4 and 3.5, we have

$$
\mathbb{E}\|\mathbf{c}+\mathbf{d}\|^2\le C\tau\int_0^\tau\mathbb{E}\|U(s)-U^{\frac{1}{2}}\|^2ds+\int_0^\tau C\tau^2h^{-4}ds\le C(\tau^3h^{-4}+\tau^3)\tag{34}
$$

and

$$
\mathbb{E}\|\mathbf{e}\|^2\le C\mathbb{E}\left[\int_0^\tau\left\|\left(I-\frac{\mathbf{i}\tau}{2h^2}A\right)^{-1}Z\left(U(s)-U^0\right)\right\|_{\mathcal{HS}}^2ds\right]\le C\tau^2.\tag{35}
$$

Rewriting $Z(U^1-U^0)\delta_1\beta = G\left(\mathbf{i}\frac{\tau}{h^2}AU^{\frac12} + \mathbf{i}\lambda\tau F(U^{\frac12})U^{\frac12} + \mathbf{i}GU^{\frac12}\right)$ together with the Hölder's inequality and (24) yields

$$\mathbb{E}\|\mathbf{f}\|^2 \le C(\tau^3 h^{-4} + \tau^2). \tag{36}$$

We then conclude from (33)–(36) and the condition $\tau = O(h^4)$ that

$$\mathbb{E}\|U(\tau) - U^1\|^2 \le C\tau^2, \tag{37}$$

which yields

$$|\mathcal{C}| = O\left(\left(\mathbb{E}\|U^1-U^0\|^4\right)^{\frac12}\left(\mathbb{E}\|U(\tau)-U^1\|^2\right)^{\frac12} + \mathbb{E}\|U(\tau)-U^1\|^2\right) \le C\tau^2. \tag{38}$$

Estimation of $\mathcal{B}$. As for $\mathcal{B} = \mathbb{E}\left[D^2\varphi(U^0)\left(U^1-U^0, \mathbf{a}+\mathbf{b}+\mathbf{c}+\mathbf{d}+\mathbf{e}+\mathbf{f}\right)\right]$, according to Hölder's inequality, (33), and (34), we have

$$\begin{aligned}
&\left|\mathbb{E}\left[D^2\varphi(U^0)\left(U^1-U^0, \mathbf{a}+\mathbf{b}+\mathbf{c}+\mathbf{d}\right)\right]\right| \\
&\le C\left(\mathbb{E}\|U^1-U^0\|^2\right)^{\frac12}\left(\mathbb{E}\|\mathbf{a}+\mathbf{b}+\mathbf{c}+\mathbf{d}\|^2\right)^{\frac12} \le C(\tau^2 h^{-2} + \tau^2).
\end{aligned}$$

Noticing that

$$\begin{aligned}
&\mathbb{E}\left[D^2\varphi(U^0)\left(U^1-U^0, \mathbf{e}+\mathbf{f}\right)\right] \\
&= \mathbb{E}\left[D^2\varphi(U^0)\left(U^1-U^0, \mathbf{i}\int_0^\tau \left(I - \frac{\mathbf{i}\tau}{2h^2}A\right)^{-1} Z(U(s)-U^1)d\beta(s)\right)\right] \\
&\quad + \frac12\mathbb{E}\left[D^2\varphi(U^0)\left(U^1-U^0, \mathbf{i}\left(I-\frac{\mathbf{i}\tau}{2h^2}A\right)^{-1} Z(U^1-U^0)\delta_1\beta\right)\right] \\
&\quad - \mathbb{E}\left[D^2\varphi(U^0)\left(U^1-U^0, \int_0^\tau e^{\mathbf{i}\frac{1}{h^2}A(\tau-s)}\hat{E}U(s)ds\right)\right] \\
&=: B_1 + B_2 + B_3,
\end{aligned}$$

where $|B_1| \le C\tau^2$ according to (37) and Lemma 3.4. Furthermore,

$$\begin{aligned}
B_2 &= \frac12\mathbb{E}\left[D^2\varphi(U^0)\left(\mathbf{i}\frac{\tau}{h^2}AU^{\frac12} + \mathbf{i}\tau\lambda F(U^{\frac12})U^{\frac12}, \mathbf{i}\left(I-\frac{\mathbf{i}\tau}{2h^2}A\right)^{-1} Z(U^1-U^0)\delta_1\beta\right)\right] \\
&\quad + \frac12\mathbb{E}\left[D^2\varphi(U^0)\left(\mathbf{i}Z\left(\frac{U^1-U^0}{2}\right)\delta_1\beta, \mathbf{i}\left(I-\frac{\mathbf{i}\tau}{2h^2}A\right)^{-1} Z(U^1-U^0)\delta_1\beta\right)\right] \\
&\quad + \frac12\mathbb{E}\left[D^2\varphi(U^0)\left(\mathbf{i}Z(U^0)\delta_1\beta, \mathbf{i}\left(I-\frac{\mathbf{i}\tau}{2h^2}A\right)^{-1} Z(U^1-U^0)\delta_1\beta\right)\right] \\
&=: B_{2,1} + B_{2,2} + B_{2,3}
\end{aligned}$$

with $|B_{2,1}+B_{2,2}| \le C(\tau^2 h^{-2} + \tau^2)$. Replacing U^1-U^0 again by (7), we obtain

$$|B_{2,3}| \le \left|\frac12\mathbb{E}\left[D^2\varphi(U^0)\left(\mathbf{i}Z(U^0)\delta_1\beta, \mathbf{i}\left(I-\frac{\mathbf{i}\tau}{2h^2}A\right)^{-1} Z\left(\mathbf{i}Z(U^{\frac12})\delta_1\beta\right)\delta_1\beta\right)\right]\right|$$

$$
\begin{aligned}
&+ C(\tau^2 h^{-2} + \tau^2) \\
&\le \left| \frac{1}{2}\mathbb{E}\left[D^2\varphi(U^0)\left(\mathbf{i}Z(U^0)\delta_1\beta, \mathbf{i}\left(I - \frac{\mathbf{i}\tau}{2h^2}A\right)^{-1} Z\left(\mathbf{i}Z(U^0)\delta_1\beta\right)\delta_1\beta \right)\right]\right| \\
&+ C(\tau^2 h^{-2} + \tau^2) \\
&= C(\tau^2 h^{-2} + \tau^2),
\end{aligned}
$$

where in the last step we used the fact $\mathbb{E}[(\delta_1\beta)^3] = 0$ and U^0 is $\mathcal{F}_0$-adapted. Also,

$$
\begin{aligned}
|B_3| \le &\left| \mathbb{E}\left[D^2\varphi(U^0)\left(\mathbf{i}\frac{\tau}{h^2}AU^{\frac{1}{2}} + \mathbf{i}\tau\lambda F(U^{\frac{1}{2}})U^{\frac{1}{2}}, \int_0^\tau e^{\mathbf{i}\frac{1}{h^2}A(\tau-s)}\hat{E}U(s)ds \right)\right]\right| \\
&+ \left| \mathbb{E}\left[D^2\varphi(U^0)\left(\mathbf{i}Z(U^{\frac{1}{2}})\delta_1\beta, \int_0^\tau e^{\mathbf{i}\frac{1}{h^2}A(\tau-s)}\hat{E}\left(U(s) - U^0\right)ds \right)\right]\right| \\
&+ \left| \mathbb{E}\left[D^2\varphi(U^0)\left(\mathbf{i}Z(U^{\frac{1}{2}})\delta_1\beta, \int_0^\tau e^{\mathbf{i}\frac{1}{h^2}A(\tau-s)}\hat{E}U^0 ds \right)\right]\right| \\
\le\ & C\left(\tau^2 h^{-2} + \tau^2\right) + \frac{1}{2}\left| \mathbb{E}\left[D^2\varphi(U^0)\left(\mathbf{i}Z(U^1 - U^0)\delta_1\beta, \int_0^\tau e^{\mathbf{i}\frac{1}{h^2}A(\tau-s)}\hat{E}U^0 ds \right)\right]\right| \\
\le\ & C(\tau^2 h^{-2} + \tau^2),
\end{aligned}
$$

so we finally obtain

$$
|\mathcal{B}| \le C(\tau^2 h^{-2} + \tau^2) \le C_h\tau^2,
$$

which, together with (32) and (38), completes the proof.

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HIGH ORDER CONFORMAL SYMPLECTIC AND ERGODIC SCHEMES FOR THE STOCHASTIC LANGEVIN EQUATION VIA GENERATING FUNCTIONS*

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Abstract. In this paper, we consider the stochastic Langevin equation with additive noises, which possesses both conformal symplectic geometric structure and ergodicity. We propose a methodology of constructing high weak order conformal symplectic schemes by converting the equation into an equivalent autonomous stochastic Hamiltonian system and modifying the associated generating function. To illustrate this approach, we construct a specific second order numerical scheme and prove that its symplectic form dissipates exponentially. Moreover, for the linear case, the proposed scheme is also shown to inherit the ergodicity of the original system, and the temporal average of the numerical solution is a proper approximation of the ergodic limit over long time. Numerical experiments are given to verify these theoretical results.

Key words. stochastic Langevin equation, conformal symplectic scheme, generating function, ergodicity, weak convergence

AMS subject classifications. 60H35, 65C30, 65P10

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1. Introduction.

A common way to describe dissipative systems which interact with their environment, especially in the fields of molecular simulations, quantum systems, cell migrations, chemical interactions, electrical engineering, and finance (see [8, 10, 20] and references therein), is by means of the stochastic Langevin equation. The stochastic Langevin equation considered in this paper is a dissipative Hamiltonian system, whose phase flow preserves conformal symplectic geometric structure [4] as an extension of the deterministic case. Namely, its symplectic form dissipates exponentially. One can also show that the considered stochastic Langevin equation is ergodic [13, 14, 21] with a unique invariant measure, i.e., the Boltzmann–Gibbs measure [4, 6]. This dynamical behavior implies that the temporal average of the solution will converge to its spatial average, which is also known as the ergodic limit, with respect to the invariant measure over long time.

This work proposes an approach for constructing high weak order conformal symplectic schemes that accurately approximates the exact solution, while preserving both the geometric structure and the dynamical behavior of the system. We illustrate this approach by a specific case and show that the proposed scheme for this particular case inherits the ergodicity of the original system with a unique invariant measure. The weak convergence error, as well as the approximate error of the ergodic limit, is proved to be of order two.

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There have been several works concentrating on the construction of numerical schemes for the stochastic Langevin equation, mainly based on the splitting technique. For instance, [4] constructs a class of the conformal symplectic integrators to preserve the conformal symplectic structure, and [18, 19] propose quasi-symplectic methods which can degenerate into symplectic ones when the system degenerates into a stochastic Hamiltonian system. The convergence rates of these schemes depend heavily on the splitting forms. As for the ergodicity, to the best of our knowledge its numerical analysis in general contains two aspects. The first is to construct numerical schemes that inherit the ergodicity (see, e.g., [13, 21]) and to give the error between the numerical invariant measure and the original one (see, e.g., [5, 7]). The other aspect is to approximate the ergodic limit with respect to the original invariant measure via the numerical temporal averages for some empirical test functions (see, e.g., [12, 14, 19]). In the latter case, the numerical solutions may not be ergodic.

In this paper, for the considered stochastic Langevin equation, we aim to construct numerical schemes which are of high weak order and are conformally symplectic. To achieve these aims without incurring the complexity of the high order splitting technique, we introduce a transformation from the stochastic Langevin equation to an autonomous stochastic Hamiltonian system. It then suffices to construct high order symplectic schemes for the autonomous Hamiltonian system, which turn out to be conformal symplectic schemes of the original system based on the inverse transformation of the phase spaces. The discretization of the modified equations, which are constructed by modifying the drift and diffusion functions as polynomials with respect to some time step, represents a powerful tool for obtaining high weak order schemes. For example, [1] constructs high order stochastic numerical integrators for general stochastic differential equations (SDEs), but these schemes may not be symplectic when applied to the Hamiltonian systems. Based on the internal properties of the Hamiltonian systems, [2] proposes a method for constructing high weak order stochastic symplectic schemes with multiple stochastic Itô integrals, using truncated generating functions. Based on these schemes, [24] gives their associated modified equations via generating functions. To reduce the simulation cost and still get high weak order symplectic schemes, inspired by [1, 2, 24], we modify the generating function for the equivalent stochastic Hamiltonian system and derive associated symplectic numerical methods by truncating modified generating functions. We would like to mention that this class of methods reduces the simulation of multiple stochastic Itô integrals by simulating products of increments of Wiener processes instead. We illustrate this approach with the construction of a stochastic numerical scheme that has weak order two. For the proposed numerical scheme, both the discretized phase volume and symplectic form dissipate exponentially, which coincides with the behavior of their exact counterparts in the original stochastic Langevin equation. Furthermore, the proposed scheme, similar to the original system, is proved to possess a numerical invariant measure that is unique for the linear case, which implies the ergodicity of the numerical solution. Finally, we verify that both the weak convergence error of the numerical scheme and the error of ergodic limit are of order two.

An outline of this paper is as follows. Section 2 gives a review of some basic properties of the stochastic Langevin equation, as well as the generating function of the stochastic Hamiltonian system, and also the transformation between the stochastic Langevin equation and an autonomous stochastic Hamiltonian system. In section 3, a weakly convergent conformal numerical scheme, which possesses an invariant measure, is proposed by means of modified generating functions and the transformation of phase space. In section 4, we show that both the weak convergence rate of the proposed scheme and the approximation error of the ergodic limit are of order two, based on the uniform estimate of the numerical solutions. Finally, we give some numerical tests to verify the theoretical results in section 5.

2. Stochastic Langevin equations. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $\mathcal{F}_t$ be the filtration for $t \geq 0$, and $W(t) = \big(W_1(t), \dots, W_m(t)\big)^\top$ be an m-dimensional standard Wiener process associated to $\{\mathcal{F}_t\}_{t\geq 0}$. Denote the 2-norm for both matrices and vectors by $\|\cdot\|$ and the determinant of matrices by $|\cdot|$, and let C be a generic constant, independent of h, that may differ from line to line.

2.1. Stochastic conformal symplectic structure and ergodicity. In this section, we focus on the stochastic Langevin equation driven by additive noises with deterministic initial values $P(0) = p \in \mathbb{R}^d$ and $Q(0) = q \in \mathbb{R}^d$, of the following form:

$$
\begin{aligned}
&dP = -f(Q)dt - vPdt - \sum_{r=1}^{m} \sigma_r dW_r(t), \\
&dQ = MPdt, \quad t \in [0, T],
\end{aligned} \tag{1}
$$

where $f \in C^\infty(\mathbb{R}^d, \mathbb{R}^d)$, $M \in \mathbb{R}^{d\times d}$ is a positive definite symmetric matrix, $v > 0$ is the absorption coefficient, and $\sigma_r \in \mathbb{R}^d$ with $r \in \{1, \dots, m\}$, $m \geq d$, and $\mathrm{rank}\{\sigma_1, \dots, \sigma_m\} = d$. In addition, assume that there exists a scalar function $F \in C^\infty(\mathbb{R}^d, \mathbb{R})$ satisfying

$$
f_i(Q) = \frac{\partial F(Q)}{\partial Q_i}, \quad i = 1, \dots, d.
$$

To simplify the notation, we will remove any mention of the dependence on $\omega \in \Omega$ unless it is absolutely necessary to avoid confusion. Note that (1), as well as all the other SDEs in what follows, holds almost surely with respect to $\mathbb{P}$. It is well known that if $v = 0$, (1) turns out to be a separable stochastic Hamiltonian system (SHS) which possesses stochastic symplectic structure and phase volume preservation [17]. However, when $v > 0$, the symplectic form of (1) dissipates exponentially, i.e.,

$$
dP(t) \wedge dQ(t) = e^{-vt} dp \wedge dq \quad \forall\, t \geq 0,
$$

which characterizes the long-time tracking of the solutions to (1), as well as the phase volume $\mathrm{Vol}(t)$. Namely, denoting by $D_t = D_t(\omega) \subset \mathbb{R}^{2d}$ a random domain which has finite volume and is independent of Wiener processes $W(t)$ with respect to the system (1), one can obtain

$$
\begin{aligned}
\mathrm{Vol}(t) &= \int_{D_t} dP^1 \cdots dP^d dQ^1 \cdots dQ^d \\
&= \int_{D_0} \left| \frac{D(P^1, \dots, P^d, Q^1, \dots, Q^d)}{D(p^1, \dots, p^d, q^1, \dots, q^d)} \right| dp^1 \cdots dp^d dq^1 \cdots dq^d,
\end{aligned}
$$

where the determinant of Jacobian matrix $\left|\frac{D(P^1,\dots,P^d,Q^1,\dots,Q^d)}{D(p^1,\dots,p^d,q^1,\dots,q^d)}\right| = e^{-vtd}$ with d being the dimension [16, 17].

As another well-known long-time behavior, the ergodicity of (1) is shown in [13] by proving that (1) possesses a unique invariant measure μ. Noticing that (1) satisfies the hypoelliptic setting

$$
\mathrm{span}\{U_i, [U_0, U_j],\, i = 0, \dots, m,\, j = 1, \dots, m\} = \mathbb{R}^{2d} \tag{2}
$$

with vector fields $U_0 = ((-f(Q) - vP)^\top, (MP)^\top)^\top$ and $U_j = (\sigma_j^\top, 0)^\top$, $j = 1, \dots, m$, which together with the following assumption yields the ergodicity of (1).

ASSUMPTION 2.1 (see [13]). *Let $F \in C^\infty(\mathbb{R}^d, \mathbb{R})$ satisfy that*

(i) *$F(u) \geq 0$ for all $u \in \mathbb{R}^d$;*

(ii) *there exist $\alpha > 0$ and $\beta \in (0,1)$ such that for all $u \in \mathbb{R}^d$, it holds*

$$\frac{1}{2} u^\top f(u) \geq \beta F(u) + v^2 \frac{\beta(2-\beta)}{8(1-\beta)} \|u\|^2 - \alpha.$$

Intuitively speaking, the ergodicity of (1) reads that the temporal averages of $P(t)$ and $Q(t)$ starting from different initial values will converge almost everywhere to its spatial average with respect to the invariant measure μ. More precisely,

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T \mathbf{E}^{(p,q)} \left[\psi(P(t), Q(t)) \right] dt = \int_{\mathbb{R}^{2d}} \psi d\mu \quad \forall\, \psi \in C_b(\mathbb{R}^{2d}, \mathbb{R}) \tag{3}$$

in $L^2(\mathbb{R}^{2d}, \mu)$, where $\mathbf{E}^{(p,q)}[\cdot]$ denotes the expectation starting from $P(0) = p$ and $Q(0) = q$. In the following, we use the notation $\mathbf{E}$ instead of $\mathbf{E}^{(p,q)}$ to simplify the notation.

Next, we aim to convert (1) into an equivalent homogenous SHS via a transformation of phase space, such that one can construct conformal symplectic schemes for (1) based on symplectic schemes of the homogenous SHS. To this end, denoting $X_i(t) = e^{vt} P_i(t)$ and $Y_i(t) = Q_i(t)$ and applying Itô's formula to $X_i(t)$ and $Y_i(t)$ for $i = 1, \dots, d$, one can rewrite (1) as

$$dX_i = -e^{vt} f_i(Y_1, \dots, Y_d) dt - e^{vt} \sum_{r=1}^m \sigma_r dW_r(t), \quad dY_i = e^{-vt} \sum_{j=1}^d M_{ij} X_j dt \tag{4}$$

with $X_i(0) = p_i$ and $Y_i(0) = q_i$. It is obvious that (4) is a nonautonomous SHS with time-dependent Hamiltonian functions

$$\tilde{H}_0 = e^{vt} F(Y_1, \dots, Y_d) + \frac{1}{2} e^{-vt} \sum_{i,j=1}^d X_i M_{ij} X_j, \quad \tilde{H}_r = e^{vt} \sum_{i=1}^d \sigma_r^i Y_i.$$

To obtain an autonomous SHS we introduce two new variables $X_{d+1} \in \mathbb{R}$ and $Y_{d+1} \in \mathbb{R}$ as the $(d+1)$th components of X and Y, respectively, satisfying

$$dY_{d+1} = dt, \quad dX_{d+1} = -\frac{\partial \tilde{H}_0}{\partial t} dt - \sum_{r=1}^m \frac{\partial \tilde{H}_r}{\partial t} \circ dW_r(t)$$

with $Y_{d+1}(0) = 0$ and $X_{d+1}(0) = F(q_1, \dots, q_d) + \frac{1}{2} \sum_{i,j=1}^d p_i M_{ij} p_j + \sum_{r=1}^m \sum_{i=1}^d \sigma_r^i q_i$. Here the notation "$\circ$" means that the equation holds in the Stratonovich integral sense. Then (4) becomes the $(2d+2)$-dimensional autonomous SHS

$$dX = -\frac{\partial H_0}{\partial Y} dt - \sum_{r=1}^m \frac{\partial H_r}{\partial Y} \circ dW_r(t), \quad dY = \frac{\partial H_0}{\partial X} dt + \sum_{r=1}^m \frac{\partial H_r}{\partial X} \circ dW_r(t), \tag{5}$$

with $X(0) = (X_1(0), \dots, X_{d+1}(0)) \in \mathbb{R}^{d+1}$, $Y(0) = (Y_1(0), \dots, Y_{d+1}(0)) \in \mathbb{R}^{d+1}$, and new Hamiltonian functions

$$H_0(X,Y) = e^{vY_{d+1}} F(Y_1,\ldots,Y_d) + \frac{1}{2} e^{-vY_{d+1}} \sum_{i,j=1}^{d} X_i M_{ij} X_j + X_{d+1},$$
$$H_r(X,Y) = e^{vY_{d+1}} \sum_{i=1}^{d} \sigma_r^i Y_i.$$

Here, (5) is called the associated autonomous SHS of (1), and its phase flow preserves the stochastic symplectic structure. Notice that the motion of the system can be described by different kinds of generating functions (see [2, 23] and references therein). We consider only the first kind of generating function S in this article.

2.2. Generating functions. For convenience, we denote $X(0) = x$ and $Y(0) = y$. It is revealed in [22] that the generating function $S(X,y,t)$ related to (5) is the solution of the following stochastic Hamilton–Jacobi partial differential equation:

$$d_t S(X,y,t) = H_0\left(X, y + \frac{\partial S}{\partial X}\right) dt + \sum_{r=1}^{m} H_r\left(X, y + \frac{\partial S}{\partial X}\right) \circ dW_r(t). \tag{6}$$

Moreover, the mapping $(x,y) \mapsto (X(t), Y(t))$ defined by

$$X(t) = x - \frac{\partial S(X(t),y,t)}{\partial y}, \quad Y(t) = y + \frac{\partial S(X(t),y,t)}{\partial X} \tag{7}$$

is the stochastic flow of (5). Based on the Itô representation theorem and stochastic Taylor–Stratonovich expansion, $S(X,y,t)$ has a series expansion (see, e.g., [2, 3])

$$S(X,y,t) = \sum_{\boldsymbol{\alpha}} G_{\boldsymbol{\alpha}}(X,y) J_{\boldsymbol{\alpha}}^t, \tag{8}$$

where

$$J_{\boldsymbol{\alpha}}^t = \int_0^t \int_0^{s_l} \cdots \int_0^{s_2} \circ dW_{j_1}(s_1) \circ dW_{j_2}(s_2) \circ \cdots \circ dW_{j_l}(s_l)$$

with multi-index $\boldsymbol{\alpha} = (j_1, j_2, \ldots, j_l) \in \{0,1,\ldots,m\}^{\otimes l}$, $l \geq 1$, and $dW_0(s) := ds$. Before calculating coefficients $G_{\boldsymbol{\alpha}}(X,y)$ in (8), we first specify some notation. Let $l(\boldsymbol{\alpha})$ denote the length of $\boldsymbol{\alpha}$, and let $\boldsymbol{\alpha}-$ be the multi-index resulting from discarding the last index of $\boldsymbol{\alpha}$. Define $\boldsymbol{\alpha} * \boldsymbol{\alpha}' = (j_1,\ldots,j_l,j_1',\ldots,j_{l'}')$, where $\boldsymbol{\alpha} = (j_1,\ldots,j_l)$ and $\boldsymbol{\alpha}' = (j_1',\ldots,j_{l'}')$. The concatenation "$*$" between a set of multi-indices Λ and $\boldsymbol{\alpha}$ is $\Lambda * \boldsymbol{\alpha} = \{\boldsymbol{\beta} * \boldsymbol{\alpha} | \boldsymbol{\beta} \in \Lambda\}$. Furthermore, define

$$\Lambda_{\boldsymbol{\alpha},\boldsymbol{\alpha}'} = \begin{cases} \{(j_1,j_1'),(j_1',j_1)\} & \text{if} \quad l = l' = 1, \\ \{\Lambda_{(j_1),\boldsymbol{\alpha}'-} * (j_{l'}'), \boldsymbol{\alpha}' * (j_1)\} & \text{if} \quad l = 1, l' \neq 1, \\ \{\Lambda_{\boldsymbol{\alpha}-,(j_1')} * (j_l), \boldsymbol{\alpha} * (j_1')\} & \text{if} \quad l \neq 1, l' = 1, \\ \{\Lambda_{\boldsymbol{\alpha}-,\boldsymbol{\alpha}'} * (j_l), \Lambda_{\boldsymbol{\alpha},\boldsymbol{\alpha}'-} * (j_{l'}')\} & \text{if} \quad l \neq 1, l' \neq 1. \end{cases}$$

For $k > 2$, let $\Lambda_{\boldsymbol{\alpha}_1,\ldots,\boldsymbol{\alpha}_k} = \{\Lambda_{\boldsymbol{\beta},\boldsymbol{\alpha}_k} | \boldsymbol{\beta} \in \Lambda_{\boldsymbol{\alpha}_1,\ldots,\boldsymbol{\alpha}_{k-1}}\}$. We refer the reader to [2] for more details about this notation. Substituting (8) into (6) and applying Taylor expansions to H_r $(r = 0,1,\ldots,m)$ at (X,y), we obtain $G_{\boldsymbol{\alpha}} = H_r$ with $\boldsymbol{\alpha} = (r)$ being a single index and

$$G_{\boldsymbol{\alpha}}=\sum_{i=1}^{l(\boldsymbol{\alpha})-1}\frac{1}{i!}\sum_{k_1,\ldots,k_i=1}^{d+1}\frac{\partial^i H_{j_l}(X,y)}{\partial y_{k_1}\cdots\partial y_{k_i}}\sum_{\substack{l(\boldsymbol{\alpha}_1)+\cdots+l(\boldsymbol{\alpha}_i)=l(\boldsymbol{\alpha})-1\\ \boldsymbol{\alpha}-\in\Lambda_{\boldsymbol{\alpha}_1,\ldots,\boldsymbol{\alpha}_i}}}\frac{\partial G_{\boldsymbol{\alpha}_1}}{\partial X_{k_1}}\cdots\frac{\partial G_{\boldsymbol{\alpha}_i}}{\partial X_{k_i}}$$

for any $\boldsymbol{\alpha}=(j_1,j_2,\ldots,j_l)$ with $l\geq 2$ (see, e.g., [2, 3]). To make it clear, the simplified expressions of $G_{\boldsymbol{\alpha}}$ are given when $l=2$ or 3: $G_{(j_1,j_2)}=\sum_{i=1}^{d+1}\frac{\partial H_{j_2}}{\partial y_i}\frac{\partial H_{j_1}}{\partial X_i}$ and

$$G_{(j_1,j_2,j_3)}=\sum_{i=1}^{d+1}\frac{\partial H_{j_3}}{\partial y_i}\frac{\partial G_{(j_1,j_2)}}{\partial X_i}+\frac{1}{2}\sum_{i,j=1}^{d+1}\frac{\partial^2 H_{j_3}}{\partial y_i\partial y_j}\left(\frac{\partial H_{j_1}}{\partial X_i}\frac{\partial H_{j_2}}{\partial X_j}+\frac{\partial H_{j_2}}{\partial X_i}\frac{\partial H_{j_1}}{\partial X_j}\right).$$

Let $C_1:=e^{vy_{d+1}}$ and $C_2:=e^{-vy_{d+1}}$. Here y_{d+1} denotes the $(d+1)$th component of y. Note that y is the initial point of the considered interval; that is, if we consider the problem on the interval $[s,t]$, then $y=Y(s)$. For $r_1,r_2,r_3\in\{1,\ldots,m\}$, we have

$$\begin{aligned}
&G_{(r_1,r_2)}=G_{(r_1,0)}=G_{(r_1,r_2,r_3)}=G_{(r_1,r_2,0)}=G_{(r_1,0,r_2)}=0,\\
&G_{(0,r_1)}=\sum_{i,j=1}^{d}\sigma_{r_1}^i M_{ij}X_j+vC_1\sum_{i=1}^{d}\sigma_{r_1}^i q_i,\quad G_{(0,r_1,r_2)}=C_1\sigma_{r_1}^\top M\sigma_{r_2},\\
&G_{(0,0)}=\sum_{i,j=1}^{d}f_i(y)M_{ij}X_j+vC_1F(y)-\frac{1}{2}vC_2\sum_{i,j=1}^{d}X_iM_{ij}X_j.
\end{aligned}$$

For a fixed small time step h, using (8) and applying Taylor expansion to $\frac{\partial S}{\partial y_i}:=\frac{\partial S}{\partial y_i}(X,y,h)$ and $\frac{\partial S}{\partial X_i}:=\frac{\partial S}{\partial X_i}(X,y,h)$ at point (x,y,h) for $i=1,\ldots,d$, we obtain

$$\begin{aligned}
\frac{\partial S}{\partial y_i}&=C_1\left[\sum_{r=1}^{m}\sigma_r^i(J_{(r)}^h+vJ_{(0,r)}^h)+f_i(y)\left(h+\frac{vh^2}{2}\right)\right]+\frac{h^2}{2}\sum_{j,k=1}^{d}\frac{\partial^2F(y)}{\partial y_i\partial y_j}M_{jk}x_k+R_1,\\
\frac{\partial S}{\partial X_i}&=C_2\sum_{j=1}^{d}M_{ij}x_j\left(h-\frac{vh^2}{2}\right)-\sum_{j=1}^{d}\sum_{r=1}^{m}M_{ij}\sigma_r^jJ_{(r,0)}^h-\frac{h^2}{2}\sum_{j=1}^{d}M_{ij}f_j(y)+R_2,
\end{aligned}$$

where every term in R_1 and R_2 contains the product of multiply stochastic integrals whose lowest order is at least $\frac{5}{2}$, as do the remainder terms R_l with $l=3,\ldots,7$ in what follows. Furthermore, $\frac{\partial S}{\partial X_{d+1}}(X,y,h)=h$ and

$$\begin{aligned}
\frac{\partial S}{\partial y_{d+1}}=\ &vh\left(C_1F(y)-\frac{C_2}{2}\sum_{i,j=1}^{d}x_iM_{ij}x_j\right)\left(1+\frac{vh}{2}\right)+vC_1\sum_{r=1}^{m}\sum_{i=1}^{d}\sigma_r^iy_i(J_{(r)}^h+vJ_{(0,r)}^h)\\
&+\sum_{i,j=1}^{d}\sum_{r=1}^{m}v\sigma_r^iM_{ij}x_jhJ_{(r)}^h+vC_1\sum_{r_1,r_2=1}^{m}\sigma_{r_1}^\top M\sigma_{r_2}J_{(0,r_1,r_2)}^h\\
&+v\sum_{i,j=1}^{d}\left(C_2\frac{\partial F(y)}{\partial y_i}M_{ij}x_jh^2-\frac{1}{2}C_1\sum_{r_1,r_2=1}^{m}\sigma_{r_1}^iM_{ij}\sigma_{r_2}^jhJ_{(r_1)}^hJ_{(r_2)}^h\right)+R_3,
\end{aligned}$$

where $\frac{\partial S}{\partial y_{d+1}}$ takes the value at (X,y,h).

By truncating the generating function, the weakly convergent stochastic symplectic numerical schemes have been proposed by several authors (see, e.g., [2, 17, 22]). In these approaches, some techniques are applied to simulate the multiple integrals in the truncated generating functions and obtain high weak order schemes. To reduce the simulation of multiple integrals, we introduce a modified generating function to construct more concise symplectic schemes in section 3, from which conformal symplectic and ergodic schemes for stochastic dynamical systems (1) are deduced by using the transformation of the phase space.

3. High order conformal symplectic and ergodic schemes. To construct high order symplectic numerical integrators for (5), we modify the stochastic Hamiltonian functions first. Namely, we consider the following $(2d+2)$-dimensional stochastic Hamiltonian system:

$$\begin{aligned} dX^M &= -\frac{\partial H_0^M(X^M,Y^M)}{\partial Y^M}dt - \sum_{r=1}^{m}\frac{\partial H_r^M(X^M,Y^M)}{\partial Y^M}\circ dW_r(t), \quad X^M(0)=x,\\ dY^M &= \frac{\partial H_0^M(X^M,Y^M)}{\partial X^M}dt + \sum_{r=1}^{m}\frac{\partial H_r^M(X^M,Y^M)}{\partial X^M}\circ dW_r(t), \quad Y^M(0)=y, \end{aligned} \tag{9}$$

where

$$\begin{aligned} H_0^M(X^M,Y^M) &= H_0(X^M,Y^M) + H_0^{[1]}(X^M,Y^M)h + \cdots + H_0^{[\tau]}(X^M,Y^M)h^\tau,\\ H_r^M(X^M,Y^M) &= H_r(X^M,Y^M) + H_r^{[1]}(X^M,Y^M)h + \cdots + H_r^{[\tau]}(X^M,Y^M)h^\tau \end{aligned} \tag{10}$$

with functions $H_i^{[j]}$, $i=0,\dots r$, $j=1,\dots,\tau$, $\tau\in\mathbb{N}_+$ to be determined. Meanwhile, according to the definition of $G_{\boldsymbol{\alpha}}$ in subsection 2.2, we get the associated generating function of (9), which is called the modified generating function of (5). Our goal is to choose undetermined functions in (10) such that the proposed scheme is of weak order $k+k'$ when approximating (5), even though it is only a kth order approximation of (9) for some positive integers k and k'. Now we first give a symplectic numerical approximation to (9) via its generating function, such that this scheme shows weak order k for (9) without specific choices of $H_i^{[j]}$ (see [2] and references therein). In detail, we replace the multiple Stratonovich integrals J_{α}^t in the modified generating function by an equivalent linear combination of multiple Itô integrals

$$I_{\boldsymbol{\beta}}^t := \int_0^t\int_0^{s_l}\cdots\int_0^{s_2} dW_{i_1}(s_1)dW_{i_2}(s_2)\cdots dW_{i_l}(s_l)$$

with multi-index $\boldsymbol{\beta}=(i_1,i_2,\dots,i_l)\in\{0,1,\dots,m\}^{\otimes l}$, $l\geq 1$, based on the relation

$$J_{\boldsymbol{\alpha}}^t = \begin{cases} \sum_{\boldsymbol{\beta}} C_{\boldsymbol{\alpha}}^{\boldsymbol{\beta}} I_{\boldsymbol{\beta}}^t, & l(\boldsymbol{\alpha})\geq 2,\\ I_{\boldsymbol{\alpha}}^t, & l(\boldsymbol{\alpha})=1, \end{cases}$$

where $C_{\boldsymbol{\alpha}}^{\boldsymbol{\beta}}$ are certain constants given in [11]. Denote by

$$S^G(X^G,y,t) = \sum_{\boldsymbol{\alpha}} G_{\boldsymbol{\alpha}}^G(X^G,y)\sum_{l(\boldsymbol{\beta})\leq k} C_{\boldsymbol{\alpha}}^{\boldsymbol{\beta}} I_{\boldsymbol{\beta}}^t \tag{11}$$

the truncated modified generating function (see, e.g., [2, 3, 11]), where

$$G^G_{\boldsymbol{\alpha}} = \sum_{i=1}^{l(\boldsymbol{\alpha})-1} \frac{1}{i!} \sum_{k_1,\ldots,k_i=1}^{d+1} \frac{\partial^i H^M_{j_l}(X^G,y)}{\partial y_{k_1}\cdots\partial y_{k_i}} \sum_{\substack{l(\boldsymbol{\alpha}_1)+\cdots+l(\boldsymbol{\alpha}_i)=l(\boldsymbol{\alpha})-1\\ \boldsymbol{\alpha}-\in\Lambda_{\boldsymbol{\alpha}_1,\ldots,\boldsymbol{\alpha}_i}}} \frac{\partial G^G_{\boldsymbol{\alpha}_1}}{\partial X^G_{k_1}}\cdots\frac{\partial G^G_{\boldsymbol{\alpha}_i}}{\partial X^G_{k_i}}$$

for $l(\boldsymbol{\alpha}) \geq 2$, and $G^G_{(r)} = H^M_r$ for $r = 0,1,\ldots,m$. Then we get the following one-step approximation:

$$X^G = x - \frac{\partial S^G(X^G,y,h)}{\partial y}, \quad Y^G = y + \frac{\partial S^G(X^G,y,h)}{\partial X^G}, \tag{12}$$

which preserves symplectic structure and is of weak order k for (9). Notice that the truncated modified generating function contains undetermined functions $H_i^{[j]}$, $i = 0,\ldots r$, $j = 1,\ldots,\tau$ in (10). To specify high weak order symplectic schemes, we need to determine all the terms $H_i^{[j]}$ such that the numerical scheme based on (12) satisfies

$$\left|\mathbf{E}\phi(X(h),Y(h)) - \mathbf{E}\phi(X^G,Y^G)\right| = O(h^{k+k'+1}) \tag{13}$$

for all κ times continuously differentiable functions $\phi \in C_P^\kappa(\mathbb{R}^{2d+2},\mathbb{R})$ with polynomial growth; that is, the numerical scheme based on (12) is of weak order $k+k'$ for (5). Conditions on κ will be specified later. The detailed approach of choosing the undetermined functions will be illustrated with the case $k = k' = 1$ in the next section.

3.1. Numerical schemes via modified generating function. For $k = k' = 1$, it is sufficient to consider $\tau = 1$ in (10). Based on the fact that $G^G_{(r)} = H^M_r$ for $r = 0,1,\ldots,m$, we rewrite the truncated generating function (11) as

$$S^G(X^G,y,h) = \left(H^M_0(X^G,y) + \frac{1}{2}\sum_{r=1}^m G^G_{(r,r)}(X^G,y)\right)h + \sum_{r=1}^m H^M_r(X^G,y)I^h_{(r)}, \tag{14}$$

where

$$G^G_{(r,r)} = C_1\sum_{i=1}^d \sigma^i_r\left(\frac{\partial H_r^{[1]}}{\partial X^G_i} + vy_i\frac{\partial H_r^{[1]}}{\partial X^G_{d+1}}\right)h + \sum_{i=1}^{d+1}\frac{\partial H_r^{[1]}}{\partial y_i}\frac{\partial H_r^{[1]}}{\partial X^G_i}h^2.$$

According to (14), the one-step approximation (12) turns out to be

$$\begin{aligned} X^G &= x - \left(\frac{\partial H^M_0(X^G,y)}{\partial y} + \frac{1}{2}\sum_{r=1}^m\frac{\partial G^G_{(r,r)}(X^G,y)}{\partial y}\right)h - \sum_{r=1}^m\frac{\partial H^M_r(X^G,y)}{\partial y}J^h_{(r)},\\ Y^G &= y + \left(\frac{\partial H^M_0(X^G,y)}{\partial X^G} + \frac{1}{2}\sum_{r=1}^m\frac{\partial G^G_{(r,r)}(X^G,y)}{\partial X^G}\right)h + \sum_{r=1}^m\frac{\partial H^M_r(X^G,y)}{\partial X^G}J^h_{(r)}. \end{aligned} \tag{15}$$

In what follows, let $\frac{\partial S^G}{\partial y_j} := \frac{\partial S^G}{\partial y_j}(X^G,y,h)$, $\frac{\partial S^G}{\partial X^G_j} := \frac{\partial S^G}{\partial X^G_j}(X^G,y,h)$, $\frac{\partial H_r^{[1]}}{\partial y_j} := \frac{\partial H_r^{[1]}}{\partial y_j}(x,y)$, and $\frac{\partial H_r^{[1]}}{\partial x_j} := \frac{\partial H_r^{[1]}}{\partial x_j}(x,y)$ for $j = 1,\ldots,d+1$ and $r = 0,1,\ldots,m$. Applying Taylor expansion to $\frac{\partial S^G}{\partial y_i}$ and $\frac{\partial S^G}{\partial X_i^G}$ at (x,y,h), for $i=1,\dots,d$, we obtain

$$\begin{aligned}\frac{\partial S^G}{\partial X_i^G} &= C_2\sum_{j=1}^{d} M_{ij}x_jh + \sum_{r=1}^{m}\left(\frac{\partial H_r^{[1]}}{\partial x_i} - \sum_{j=1}^{d} M_{ij}\sigma_r^j\right)I_{(r)}^h h - \sum_{j=1}^{d} M_{ij}f_j(y)h^2 + \frac{\partial H_0^{[1]}}{\partial x_i}h^2\\ &\quad + \sum_{r=1}^{m}\frac{\partial^2 H_r^{[1]}}{\partial x_i\partial x_{d+1}}(X_{d+1}^G - x_{d+1})I_{(r)}^h h - C_1\sum_{r_1,r_2=1}^{m}\sum_{j=1}^{d}\frac{\partial^2 H_{r_1}^{[1]}}{\partial x_i\partial x_j}\sigma_{r_2}^j I_{(r_1)}^h I_{(r_2)}^h h\\ &\quad + \frac{1}{2}C_1\sum_{j=1}^{d}\sum_{r=1}^{m}\sigma_r^j\left(\frac{\partial^2 H_r^{[1]}}{\partial x_i\partial x_j} + vy_i\frac{\partial^2 H_r^{[1]}}{\partial x_i\partial x_{d+1}}\right)h^2 + R_4\end{aligned}$$

and

$$\begin{aligned}\frac{\partial S^G}{\partial y_i} &= C_1\sum_{r=1}^{m}\left(\sigma_r^i I_{(r)}^h + f_i(y)h\right) + \sum_{r=1}^{m}\frac{\partial H_r^{[1]}}{\partial y_i}I_{(r)}^h h + \sum_{r=1}^{m}\sum_{j=1}^{d+1}\frac{\partial^2 H_r^{[1]}}{\partial y_i\partial x_j}(X_j^G - x_j)I_{(r)}^h h\\ &\quad + \left(\frac{\partial H_0^{[1]}}{\partial y_i} + \frac{C_1}{2}\sum_{r=1}^{m}\left[\sum_{j=1}^{d}\sigma_r^j\frac{\partial^2 H_r^{[1]}}{\partial y_i\partial x_j} + v\sigma_r^i\left(\frac{\partial H_r^{[1]}}{\partial x_{d+1}} + y_i\frac{\partial^2 H_r^{[1]}}{\partial x_{d+1}\partial y_i}\right)\right]\right)h^2 + R_5.\end{aligned}$$

Similarly,

$$\begin{aligned}\frac{\partial S^G}{\partial X_{d+1}^G} &= h + \sum_{r=1}^{m}\frac{\partial H_r^{[1]}}{\partial x_{d+1}}I_{(r)}^h h + \sum_{j=1}^{d+1}\frac{\partial^2 H_r^{[1]}}{\partial x_{d+1}\partial x_j}\left(X_j^G - x_j\right)I_{(r)}^h h + \frac{\partial H_0^{[1]}}{\partial x_{d+1}}h^2\\ &\quad + C_1\sum_{i=1}^{d}\sigma_r^i\frac{\partial^2 H_r^{[1]}}{\partial x_i\partial x_{d+1}}h^2 + C_1\sum_{i=1}^{d}v\sigma_r^i y_i\frac{\partial^2 H_r^{[1]}}{\partial x_{d+1}^2}h^2 + R_6,\end{aligned}$$

and

$$\begin{aligned}\frac{\partial S^G}{\partial y_{d+1}} &= v\left(C_1F(y) - \frac{1}{2}C_2\sum_{i,j=1}^{d}x_iM_{ij}x_j\right)h + vC_1\sum_{r=1}^{m}\sum_{i=1}^{d}\sigma_r^i y_i I_{(r)}^h + \sum_{r=1}^{m}\frac{\partial H_r^{[1]}}{\partial y_{d+1}}hI_{(r)}^h\\ &\quad + \sum_{i,j=1}^{d}\sum_{r=1}^{m}v\sigma_r^i M_{ij}x_jhI_{(r)}^h + \sum_{r=1}^{m}\sum_{i=1}^{d+1}\frac{\partial^2 H_r^{[1]}}{\partial y_{d+1}\partial x_i}(X_i^G - x_i)hI_{(r)}^h + \frac{\partial H_0^{[1]}}{\partial y_{d+1}}h^2\\ &\quad + \frac{C_1}{2}\sum_{i=1}^{d}\sum_{r=1}^{m}\sigma_r^i\left(v\frac{\partial H_r^{[1]}}{\partial x_i} + v^2y_i\frac{\partial H_r^{[1]}}{\partial x_{d+1}} + \frac{\partial^2 H_r^{[1]}}{\partial x_i\partial y_{d+1}} + vy_i\frac{\partial^2 H_r^{[1]}}{\partial x_{d+1}\partial y_{d+1}}\right)h^2\\ &\quad + v\sum_{i,j=1}^{d}\left(C_2\frac{\partial F(y)}{\partial y_i}M_{ij}x_jh^2 - \frac{C_1}{2}\sum_{r_1,r_2=1}^{m}\sigma_{r_1}^i M_{ij}\sigma_{r_2}^j hI_{(r_1)}^h I_{(r_2)}^h\right) + R_7.\end{aligned}$$

Applying Taylor expansion to $\phi(X(h),Y(h))$ and $\phi(X^G,Y^G)$ at (x,y) and taking expectations, we have

$$
\begin{aligned}
&\mathbf{E}\phi(X(h),Y(h)) - \mathbf{E}\phi(X^G,Y^G)\\
=&\sum_{i=1}^{d+1}\frac{\partial\phi(x,y)}{\partial x_i}\mathbf{E}\left(\frac{\partial S^G}{\partial y_i}-\frac{\partial S}{\partial y_i}\right)+\sum_{i=1}^{d+1}\frac{\partial\phi(x,y)}{\partial y_i}\mathbf{E}\left(\frac{\partial S}{\partial X_i}-\frac{\partial S^G}{\partial X_i^G}\right)\\
&+\frac{1}{2}\sum_{i,j=1}^{d+1}\frac{\partial^2\phi(x,y)}{\partial x_i\partial x_j}\mathbf{E}\left(\frac{\partial S}{\partial y_i}\frac{\partial S}{\partial y_j}-\frac{\partial S^G}{\partial y_i}\frac{\partial S^G}{\partial y_j}\right)\\
&+\sum_{i,j=1}^{d+1}\frac{\partial^2\phi(x,y)}{\partial y_i\partial x_j}\mathbf{E}\left(\frac{\partial S^G}{\partial X_i^G}\frac{\partial S^G}{\partial y_j}-\frac{\partial S}{\partial X_i}\frac{\partial S}{\partial y_j}\right)\\
&+\frac{1}{2}\sum_{i,j=1}^{d+1}\frac{\partial^2\phi(x,y)}{\partial y_i\partial y_j}\mathbf{E}\left(\frac{\partial S}{\partial X_i}\frac{\partial S}{\partial X_j}-\frac{\partial S^G}{\partial X_i^G}\frac{\partial S^G}{\partial X_j^G}\right)+\cdots.
\end{aligned}
\tag{16}
$$

To make the symplectic numerical approximation be of higher weak order, we choose $H_i^{[j]}$, $i=0,\ldots,r$, $j=1,\ldots,\tau$, such that the terms containing h and h^2 in the right-hand side of (16) vanish. Note that the coefficients of $J_{(r)}^h$ and h in $\frac{\partial S^G}{\partial X_i^G}$ and $\frac{\partial S^G}{\partial y_i}$ are the same as those in $\frac{\partial S}{\partial X_i}$ and $\frac{\partial S}{\partial y_i}$ with $i=1,\ldots,d+1$, respectively. Then we get

$$
\mathbf{E}\left(\frac{\partial S^G}{\partial X_{d+1}^G}\frac{\partial S^G}{\partial y_{d+1}}-\frac{\partial S}{\partial X_{d+1}}\frac{\partial S}{\partial y_{d+1}}\right)=\sum_{r=1}^{m}\sum_{i=1}^{d}vC_1\sigma_r^i y_i\frac{\partial H_r^{[1]}}{\partial x_{d+1}}h^2+h^3e_1(x,y),
$$

where $e_1(x,y)$ denotes the coefficient of the term containing h^3 and can be calculated based on the expression of the partial derivatives of S^G and S, as do the other remainder terms e_l, $l=2,\ldots,7$, in what follows. Thus, we choose $\frac{\partial H_r^{[1]}}{\partial x_{d+1}}=0$ for $r=1,\ldots,m$. Substituting $\frac{\partial H_r^{[1]}}{\partial x_{d+1}}=0$ into $\frac{\partial S^G}{\partial X_{d+1}^G}$, we have

$$
\mathbf{E}\left(\frac{\partial S^G}{\partial X_{d+1}^G}-\frac{\partial S}{\partial X_{d+1}}\right)=\frac{\partial H_0^{[1]}}{\partial x_{d+1}}h^2+\mathbf{E}(R_6)=\frac{\partial H_0^{[1]}}{\partial x_{d+1}}h^2+h^3e_2(x,y),
$$

which leads us to make $\frac{\partial H_0^{[1]}}{\partial x_{d+1}}=0$. In the same way, using $\frac{\partial H_r^{[1]}}{\partial x_{d+1}}=0$ for $r=0,1,\ldots,m$, we derive

$$
\mathbf{E}\left(\frac{\partial S}{\partial y_i}\frac{\partial S}{\partial y_j}-\frac{\partial S^G}{\partial y_i}\frac{\partial S^G}{\partial y_j}\right)=C_1\sum_{r=1}^{m}\left(vC_1\sigma_r^i\sigma_r^j-\sigma_r^i\frac{\partial H_r^{[1]}}{\partial y_j}-\sigma_r^j\frac{\partial H_r^{[1]}}{\partial y_i}\right)h^2+h^3e_3(x,y)
$$

and

$$
\mathbf{E}\left(\frac{\partial S}{\partial y_i}\frac{\partial S}{\partial X_j}-\frac{\partial S^G}{\partial y_i}\frac{\partial S^G}{\partial X_j^G}\right)=C_1\sum_{r=1}^{m}\sigma_r^i\left(\frac{1}{2}\sum_{k=1}^{d}M_{jk}\sigma_r^k-\frac{\partial H_r^{[1]}}{\partial x_j}\right)h^2+h^3e_4(x,y)
$$

with $i,j=1,\ldots,d$, and hence choose

$$
\frac{\partial H_r^{[1]}}{\partial y_i}=\frac{1}{2}vC_1\sigma_r^i,\quad \frac{\partial H_r^{[1]}}{\partial x_i}=\frac{1}{2}\sum_{j=1}^{d}M_{ij}\sigma_r^j,\quad r=1,\ldots,m.
$$

The last term in (16) is of order 3 due to the following estimate:

$$\mathbf{E}\left(\frac{\partial S}{\partial X_i}\frac{\partial S}{\partial X_j} - \frac{\partial S^G}{\partial X_i^G}\frac{\partial S^G}{\partial X_j^G}\right) = h^3 e_5(x,y), \quad i,j = 1,\dots,d+1.$$

Since both $\frac{\partial H_r^{[1]}}{\partial y_i}$ and $\frac{\partial H_r^{[1]}}{\partial x_i}$, with $r = 0,1,\dots,m$, are independent of x_i and y_i, we have

$$\mathbf{E}\left(\frac{\partial S}{\partial y_i} - \frac{\partial S^G}{\partial y_i}\right) = \left(\frac{1}{2}\sum_{j,k=1}^{d}\frac{\partial^2 F(y)}{\partial y_i \partial y_j}M_{jk}x_k + \frac{1}{2}vC_1 f_i(y) - \frac{\partial H_0^{[1]}}{\partial y_i}\right)h^2 + h^3 e_6(x,y),$$

$$\mathbf{E}\left(\frac{\partial S}{\partial X_i} - \frac{\partial S^G}{\partial X_i^G}\right) = \left(\frac{1}{2}\sum_{j=1}^{d}M_{ij}f_j(y) - \frac{1}{2}\sum_{j=1}^{d}vC_2 M_{ij}x_j - \frac{\partial H_0^{[1]}}{\partial x_i}\right)h^2 + h^3 e_7(x,y)$$

for $i = 1,\dots,d$. We choose $H_0^{[1]}$ such that the above terms containing h^2 vanish, i.e.,

$$\frac{\partial H_0^{[1]}}{\partial y_i} = \frac{1}{2}\sum_{j,k=1}^{d}\frac{\partial^2 F(y)}{\partial y_i \partial y_j}M_{jk}x_k + \frac{1}{2}vC_1 f_i(y),$$

$$\frac{\partial H_0^{[1]}}{\partial x_i} = \frac{1}{2}\sum_{j=1}^{d}M_{ij}\left(f_j(y) - vC_2 x_j\right).$$

Substituting the above results on the partial derivatives of $H_r^{[1]}$, $r = 0,1,\dots,m$, into (15), we have the following scheme of (9):

$$\begin{aligned}
X_i^G &= x_i - \sum_{r=1}^{m} e^{vt_n}\sigma_r^i I_{(r)}^h - e^{vt_n} f_i(y)h - \frac{1}{2}\sum_{r=1}^{m} v e^{vt_n}\sigma_r^i h I_{(r)}^h \\
&\quad - \frac{1}{2}\sum_{j,k=1}^{d}\frac{\partial^2 F(y)}{\partial y_i \partial y_j}M_{jk}X_k^G h^2 - \frac{1}{2}v e^{vt_n} f_i(y)h^2,
\end{aligned} \tag{17}$$
$$\begin{aligned}
Y_i^G &= y_i + \sum_{j=1}^{d} e^{-vt_n}M_{ij}X_j^G h + \frac{1}{2}\sum_{r=1}^{m}\sum_{j=1}^{d}M_{ij}\sigma_r^j I_{(r)}^h h \\
&\quad + \frac{1}{2}\sum_{j=1}^{d}M_{ij}\left(f_j(y) - v e^{-vt_n}X_j^G\right)h^2,
\end{aligned}$$

which is started at time $t_n = nh$ for $n = 1,\dots,N = T/h$. That is, $x_i = X_i(t_n)$, $y_i = Y_i(t_n)$ for $i = 1,\dots,d$, and $y_{d+1} = t_n$.

To transform scheme (17) into an equivalent scheme of (1), we denote $P_i^h[n] := e^{-vt_n}x_i$, $Q_i^h[n] := y_i$, $P_i^h[n+1] := e^{-vt_{n+1}}X_i^G$, and $Q_i^h[n+1] := Y_i^G$ for $i = 1,\dots,d$. Based on the transformation between two phase spaces of (1) and (5), we get

(18)

$$\begin{aligned}
P^h[n+1] &= e^{-vh}P^h[n] - \frac{h^2}{2}\nabla^2 F(Q^h[n])MP^h[n+1] - h\Big(1 + \frac{vh}{2}\Big)e^{-vh}f(Q^h[n]) \\
&\quad - \Big(1 + \frac{vh}{2}\Big)e^{-vh}\sigma\Delta_{n+1}W, \\
Q^h[n+1] &= Q^h[n] + h\left(1 - \frac{vh}{2}\right)e^{vh}MP^h[n+1] + \frac{h^2}{2}Mf(Q^h[n]) + \frac{h}{2}M\sigma\Delta_{n+1}W,
\end{aligned}$$

where $\sigma = (\sigma_1, \dots, \sigma_r)$ and $\Delta_{n+1}W = W(t_{n+1}) - W(t_n)$. Notice that $\Delta_n W$ can be simulated by $\xi^n\sqrt{h}$ with $\xi^n = (\xi_1^n, \dots, \xi_d^n)^\top$ being an $\mathcal{F}_{t_n}$-adapted d-dimensional normal distributed random vector.

Remark 3.1. The proposed scheme (18) also has exponentially dissipative phase volume. More precisely, denoting $D(q) = (I_d + \frac{h^2}{2}\nabla^2 F(q)M)^{-1}$, the determinant of Jacobian matrix

$$\begin{aligned}\begin{vmatrix} \frac{\partial P^h[1]}{\partial p} & \frac{\partial P^h[1]}{\partial q} \\ \frac{\partial Q^h[1]}{\partial p} & \frac{\partial Q^h[1]}{\partial q} \end{vmatrix} &= \begin{vmatrix} e^{-vh}D(q) & \frac{\partial P^h[1]}{\partial q} \\ h(1-\frac{vh}{2})MD(q) & D(q)^{-\top} + h(1-\frac{vh}{2})e^{vh}M\frac{\partial P^h[1]}{\partial q} \end{vmatrix} \\ &= |e^{-vh}I_d||D(q)||D(q)^{-\top}| = e^{-vhd}.\end{aligned}$$

Furthermore,

$$\begin{vmatrix} \frac{\partial P^h[n]}{\partial p} & \frac{\partial P^h[n]}{\partial q} \\ \frac{\partial Q^h[n]}{\partial p} & \frac{\partial Q^h[n]}{\partial q} \end{vmatrix} = e^{-vt_n d}.$$

3.2. Conformal symplectic structure and ergodicity. In this subsection, we prove the conformal symplecticity of the proposed scheme (18) as well as its ergodicity.

THEOREM 3.2. *The proposed scheme* (18) *preserves conformal symplectic structure, i.e,*

$$dP^h[n+1] \wedge dQ^h[n+1] = e^{-vh}dP^h[n] \wedge dQ^h[n].$$

Proof. Based on (18), we obtain

$$\begin{aligned}&dP^h[n+1] \wedge dQ^h[n+1] \\ &= dP^h[n+1] \wedge dQ^h[n] + \frac{1}{2}h^2 dP^h[n+1] \wedge M\nabla^2 F dQ^h[n] \\ &= e^{-vh}dP^h[n] \wedge dQ^h[n] - \frac{h^2}{2}d\left[\nabla^2 F(Q^h[n])MP^h[n+1]\right] \wedge dQ^h[n] \\ &\quad + \frac{h^2}{2}dP^h[n+1] \wedge M\nabla^2 F(Q^h[n])dQ^h[n].\end{aligned}$$

Denote $\tilde{P}^h := MP^h[n+1]$; then the second term becomes

$$\begin{aligned}&\frac{h^2}{2}d\left[\nabla^2 F(Q^h[n])\tilde{P}^h\right] \wedge dQ^h[n] \\ =&\frac{h^2}{2}\sum_{i,j,l=1}^{d}\frac{\partial^3 F}{\partial q_i \partial q_j \partial q_l}\tilde{P}_j^h dQ_l^h[n] \wedge dQ_i^h[n] - \frac{h^2}{2}\nabla^2 F(Q^h[n])MdP^h[n+1] \wedge dQ^h[n].\end{aligned}$$

Since matrix M is symmetric and the first term in the right-hand side of the above equation vanishes, we finally get

$$dP^h[n+1] \wedge dQ^h[n+1] = e^{-vh}dP^h[n] \wedge dQ^h[n]. \qquad \square$$

To show the ergodicity of (18), we first introduce the following conditions which are sufficient to ensure the existence and uniqueness of the invariant measure (see [13] and references therein). Then we will show that these conditions are exactly satisfied by the proposed scheme.

CONDITION 3.3. *The Markov chain* $Z_n := (P^h[n]^\top, Q^h[n]^\top)^\top$ *with* $Z_0 = z$ *satisfies*

(i) *for any* $\gamma \ge 1$, *there exists* $C_2 = C(\gamma) > 0$ *which is independent of* h, *such that* $\mathbf{E}\|Z_1\|^\gamma \le C_2(1 + \|z\|^\gamma)$ *for all* $z \in \mathbb{R}^{2d}$*;*

(ii) *there exist* $C_1 > 0$ *and* $\epsilon > 0$ *which are independent of* h, *such that* $\mathbf{E}\|Z(h) - Z_1\|^2 \le C_1(1 + \|z\|^2)h^{\epsilon+2}$ *for all* $z \in \mathbb{R}^{2d}$, *where* $Z(h) = (P(h)^\top, Q(h)^\top)^\top$.

CONDITION 3.4. *For some fixed compact set* $G \in \mathcal{B}(\mathbb{R}^{2d})$ *with* $\mathcal{B}(\mathbb{R}^{2d})$ *denoting the Borel* σ*-algebra on* $\mathbb{R}^{2d}$, *the Markov chain* $Z_n := (P^h[n]^\top, Q^h[n]^\top)^\top \in \mathcal{F}_{t_n}$ *with transition kernel* $\mathcal{P}_n(z, A)$ *satisfies*

(i) *for some* $z^* \in \operatorname{int}(G)$ *and for any* $\delta > 0$, *there exists a positive integer* n *such that*

$$\mathcal{P}_n(z, B_\delta(z^*)) > 0 \quad \forall\, y \in G,$$

where $B_\delta(z^*)$ *denotes the open ball of radius* δ *centered at* z^**;*

(ii) *for any* $n \in \mathbb{N}$, *the transition kernel* $\mathcal{P}_n(z, A)$ *possesses a density* $\rho_n(z, w)$ *which is jointly continuous in* $(z, w) \in G \times G$.

THEOREM 3.5 (see [13, Theorem 7.3]). *For some* $K \in \mathbb{N}$, *if Conditions* 3.3 *and* 3.4 *are satisfied by a Markov chain* Z_n *when sampled at rate* K, *that is, these conditions hold for the chain* $\tilde{Z}_n := Z_{nK}$, *then* Z_n *has a unique invariant measure.*

THEOREM 3.6. *Assume that the vector field* f *is globally Lipschitz. The solution* $(P^h[n], Q^h[n])$ *of* (18), *which is an* $\mathcal{F}_{t_n}$*-adapted Markov chain, satisfies Condition* 3.3 *and hence admits an invariant measure* μ_h *on* $\mathbb{R}^{2d}$. *In addition, if* f *is a linear function, then Condition* 3.4 *is also satisfied and the invariant measure is unique, that is,* (18) *is ergodic.*

Proof. Step 1. We first show that scheme (18) satisfies Condition 3.3. Denote $Z(t) = (P(t)^\top, Q(t)^\top)^\top \in \mathbb{R}^{2d}$, $Z_n = (P^h[n]^\top, Q^h[n]^\top)^\top \in \mathbb{R}^{2d}$, $\sigma = (\sigma_1, \ldots, \sigma_r) \in \mathbb{R}^{d\times r}$, $W = (W_1, \ldots, W_r)^\top \in \mathbb{R}^r$, and $D(q) = (I_d + \frac{h^2}{2}\nabla^2 F(q)M)^{-1}$. We rewrite (18) as

$$\begin{aligned} P^h[1] &= D(q)\left(e^{-vh}p - \Big(1 + \frac{vh}{2}\Big)e^{-vh}\sigma\Delta_1 W - h\Big(1 + \frac{vh}{2}\Big)e^{-vh}f(q)\right), \\ Q^h[1] &= q + h\Big(1 - \frac{vh}{2}\Big)e^{vh}MP^h[1] + \frac{h^2}{2}Mf(q) + \frac{h}{2}M\sigma\Delta_1 W \end{aligned} \tag{19}$$

with $z := (P_0^\top, Q_0^\top)^\top = (p^\top, q^\top)^\top$, which yields

$$\begin{aligned} \mathbf{E}\|P^h[1]\|^\gamma + \mathbf{E}\|Q^h[1]\|^\gamma &\le C(1 + \|p\|^\gamma + \|q\|^\gamma) + C(1 + \|q\|^\gamma + \mathbf{E}\|P^h[1]\|^\gamma) \\ &\le C(1 + \|p\|^\gamma + \|q\|^\gamma) \end{aligned} \tag{20}$$

based on the fact that vector field f is globally Lipschitz, the matrix $I + \frac{h^2}{2}\nabla^2 F(q)M$ is positive definite, and $\|D(q)\| \le 1$ for any $q \in \mathbb{R}^d$ and $h \in (0, 1)$. As the norm $\|Z_1\| = (\|P^h[1]\|^2 + \|Q^h[1]\|^2)^{\frac{1}{2}}$ is equivalent to the norm $(\|P^h[1]\|^\gamma + \|Q^h[1]\|^\gamma)^{\frac{1}{\gamma}}$, Condition 3.3(i) holds.

Rewrite (1) into the following mild solution form:

$$\begin{aligned} P(h) &= p - \int_0^h e^{-v(h-s)}f(Q(s))ds - \int_0^h e^{-v(h-s)}\sigma dW(s), \\ Q(h) &= q + \int_0^h MP(s)ds \end{aligned}$$

with $P(0) = p$ and $Q(0) = q$. Based on (18), we have

$$
\begin{aligned}
P(h) - P^h[1] =& \left[h\Big(1 + \frac{vh}{2}\Big) e^{-vh} f(q) + \frac{h^2}{2} \nabla^2 F(q) M P^h[1] - \int_0^h e^{-v(h-s)} f(Q(s)) ds \right] \\
&+ \left[\Big(1 + \frac{vh}{2}\Big) e^{-vh} \sigma \Delta_1 W - \int_0^h e^{-v(h-s)} \sigma dW(s) \right] \\
=& : I + II, \\
Q(h) - Q^h[1] =& \left[\int_0^h MP(s) ds - h\Big(1 - \frac{vh}{2}\Big) e^{vh} M P^h[1] \right] - \left[\frac{h}{2} M \sigma \Delta_1 W + \frac{h^2}{2} M f(q) \right] \\
=& : III + IV.
\end{aligned}
$$

Now we estimate terms I, II, III, and IV, respectively:

$$
\begin{aligned}
\mathbf{E}\|I\|^2 \leq& \, C\mathbf{E} \left\| \frac{h^2}{2} \nabla^2 F(q) P^h[1] \right\|^2 + C\mathbf{E} \left\| \int_0^h e^{-v(h-s)} \left(f(Q(s)) - f(q) \right) ds \right\|^2 \\
&+ C \left\| \int_0^h e^{-v(h-s)} ds f(q) - h\Big(1 + \frac{vh}{2}\Big) e^{-vh} f(q) \right\|^2 \\
\leq& \, Ch^4(1 + \|z\|^2) + C \int_0^h e^{-2v(h-s)} ds \int_0^h \left(\|Q(s) - Q^h[1]\|^2 + \|Q^h[1] - q\|^2 \right) ds \\
&+ C \left(\frac{1 - e^{-vh}}{v} - h\Big(1 + \frac{vh}{2}\Big) e^{-vh} \right)^2 (1 + \|q\|^2) \\
\leq& \, Ch^3(1 + \|z\|^2) + C \int_0^h \|Q(s) - Q^h[1]\|^2 ds, \qquad (21)
\end{aligned}
$$

where in the last step we have used (20). For the term II, based on the Itô isometry,

$$
\mathbf{E}\|II\|^2 \leq \int_0^h \left(\Big(1 + \frac{vh}{2}\Big) e^{-vh} - e^{-v(h-s)} \right)^2 ds \mathrm{Tr} \left(\sigma \sigma^\top \right) \leq Ch^3. \qquad (22)
$$

Similarly, we have

(23)

$$
\begin{aligned}
\mathbf{E}\|III\|^2 \leq& \, C\mathbf{E} \left\| \int_0^h M \left(P(s) - P^h[1] \right) ds \right\|^2 + C\mathbf{E} \left\| h \left(1 - \Big(1 - \frac{vh}{2}\Big) e^{vh} \right) M P^h[1] \right\|^2 \\
\leq& \, C \int_0^h \|P(s) - P^h[1]\|^2 ds + Ch^4(1 + \|z\|^2)
\end{aligned}
$$

and

$$
\mathbf{E}\|IV\|^2 \leq Ch^3(1 + \|q\|^2). \qquad (24)
$$

From (21)–(24), we conclude

$$
\mathbf{E}\|Z(h) - Z_1\|^2 \leq C \int_0^h \mathbf{E}\|Z(s) - Z_1\|^2 ds + Ch^3(1 + \|z\|^2),
$$

which together with Gronwall's inequality yields Condition 3.3(ii) with $\epsilon = 1$. In this case, there exist real numbers $\tilde{\alpha} \in (0,1)$ and $\tilde{\beta} \in [0,\infty)$ such that $\mathbf{E}[V(Z_{n+1})|\mathcal{F}_{t_n}] \leq \tilde{\alpha} V(Z_n) + \tilde{\beta}$ for $V(z) = \frac{1}{2}\|p\|^2 + F(q) + \frac{v}{2}p^\top q + \frac{v^2}{4}\|q\|^2 + 1$ with $z = (p^\top, q^\top)^\top$ (see Theorem 7.2 in [13]). Hence,

$$\mathbf{E}[V(Z_{n+1})] \leq \tilde{\alpha}\mathbf{E}[V(Z_n)] + \tilde{\beta} \leq \tilde{\alpha}^{n+1}\mathbf{E}[V(Z_0)] + \tilde{\beta}\frac{1-\tilde{\alpha}^n}{1-\tilde{\alpha}} \leq C(Z_0),$$

which induces the existence of invariant measures (see Proposition 7.10 in [9]).

Step 2. We now consider the chain Z_{2n} sampled at rate $K = 2$ and verify Condition 3.4 when f is linear with a constant $C_f := \nabla f = \nabla^2 F$. Let $G := \{(P^\top, Q^\top)^\top \in \mathbb{R}^{2d} : Q = 0, \|P\| \leq 1\}$, which is a compact set. For any $z = (p^\top, 0)^\top \in G$ and $w = (w_1^\top, w_2^\top)^\top \in B$ with $B \in \mathcal{B}(\mathbb{R}^{2d})$, we aim to show that $\Delta_1 W$ and $\Delta_2 W$ can be properly chosen to ensure that $P^h[2] = w_1$ and $Q^h[2] = w_2$ starting from $(P_0^\top, Q_0^\top)^\top = z$. Denoting $L_h = h(1 - \frac{vh}{2})e^{vh}M$, from (18), we have

(25)
$$w_1 = e^{-vh}P^h[1] - \frac{h^2}{2}C_f M w_1 - h\Big(1 + \frac{vh}{2}\Big)e^{-vh}f(Q^h[1]) - \Big(1 + \frac{vh}{2}\Big)e^{-vh}\sigma\Delta_2 W,$$

(26)
$$\begin{aligned} w_2 &= Q^h[1] + L_h w_1 + \frac{h^2}{2}Mf(Q^h[1]) + \frac{h}{2}M\sigma\Delta_2 W \\ &= Q^h[1] + L_h w_1 + \frac{h}{2}\Big(1 + \frac{vh}{2}\Big)^{-1}e^{vh}M\left(e^{-vh}P^h[1] - w_1 - \frac{h^2}{2}C_f M w_1\right), \end{aligned}$$

(27)
$$P^h[1] = e^{-vh}p - \frac{h^2}{2}C_f M P^h[1] - h\Big(1 + \frac{vh}{2}\Big)e^{-vh}f(0) - \Big(1 + \frac{vh}{2}\Big)e^{-vh}\sigma\Delta_1 W,$$

(28)
$$\begin{aligned} Q^h[1] &= L_h P^h[1] + \frac{h^2}{2}Mf(0) + \frac{h}{2}M\sigma\Delta_1 W \\ &= L_h P^h[1] + \frac{h}{2}\Big(1 + \frac{vh}{2}\Big)^{-1}e^{vh}M\left(e^{-vh}p - P^h[1] - \frac{h^2}{2}C_f M P^h[1]\right). \end{aligned}$$

Notice that (26) and (28) form a linear system, from which we can get the solution $P^h[1]$ and $Q^h[1]$ based on the positive definite coefficient matrix. Then $\Delta_2 W$ and $\Delta_1 W$ can be uniquely determined by (25) and (27), respectively. Condition 3.4(i) is then ensured according to the property that Brownian motions hit a cylinder set with positive probability. For Condition 3.4(ii), from (19), we can find out that $P^h[1]$ has a C^∞ density based on the facts that $\Delta_1 W$ has a C^∞ density, σ is full rank, and $D(q)$ is positive definite for any $q \in \mathbb{R}^d$. Thus, $Q^h[1]$ also has a C^∞ density, and Theorem 3.5 is applied to complete the proof. ∎

Remark 3.7. For the nonlinear case, the uniqueness of the invariant measure is unsolved since both equations in (18) contain the same noise, which is totally different from the continuous case and brings essential difficulties when showing the irreducible property. For higher k and k', following the same procedure as for the case $k = k' = 1$ (see also [1]), choosing undetermined functions such that the error in (13) is of higher order, we can also get higher weak order symplectic schemes for (5), which turn out to be high weak order conformal symplectic schemes for the original system (1) based on the inverse transformation $(X, Y) \mapsto (P, Q)$. It is worth mentioning that the solvability of undetermined functions, as well as the ergodicity of the schemes, is unknown for high order cases, as far as we know.

4. Approximation error. In this section, we consider the weak convergence order of (18) by investigating the local convergence error first. Furthermore, based on the local convergence error and the hypoelliptic setting (2), we can also get the approximation error of the ergodic limit. Denote the exact solution of (1) and the numerical solution by $Z(t) = (P(t)^\top, Q(t)^\top)^\top$ and $Z_n = (P^h[n]^\top, Q^h[n]^\top)^\top$, respectively. The next theorem gives that the moments of (1) are uniformly bounded, and its proof follows the same procedure as that of Lemma 3.3 in [13].

THEOREM 4.1. *Let Assumption* 2.1 *hold. Then for any $k \in \mathbb{N}_+$, the kth moments of $P(t)$ and $Q(t)$ are uniformly bounded with respect to $t \in \mathbb{R}_+$.*

Before proving the main convergence theorem, we first show the boundedness of the numerical solution to (18) in the following theorem.

THEOREM 4.2. *Assume that the coefficient f of* (1) *is globally Lipschitz and satisfies the linear growth condition, i.e.,*

$$\|f(u) - f(w)\| \le L\|u - w\|, \quad \|f(u)\| \le C_f(1 + \|u\|) \tag{29}$$

for some constants $L > 0$ and $C_f \ge 0$, and any $u, w \in \mathbb{R}^d$. Then there exists a positive constant h_0 such that for any $h \le h_0$, it holds that

$$\sup_{n\in\{1,\ldots,N\}} \mathbf{E}\left[\|P^h[n]\|^k + \|Q^h[n]\|^k\right] < \infty.$$

Proof. For any fixed initial value $z = (p^\top, q^\top)^\top$, random variable $\xi := \xi^1$, and h, we have based on (18) that

$$\begin{aligned}\|P^h[1] - p\| \le |e^{-vh} - 1|\|p\| + h\Big(1 + \frac{vh}{2}\Big)\|f(q)\| + \sqrt{h}\Big(1 + \frac{vh}{2}\Big)\|\sigma\xi\| \\ + \frac{h^2}{2}\|\nabla^2 F(q)\|\|M\|\|p\| + \frac{h^2}{2}\|\nabla^2 F(q)\|\|M\|\|P^h[1] - p\|.\end{aligned}$$

Denote $C_v := 1 + \frac{vh}{2}$. Using the global Lipschitz condition and mean value theorem, there exists some $\theta \in (0, 1)$ such that

$$\begin{aligned}\|P^h[1] - p\| &\le | - vhe^{-v\theta h}|\|p\| + hC_f(1 + \|z\|) + \sqrt{h}C_v\|\sigma\xi\| \\ &\quad + \frac{h^2}{2}L\|M\|\|z\| + \frac{h^2}{2}L\|M\|\|P^h[1] - p\| \\ &\le C(1 + \|z\|)(\|\xi\|\sqrt{h} + h) + L\|M\|\|P^h[1] - p\|\frac{h^2}{2}.\end{aligned}$$

It is obvious that there exists a positive constant h_0 such that for any $h \le h_0$,

$$L\|M\|\frac{h^2}{2} \le \frac{1}{2}.$$

It then yields

$$\|P^h[1] - p\| \le 2C(1 + \|z\|)(\|\xi\|\sqrt{h} + h).$$

On the other hand, for $h \le h_0$, we have

$$
\begin{aligned}
&\|\mathbf{E}(P^h[1]-p)\| \\
&\le \left\|(e^{-vh}-1)p - \frac{h^2}{2}\nabla^2 F(q)Mp - hC_v e^{-vh} f(q)\right\| + \left\|\frac{h^2}{2}\nabla^2 F(q)M\mathbf{E}(P^h[1]-p)\right\| \\
&\le vh\|p\| + hL\|M\|\|p\| + hC_f C_v(1+\|z\|) + \frac{h^2}{2}L\|M\|\|\mathbf{E}(P^h[1]-p)\|,
\end{aligned}
$$

which leads to

$$\|\mathbf{E}(P^h[1]-p)\| \le C(1+\|z\|)h.$$

Based on the estimate of $P^h[1]-p$, similarly, we have

$$\|Q^h[1]-q\| \le C(1+\|z\|)(\|\xi\|\sqrt{h}+h), \quad \|\mathbf{E}(Q^h[1]-q)\| \le C(1+\|z\|)h.$$

We can conclude that, for $Z_1 = (P^h[1]^\top, Q^h[1]^\top)^\top$,

$$\|Z_1 - z\| \le C(\|\xi\| + \sqrt{h})(1+\|z\|)\sqrt{h} \le C(\|\xi\|+1)(1+\|z\|)\sqrt{h}. \tag{30}$$

Thus, we complete the proof according to Lemma 9.1 in [15]. ∎

Based on the above preliminaries, our result concerning the weak convergence order of the proposed scheme is as follows.

THEOREM 4.3. *Under the assumptions in Theorem* 4.2*, the proposed scheme* (18) *is of weak order* 2*. More precisely,*

$$\left|\mathbf{E}\psi\left(P(T),Q(T)\right) - \mathbf{E}\psi\left(P^h[N],Q^h[N]\right)\right| = O(h^2)$$

for all $\psi \in C_P^6(\mathbb{R}^{2d},\mathbb{R})$ *and* $T = Nh$.

Proof. Without loss of generality, we consider the case of $d = 1$. Based on Itô's formula and Theorems 4.1 and 4.2, we obtain

$$
\begin{aligned}
P(h) &= p - \int_0^h \left(f(Q(s)) + vP(s)\right) ds - \sum_{r=1}^m \int_0^h \sigma_r dW_r(s) \\
&= p - \int_0^h \left(f(q) + \int_0^s \nabla^2 F(Q(\theta))MP(\theta)d\theta\right) ds - \sum_{r=1}^m \int_0^h \sigma_r dW_r(s) \\
&\quad - v\int_0^h \left(p - \int_0^s f(Q(\theta))d\theta - \int_0^s vP(\theta)d\theta - \sum_{r=1}^m \sigma_r dW_r(\theta)\right) ds,
\end{aligned}
$$

which leads to

$$
\begin{aligned}
P(h) = p - f(q)h - vph - \frac{1}{2}\nabla^2 F(q)Mph^2 - \sum_{r=1}^m \int_0^h \sigma_r dW_r(s) \\
+ \frac{1}{2}vf(q)h^2 + \frac{1}{2}v^2ph^2 + v\sum_{r=1}^m \int_0^h \int_0^s \sigma_r dW_r(\theta)ds + \delta_1,
\end{aligned}
\tag{31}
$$

where $\mathbf{E}\|\delta_1\| = O(h^3)$ and $\mathbf{E}\|\delta_1\|^2 = O(h^5)$. Analogously, it also holds that

$$
\begin{aligned}
Q(h) &= q + \int_0^h M\left(p - \int_0^s f(Q(\theta))d\theta - v\int_0^s P(\theta)d\theta - \sum_{r=1}^m \int_0^s \sigma_r dW_r(\theta)\right) ds \\
&= q + Mph - \frac{1}{2}f(q)h^2 - \frac{1}{2}vMph^2 - \sum_{r=1}^m M\sigma_r \int_0^h \int_0^s dW_r(\theta)ds + \delta_2
\end{aligned}
\tag{32}
$$

with $\mathbf{E}\|\delta_2\| = O(h^3)$ and $\mathbf{E}\|\delta_2\|^2 = O(h^5)$. For (18), applying Taylor expansion to $P^h[1]$ and $Q^h[1]$ at (p,q), we obtain

$$\begin{aligned} P^h[1] &= p - f(q)h - vph - \frac{1}{2}\nabla^2 F(q)Mph^2 - \sum_{r=1}^m \sigma_r \Delta_1 W \\ &\quad + \frac{1}{2}vf(q)h^2 + \frac{1}{2}v^2ph^2 + \frac{1}{2}v\sum_{r=1}^m \sigma_r h\Delta_1 W + \delta_3, \end{aligned} \tag{33}$$

$$Q^h[1] = q + Mph - \frac{1}{2}f(q)h^2 - \frac{1}{2}vMph^2 - \frac{1}{2}\sum_{r=1}^m M\sigma_r h\Delta_1 W + \delta_4, \tag{34}$$

where $\mathbf{E}\|\delta_i\| = O(h^3)$ and $\mathbf{E}\|\delta_i\|^2 = O(h^5)$ with $i = 3,4$. Due to (31) and (33), we know that

$$P(h) - P^h[1] = v\sum_{r=1}^m \sigma_r \left(\int_0^h \int_0^s dW_r(\theta)ds - \frac{1}{2}h\Delta_1 W\right) + (\delta_1 - \delta_3),$$

and thus $\|\mathbf{E}(P(h) - P^h[1])\| = O(h^3)$. Similarly, based on (32) and (34), we have $\|\mathbf{E}(Q(h) - Q^h[1])\| = O(h^3)$. For $i = 2,3,4,5$, we obtain

$$\begin{aligned} \left\|\mathbf{E}\left[(P(h)-p)^i - (P^h[1]-p)^i\right]\right\| &\le Ch^3 + O(h^4), \\ \left\|\mathbf{E}\left[(Q(h)-q)^i - (Q^h[1]-q)^i\right]\right\| &\le Ch^3 + O(h^4). \end{aligned}$$

Moreover, for $i_1 + i_2 = 2,3,4,5$ and $i_1 \ge 1$,

$$\left\|\mathbf{E}\left[(P(h)-p)^{i_1}(Q(h)-q)^{i_2} - (P^h[1]-p)^{i_1}(Q^h[1]-q)^{i_2}\right]\right\| \le Ch^3 + O(h^4).$$

By Taylor expansion and the mean value theorem, we obtain

$$\begin{aligned} &\left|\mathbf{E}\left[\psi(P(h),Q(h)) - \psi(P^h[1],Q^h[1])\right]\right| \\ &\le \left|\frac{\partial\psi}{\partial p}(p,q)\right| \left\|\mathbf{E}(P(h)-P^h[1])\right\| + \left|\frac{\partial\psi}{\partial q}(p,q)\right| \left\|\mathbf{E}(Q(h)-Q^h[1])\right\| \\ &\quad + \sum_{j=2}^5\sum_{i=0}^j \left|\frac{\partial^j\psi(p,q)}{\partial p^i\partial q^{j-i}}\right| \left\|\mathbf{E}[(P(h)-p)^i(Q(h)-q)^{j-i} - (P^h[1]-p)^i(Q^h[1]-q)^{j-i}]\right\| \\ &\quad + \sum_{i=0}^6 \mathbf{E}\left(\left|\frac{\partial^6\psi(p+\theta_1 P(h), q+\theta_1 Q(h))}{\partial p^i\partial q^{6-i}}\right| \left\|(P(h)-p)^i(Q(h)-q)^{6-i}\right\|\right) \\ &\quad + \sum_{i=0}^6 \mathbf{E}\left(\left|\frac{\partial^6\psi(p+\theta_2 P^h[1], q+\theta_2 Q^h[1])}{\partial p^i\partial q^{6-i}}\right| \left\|(P^h[1]-p)^i(Q^h[1]-q)^{6-i}\right\|\right) \end{aligned} \tag{35}$$

with constants $0 \le \theta_1 \le 1$ and $0 \le \theta_2 \le 1$. Here, based on (31)–(34) and Theorems 4.1 and 4.3, we derive

$$\begin{aligned} &\mathbf{E}\left(\left|\frac{\partial^6\psi(p+\theta_1 P(h), q+\theta_1 Q(h))}{\partial p^i\partial q^{6-i}}\right| \left\|(P(h)-p)^i(Q(h)-q)^{6-i}\right\|\right) \\ &\le C\left(\mathbf{E}\left\|(P(h)-p)^{2i}(Q(h)-q)^{12-2i}\right\|\right)^{\frac{1}{2}} \le Ch^{6-\frac{i}{2}}, \end{aligned}$$

where we also use the fact that $\psi \in C_P^6(\mathbb{R}^{2d}, \mathbb{R})$. Analogously,

$$\mathbf{E}\left(\left|\frac{\partial^6\psi(p+\theta_2 P^h[1], q+\theta_2 Q^h[1])}{\partial p^i \partial q^{6-i}}\right| \left\|(P^h[1]-p)^i (Q^h[1]-q)^{6-i}\right\|\right) = O(h^{6-\frac{i}{2}})$$

for $0 \le i \le 6$. Finally, we deduce

$$\left|\mathbf{E}\psi(P(h), Q(h)) - \mathbf{E}\psi(P^h[1], Q^h[1])\right| \le O(h^3), \tag{36}$$

which, together with Theorem 9.1 in [15], yields global weak order two for the proposed scheme (18). ∎

According to the above theorem and the condition (2), we can get that the temporal average of the proposed scheme (18) is a proper approximation of the ergodic limit $\int_{\mathbb{R}^{2d}} \psi d\mu$.

THEOREM 4.4. *For any $\psi \in C_b^6(\mathbb{R}^{2d}, \mathbb{R})$ and any initial values, under assumptions in Theorems* 3.6 *and* 4.3*, the scheme* (18) *satisfies*

$$\left|\frac{1}{N}\sum_{n=1}^{N} \mathbf{E}\psi(P^h[n], Q^h[n]) - \int_{\mathbb{R}^{2d}} \psi d\mu\right| \le C\left(h^2 + \frac{1}{T}\right).$$

In fact, one can check that the assumptions in Theorem 5.6 in [14] are satisfied by (18) and thus deduce this result.

5. Numerical experiments. The first example (section 5.1) tests the numerical approximation by simulating a linear stochastic Langevin equation. In section 5.2, numerical tests of the conformal symplectic scheme for the nonlinear case are presented. In all of the experiments, the expectation is approximated by taking the average over 5000 realizations.

5.1. A linear oscillator with damping. Consider the following two-dimensional stochastic Langevin equation:

$$\begin{aligned} dP &= -aQdt - vPdt - \sigma dW(t), \quad P(0) = p, \\ dQ &= aPdt, \quad Q(0) = q, \end{aligned} \tag{37}$$

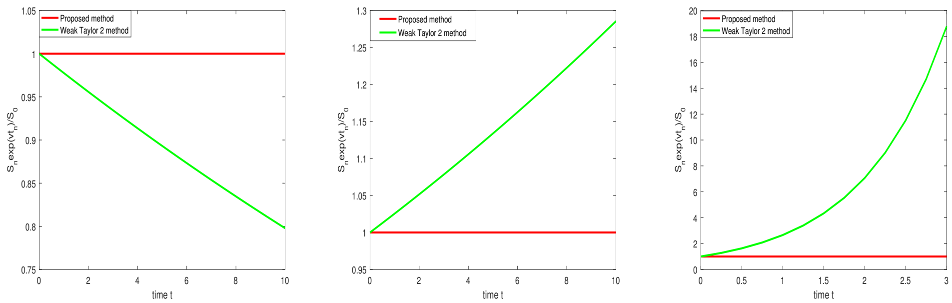
where a, $v > 0$ and $\sigma \neq 0$ are constants and $W(t)$ is a one-dimensional standard Wiener process. The solution to (37) possesses a unique invariant measure μ_1:

$$d\mu_1 = \rho_1(p, q)dpdq,$$

where $\rho_1(p, q) = \Theta \exp\left(-\frac{av(p^2+q^2)}{\sigma^2}\right)$ is known as the Boltzmann–Gibbs density and $\Theta = \left(\int_{\mathbb{R}^2} \exp\left(-\frac{av(p^2+q^2)}{\sigma^2}\right)dpdq\right)^{-1}$ is a renormalization constant. The proposed scheme applied to (37) yields

$$\begin{aligned} P_{n+1} &= e^{-vh}P_n - \frac{h^2}{2}a^2 P_{n+1} - h\Big(1+\frac{vh}{2}\Big)e^{-vh}Q_n - \Big(1+\frac{vh}{2}\Big)e^{-vh}\sigma\Delta_{n+1}W, \\ Q_{n+1} &= Q_n + h\Big(1-\frac{vh}{2}\Big)e^{vh}aP_{n+1} + \frac{h^2}{2}a^2 Q_n + \frac{h}{2}a\sigma\Delta_{n+1}W. \end{aligned} \tag{38}$$

Based on Theorems 3.2 and 3.6, scheme (38) inherits both the conformal symplecticity and ergodicity of the original system. To verify these properties numerically, we choose $p = 3$ and $q = 1$.



(a) $v = 1$. (b) $v = 2$. (c) $v = 4$.

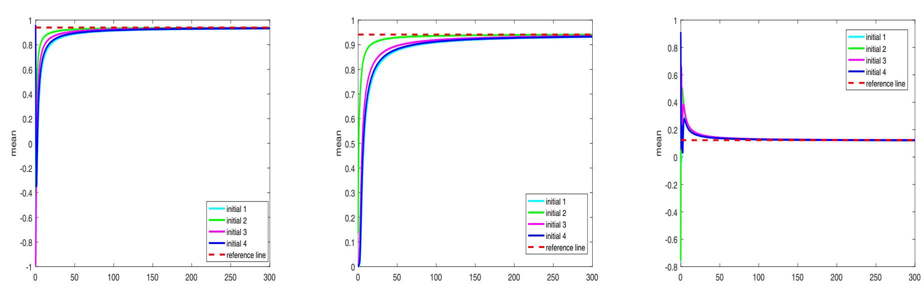
FIG. 1. *The value* $\frac{S_n \exp(vt_n)}{S_0}$ *of two numerical schemes* ($a = 1$ *and* $\sigma = 1$).

Figure 1 shows the value $\frac{S_n \exp(vt_n)}{S_0}$ of the weak Taylor 2 method and the proposed scheme, with v being different dissipative scales and S_n being the triangle square at step n. We choose the original triangle which is produced by three points $(-1, 5)^\top$, $(20, 2)^\top$, $(0, 30)^\top$. We find out that the discrete phase square of the proposed scheme exhibits exponential decay, i.e., $S_n = \exp(-vt_n)S_0$ with the same dissipative coefficient v as in the continuous case, while the weak Taylor 2 scheme does not.

For ergodicity and weak convergence of the proposed scheme, we have taken the three different kinds of test functions (a) $\psi(p, q) = \cos(p + q)$, (b) $\psi(p, q) = \exp\left(-\frac{p^2}{2} - \frac{q^2}{2}\right)$, and (c) $\psi(p, q) = \sin(p^2 + q^2)$ as the test functions. To verify that the temporal averages starting from different initial values will converge to the spatial average, i.e., the ergodic limit

$$\int_{\mathbb{R}^2} \psi(p, q) d\mu_1 = \int_{\mathbb{R}^2} \psi(p, q)\rho_1(p, q) dp dq,$$

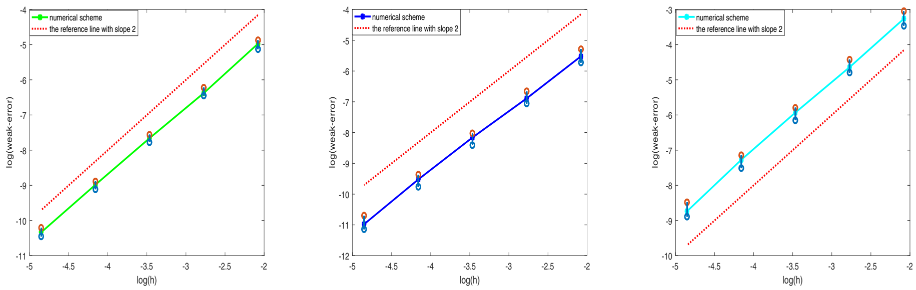
we introduce the reference value for a specific test function ψ to represent the ergodic limit: since the function ψ is uniformly bounded and the density function ρ_1 dissipates exponentially, the integrator is almost zero when $p^2 + q^2$ is sufficiently large. Thus, we choose $\int_{-10}^{10}\int_{-10}^{10} \psi(p, q)\rho_1(p, q) dp dq$ as the reference value, which appears as the dashed line in Figure 2. We can tell from Figure 2 that the tempo-



(a) $\psi(p, q) = \cos(p + q)$. (b) $\psi(p, q) = \exp(-\frac{p^2}{2} - \frac{q^2}{2})$. (c) $\psi(p, q) = \sin(p^2 + q^2)$.

FIG. 2. *The temporal averages* $\frac{1}{N}\sum_{n=1}^{N} \mathbf{E}\psi(P_n, Q_n)$ *starting from different initial values* ($a = 1$, $v = 2$, $\sigma = 0.5$, *and* $T = 300$).

ral averages $\frac{1}{N}\sum_{n=1}^{N}\mathbf{E}\psi(P_n,Q_n)$ of the proposed scheme starting from four different initial values, initial(1) $=(-10,1)^\top$, initial(2) $=(2,0)^\top$, initial(3) $=(0,3)^\top$, and initial(4) $=(4,2)^\top$, converge to the reference line with error no more than $h^2+\frac{1}{T}$, which coincides with Theorem 4.4.



(a) $\psi(p,q)=\cos(p+q)$. (b) $\psi(p,q)=\exp(-\frac{p^2}{2}-\frac{q^2}{2})$. (c) $\psi(p,q)=\sin(p^2+q^2)$.

FIG. 3. *Rate of convergence in weak sense* ($a=1$, $v=2$, *and* $\sigma=0.5$).

Figure 3 plots the value $\ln|\mathbf{E}\psi(P(T),Q(T))-\mathbf{E}\psi(P_N,Q_N)|$ against $\ln h$ for five different step sizes $h=[2^{-3},2^{-4},2^{-5},2^{-6},2^{-7}]$ at $T=1$, where $(P(T),Q(T))$ and (P_N,Q_N) represent the exact and numerical solutions at time T, respectively. It can be seen that the weak order of (38) is two, as indicated by the reference line of slope 2.

5.2. A nonlinear oscillator with linear damping. In this section, we consider the following equation:

$$\begin{aligned}&dP=-(4Q^3-6Q)dt-vPdt+\sqrt{2\beta^{-1}v}dW(t),\quad P(0)=p,\\&dQ=Pdt,\quad Q(0)=q,\end{aligned}\tag{39}$$

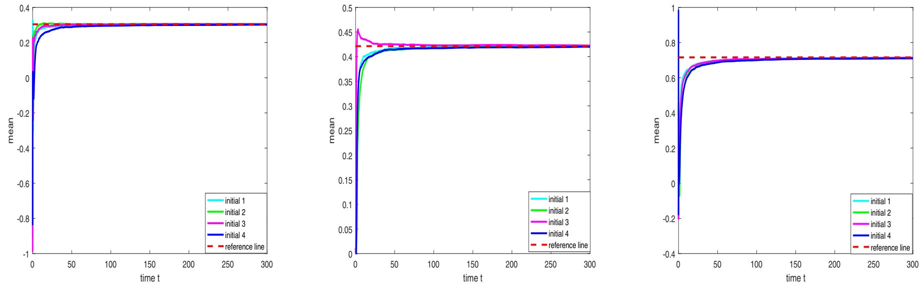
where v, $\beta>0$ are fixed constants and $W(t)$ denotes a one-dimensional standard Wiener process. Similarly to (37), [14] shows that the dynamics generated by (39) is ergodic with the invariant measure μ_2, which can be characterized by the Boltzmann–Gibbs density

$$\rho_2(p,q)=\Theta\exp\bigg(-\beta\Big(\frac{1}{2}p^2+\Big(\frac{3}{2}-q^2\Big)^2\Big)\bigg)$$

with the renormalization constant $\Theta=\big(\int_{\mathbb{R}^2}e^{-\beta(\frac{1}{2}p^2+(\frac{3}{2}-q^2)^2)}dpdq\big)^{-1}$. Based on (18), we get the associated conformal symplectic scheme

$$\begin{aligned}P_{n+1}&=e^{-vh}P_n-\frac{h^2}{2}P_{n+1}\big(12Q_n^2-4\big)-he^{-vh}\Big(1+\frac{vh}{2}\Big)\Big(4Q_n^3-6Q_n\Big)\\&\quad+e^{-vh}\Big(1+\frac{vh}{2}\Big)\sqrt{2\beta^{-1}v}\Delta_{n+1}W,\\Q_{n+1}&=Q_n+he^{vh}\Big(1-\frac{vh}{2}\Big)P_{n+1}+\frac{h^2}{2}\Big(4Q_n^3-6Q_n\Big)-\frac{h}{2}\sqrt{2\beta^{-1}v}\Delta_{n+1}W.\end{aligned}\tag{40}$$

Since this nonglobal Lipschitz case is not included in Theorems 3.6 and 4.3, we investigate its ergodicity and weak convergence order in view of numerical tests.



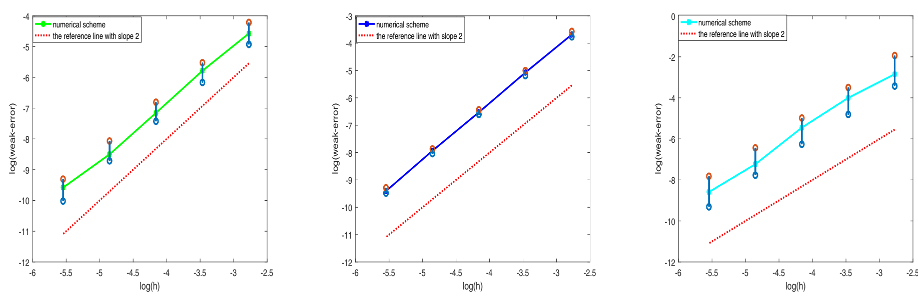
(a) $\psi(p,q) = \cos(p+q)$. (b) $\psi(p,q) = \exp(-\frac{p^2}{2} - \frac{q^2}{2})$. (c) $\psi(p,q) = \sin(p^2+q^2)$.

FIG. 4. *The temporal averages* $\frac{1}{N}\sum_{n=1}^{N} \mathbf{E}\psi(P_n, Q_n)$ *starting from different initial values with* $T = 300$.

Let $v = 4$, $\beta = 2$, and test functions ψ be the same as those in section 5.1. Figure 4 shows the temporal averages $\frac{1}{N}\sum_{n=1}^{N} \mathbf{E}\psi(P_n, Q_n)$ of (40) starting from different initial values initial(1) $= (-10, 1)^\top$, initial(2) $= (2, 7)^\top$, initial(3) $= (0, 3)^\top$, and initial(4) $= (4, 6)^\top$. We also use $\int_{-10}^{10}\int_{-10}^{10} \psi(p,q)\rho_2(p,q)dpdq$ as an approximation of the reference value, i.e., the ergodic limit

$$\int_{\mathbb{R}^2} \psi(p,q)d\mu = \int_{\mathbb{R}^2} \psi(p,q)\rho_2(p,q)dpdq.$$

Figure 4 indicates that the proposed scheme also converges to the reference line when time goes to infinity.



(a) $\psi(p,q) = \cos(p+q)$. (b) $\psi(p,q) = \exp(-\frac{p^2}{2} - \frac{q^2}{2})$. (c) $\psi(p,q) = \sin(p^2+q^2)$.

FIG. 5. *Rate of convergence in weak sense* ($p = -2$ *and* $q = -2$).

The value $\ln |\mathbf{E}\psi(P(T), Q(T)) - \mathbf{E}\psi(P_N, Q_N)|$ against $\ln h$ for five different step sizes $h = [2^{-4}, 2^{-5}, 2^{-6}, 2^{-7}, 2^{-8}]$ at $T = 0.5$ is shown in Figure 5, similarly to Figure 3. Compared with the reference line of slope 2 in Figure 5, it can be seen that (40) has order two in the sense of weak approximations.

6. Conclusion. In this paper, an approach for constructing high weak order conformal symplectic schemes for stochastic Langevin equations is developed, motivated by the ideas in [1, 2, 18, 24]. The key points are that the generating function is applied to ensure that the proposed scheme preserves the geometric structure, while the modified technique is used to reduce the simulation of multiple integrations. We show that, for the case $k = k' = 1$, the proposed scheme could inherit both the conformal symplectic geometric structure (under Lipschitz assumption) and the ergodicity (under linear assumption) of the stochastic Langevin equation. Numerical experiments verify our theoretical results. In addition, the numerical tests of an oscillator with nonglobal Lipschitz coefficients indicate that the proposed scheme could also inherit the internal properties of the original system, which implies that our results may possibly be extended to the nonglobal Lipschitz case. The theoretical analysis of this extension is also ongoing.

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Approximation of Invariant Measure for Damped Stochastic Nonlinear Schrödinger Equation via an Ergodic Numerical Scheme

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Abstract In order to inherit numerically the ergodicity of the damped stochastic nonlinear Schrödinger equation with additive noise, we propose a fully discrete scheme, whose spatial direction is based on spectral Galerkin method and temporal direction is based on a modification of the implicit Euler scheme. We not only prove the unique ergodicity of the numerical solutions of both spatial semi-discretization and full discretization, but also present error estimations on invariant measures, which gives order 2 in spatial direction and order $\frac{1}{2}$ in temporal direction under certain hypotheses.

Keywords Stochastic Schrödinger equation · Numerical scheme · Ergodicity · Invariant measure · Error estimation

Mathematics Subject Classifications (2010) 37M25 · 60-08 · 60H35 · 65C30

1 Introduction

The ergodicity of stochastic differential equations (SDEs) and stochastic partial differential equations (SPDEs) characterizes the longtime behavior of the solutions (see [5, 8, 14] and references therein), and it is natural to construct proper numerical schemes which could inherit the ergodicity. For ergodic SDEs with bounded or global Lipschitz coefficients,

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the ergodicity of several schemes were studied in [15]. It also gave an error estimation of invariant measures

$$e(\phi) = \left| \int \phi(y) d\mu(y) - \int \phi(y) d\tilde{\mu}(y) \right|$$

via the exponential decay property of the solution of Kolmogorov equation, where μ and $\tilde{\mu}$ denote the original invariant measure and the numerical one respectively. In the local Lipschitz case, the ergodicity is inherited by specially constructed implicit discretizations (see [14] and references therein). For SDEs, there are also various works related to the study of error $e(\phi)$ by assuming the ergodicity of the schemes (see [1] and references therein). For SPDEs, there have also been some significant results concentrating on invariant laws, e.g., [3] studied a semi-implicit Euler scheme in temporal direction with respect to parabolic type SPDEs with bounded nonlinearity and space-time white noise; [4] studied a full discretization for stochastic evolution equations with global Lipschitz nonlinearity and space-time white noise. Invariant laws of the approximations are, in general, possibly not unique. To our knowledge, there has been less work on constructing a fully discrete scheme to inherit the unique ergodicity of SPDEs up to now.

In this paper, we consider an initial-boundary problem of an ergodic one-dimensional damped stochastic nonlinear Schrödinger equation

$$\begin{cases} du = \big(\mathbf{i}\Delta u - \alpha u + \mathbf{i}\lambda |u|^2 u\big)dt + Q^{\frac{1}{2}} dW \\ u(t,0) = u(t,1) = 0, \ t \geq 0 \\ u(0,x) = u_0(x), \ x \in [0,1], \end{cases} \tag{1.1}$$

where $\alpha > 0$, $\lambda = \pm 1$ and the solution u is a complex valued ($\mathbb{C}$-valued) random field on a probability space $(\Omega, \mathcal{F}, P)$. The noise term involves a cylindrical Wiener process W and a symmetric, positive, trace class operator Q such that the noise is colored in space and white in time. The operator Q is supposed to commute with Laplacian Δ, and the noise has the following Karhunen-Loeve expansion

$$Q^{\frac{1}{2}} dW = \sum_{m=1}^{\infty} \sqrt{\eta_m} e_m(x) d\beta_m(t), \quad \eta_m \in \mathbb{R}^+ \ \text{ and } \ \eta := \sum_{m=1}^{\infty} \eta_m < \infty,$$

where $\{\beta_m(t)\}_{m\geq 1}$, associated to a filtration $\{\mathcal{F}_t\}_{t\geq 0}$, is a family of independent and identically distributed $\mathbb{C}$-valued Wiener processes and $\{e_m\}_{m\geq 1}$ is the eigenbasis of the Dirichlet Laplacian. This model has many applications in statistical physics and has been studied by many authors. For instance, it can describe the transmission of the signal along the fiber line with signal loss (see [11, 12] and references therein). The ergodicity for Eq. 1.1 with $\lambda = 1$ has been studied in [8] based on a coupling method, Foias-Prodi type estimates and a priori estimates for a modified Hamiltonian $\mathcal{H} = \frac{1}{2}\|\cdot\|_1^2 - \frac{1}{4}\|\cdot\|_{L^4}^4 + c_0\|\cdot\|_0^6$. The authors showed that (1.1) possesses a unique invariant measure μ assuming that the noise is non-degenerate in the low modes, i.e., $\eta_m > 0$, $m \leq N_*$ for some sufficiently large N_*. In the same procedure, one can also show the ergodicity for the cases $\lambda = 0$ and $\lambda = -1$ by setting $\mathcal{H} = \frac{1}{2}\|\cdot\|_1^2 - \frac{\lambda}{4}\|\cdot\|_{L^4}^4 + c_0\|\cdot\|_0^6$. Note that the damped term ($\alpha > 0$) is necessary for both linear and nonlinear Schrödinger equation to be ergodic.

Our work mainly focuses on the construction of a fully discrete and uniquely ergodic numerical scheme (i.e., whose numerical solution possesses a unique invariant measure). Moreover, the estimation of error between the original invariant measure and the numerical one is also considered based on the weak error of solutions.

In order to obtain a scheme whose noise remains in an explicit expression, we apply spectral Galerkin method in spatial direction to obtain a N-dimensional SDE

$$du_N = \Big(\mathbf{i}\Delta u_N - \alpha u_N + \mathbf{i}\lambda\pi_N\left(|u_N|^2 u_N\right)\Big)dt + \pi_N Q^{\frac{1}{2}} dW \tag{1.2}$$

with π_N being a projection operator. Here the spectral Galerkin method also ensures that the semigroup operator is the same as the one of Eq. 1.1, which simplifies the error estimate in spatial direction. We find a Lyapunov function by proving the uniform boundedness of u_N in L^2-norm. It ensures the existence of the invariant measure of Eq. 1.2. We show that the solution $u_N(t)$ is a strong Feller and irreducible process via the non-degeneracy of the noise term in Eq. 1.2. Hence, $u_N(t)$ possesses a unique invariant measure μ_N, which implies the ergodicity of $u_N(t)$. We would like to emphasize that the noise in the original equation do not need to be non-degenerate. Our method is also available under the same assumption in [8], that is $\eta_m > 0$, $m < N_*$ for some sufficiently large N_*. Here N and N_* need to satisfy the condition $N < N_*$ to ensure the non-degeneracy for the truncated noise and obtain the ergodicity for numerical solutions. The error between invariant measures μ_N and μ is transferred into the weak error of the solutions, which is required to be independent of time t. Different from conservative equations, the damped term in Eqs. 1.1 and 1.2 contributes to an exponential estimate on the difference between semigroup operators $S(t)$ and $S(t)\pi_N$, where $S(t)$ is generated by the linear operator $\mathbf{i}\Delta - \alpha$. Therefore, we achieve the time-independent weak error of solutions directly which, together with the ergodicity of u and u_N, deduces the error between invariant measures μ_N and μ.

For the temporal discretization of Eq. 1.2, we propose a new scheme

$$u_N^k - e^{-\alpha\tau}u_N^{k-1} = \left(\mathbf{i}\Delta u_N^k + \mathbf{i}\lambda\pi_N\left(\frac{|u_N^k|^2 + |e^{-\alpha\tau}u_N^{k-1}|^2}{2}u_N^k\right)\right)\tau + \pi_N Q^{\frac{1}{2}}\delta W_k, \tag{1.3}$$

which is a modification of the implicit Euler scheme. In order to analyze the effect of the time discretization, we investigate both the ergodicity of u_N^k and the weak error between u_N and u_N^k. The fully discrete scheme (1.3) is specially constructed to ensure the uniform boundedness of u_N^k in L^2-, $\dot{H}^1$- and $\dot{H}^2$-norms, which is essential to obtain the existance of the invariant measure as well as the time-independence of the weak error. Together with the Brouwer fixed point theorem and properties of homogeneous Markov chains, we prove that u_N^k is uniquely ergodic. For the weak error, it is usually analyzed in a finite time interval $[0, T]$ and depends on T (see e.g. [7, 9]). In our cases, however, the weak error between $u_N(T)$ and $u_N^M(T)$ is required to be independent of time T and step M. Thus, some technical estimates are given to obtain the exponential decay of the difference between non-global Lipschitz nonlinear terms and between $S(t)$ and S_τ. Based on the time-independency of the weak error of the solutions, we show that the error of invariant measures has at least the same order as the weak error of the solutions.

This paper is organized as follows. In Section 2, some notations and definitions about ergodicity are introduced. In Section 3, we apply spectral Galerkin method to Eq. 1.1 and prove the ergodicity of the spatial semi-discrete scheme. The time-independent weak error of the solutions, together with the error between invariant measures, is given. Section 4 is devoted to the proof of ergodicity of the fully discrete scheme. Moreover, we give the approximation error of invariant measure in temporal direction via the time-independent weak error. In Section 5, numerical experiments are given to verify the time independence of the weak error as well as the weak order in temporal direction for the linear case. The last section is the appendix of some proofs.

2 Preliminaries

In this section, we present some notations and the definition of ergodicity. Moreover, we introduce a sufficient condition for a stochastic process to be ergodic, which will be used in our proof on ergodicity of the numerical solution.

2.1 Notations

We set the linear operator $A := -\mathbf{i}\Delta + \alpha$, and the semigroup $S(t) := e^{-tA} = e^{t(\mathbf{i}\Delta-\alpha)}$ is generated by A. The mild solution of Eq. 1.1 exists globally and can be written as

$$u(t) = S(t)u_0 + \mathbf{i}\lambda \int_0^t S(t-s)|u(s)|^2 u(s)ds + \int_0^t S(t-s)Q^{\frac{1}{2}}dW(s).$$

It is obvious that $\{\lambda_n\}_{n\in\mathbb{N}} := \left\{\mathbf{i}(n\pi)^2 + \alpha\right\}_{n\in\mathbb{N}}$ is a sequence of eigenvalues of A with $1 \le |\lambda_n| \to +\infty$ and $\{e_n\}_{n\in\mathbb{N}} := \left\{\sqrt{2}\sin n\pi x\right\}_{n\in\mathbb{N}}$ is the associated eigenbasis of A with Dirichlet boundary condition. Denoting $L_0^2(0,1)$ as the space $L^2(0,1)$ with homogenous Dirichlet boundary condition, then $\{e_n\}_{n\in\mathbb{N}}$ is an orthonormal basis of $L_0^2(0,1)$.

Definition 1 For all $s \in \mathbb{N}$, we define the normed linear space

$$\dot{H}^s := D(A^{\frac{s}{2}}) = \left\{u \middle| u = \sum_{n=1}^{\infty}(u, e_n)e_n \in L_0^2(0,1)\ s.t.\ \sum_{n=1}^{\infty}\left|(u, e_n)\right|^2|\lambda_n|^s < \infty\right\},$$

endowed with the s-norm

$$\|u\|_s := \left(\sum_{n=1}^{\infty}\left|\ (u, e_n)\ \right|^2 |\lambda_n|^s\right)^{\frac{1}{2}},$$

where the inner product in the complex Hilbert space $L^2(0,1)$ is defined by

$$(u, v) = \int_0^1 u(x)\overline{v}(x)dx,\ \ \forall u, v \in L^2(0,1).$$

In particular, $\|u\|_0 = \|u\|_{L^2}, \forall u \in \dot{H}^0$.

In the sequel, we use notations $L^2 := L^2(0,1)$ and $H^s := H^s(0,1)$. It's easy to check that the above norms satisfy $\|u\|_r \le \|u\|_s (\forall\, 0 \le r \le s)$ and $\|u\|_s \cong \|u\|_{H^s} (s = 0, 1, 2)$ for any $u \in \dot{H}^s$.

The operator norm is defined as

$$\|B\|_{\mathcal{L}(\dot{H}^s, \dot{H}^r)} = \sup_{u\in\dot{H}^s}\frac{\|Bu\|_r}{\|u\|_s},\ \forall r, s \in \mathbb{N},$$

hence, for $0 \le r \le s$,

$$\|S(t)\|_{\mathcal{L}(\dot{H}^s, \dot{H}^r)} = \sup_{u\in\dot{H}^s}\frac{\left(\sum_{n=1}^{\infty}\left|\ \left(e^{t(\mathbf{i}\Delta-\alpha)}u, e_n\right)\right|^2 |\lambda_n|^r\right)^{\frac{1}{2}}}{\|u\|_s} = \sup_{u\in\dot{H}^s}\frac{e^{-\alpha t}\|u\|_r}{\|u\|_s} \le e^{-\alpha t}.$$

We need $Q^{\frac{1}{2}}$ to be a Hilbert-Schmidt operator from L^2 to $\dot{H}^s$ with norm

$$\|Q^{\frac{1}{2}}\|^2_{\mathcal{HS}(L^2,\dot{H}^s)} := \sum_{m=1}^{\infty} \|Q^{\frac{1}{2}} e_m\|_s^2 = \sum_{m=1}^{\infty} |\lambda_m|^s \eta_m < \infty.$$

Assumptions on s will be given below.

2.2 Ergodicity

Let P_t be the Markov transition semigroup with an invariant measure μ and V be a Hilbert space. The Von Neumann theorem ensures that the limit

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T P_t\phi(y)dt, \quad \phi \in L^2(V,\mu)$$

always exists in $L^2(V,\mu)$, where y denotes the initial value of the stochastic process.

Definition 2 (see e.g. [5]) If P_t has an invariant measure μ, and in addition it happens that

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T P_t\phi(y)dt = \int_V \phi d\mu \quad in \quad L^2(V,\mu) \tag{2.1}$$

for all $\phi \in L^2(V,\mu)$. Then P_t is said to be ergodic.

Remark 1 In the following sections, we choose $P_t\phi(u_0) = E[\phi(u(t))|u(0) = u_0]$ for any deterministic initial value u_0, and take expectation of both sides of Eq. 2.1 to obtain

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T E[\phi(u)]dt = \int_V \phi d\mu \quad in \quad \mathbb{R}. \tag{2.2}$$

The sufficient conditions for a stochastic process to be ergodic are stated in the following theorem.

Theorem 2.1 *(see e.g. [5]) Let $F : V \to [0,\infty]$ be a Borel function (Lyapunov function) whose level sets*

$$L_a := \{x \in V : F(x) \le a\}$$

are compact for any $a > 0$. Assume that there exists $y \in V$ and $C(y) > 0$ such that

$$E\Big[F\big(u(t;y)\big)\Big] \le C(y) \quad for \quad all \quad t \in \mathbb{R}^+,$$

where $u(t;y)$ denotes a stochastic process whose start point is y. Then u has at least one invariant measure.

If in addition the associated semigroup P_t is strong Feller and irreducible, then u possesses a unique invariant measure. Thus, u is ergodic.

For Eq. 1.1, it is ergodic with a unique invariant measure.

Theorem 2.2 *(see [8]) There exists a unique stationary probability measure μ of $\{P_t\}_{t\in\mathbb{R}^+}$ on $H_0^1(0,1)$. Moreover, for any $p \in \mathbb{N}\backslash\{0\}$, μ satisfies*

$$\int_{H_0^1(0,1)} \|u\|_1^{2p} d\mu < \infty.$$

3 Spatial Semi-discretization

We apply spectral Galerkin method to problem (1.1) to get a spatial semi-discrete scheme which is a finite-dimensional SDE. We show that the solution u_N of Eq. 3.1 possesses a unique invariant measure μ_N, which leads to the ergodicity of u_N. Furthermore, we prove that the weak error of the spatial semi-discrete scheme does not depend on the time interval, which implies that μ_N converges to μ in at least the same rate.

3.1 Spectral Galerkin Method

The finite-dimensional spectral space is defined as

$$V_N := span\{e_m\}_{m=1}^{N}.$$

Let $\pi_N : \dot{H}^0 \to V_N$ be a projection operator, which is defined as

$$\pi_N u = \sum_{m=1}^{N} (u, e_m)e_m, \ \forall\, u = \sum_{m=1}^{\infty} (u, e_m)e_m \in \dot{H}^0.$$

We use u_N as an approximation to the original solution u, and the spatial semi-discrete scheme is expressed as

$$\begin{cases} du_N = \left(\mathbf{i}\Delta u_N - \alpha u_N + \mathbf{i}\lambda\pi_N\left(|u_N|^2 u_N\right)\right)dt + \pi_N Q^{\frac{1}{2}} dW \\ u_N(0, x) = \pi_N u_0(x), \end{cases} \tag{3.1}$$

where $\pi_N Q^{\frac{1}{2}} dW = \sum_{m=1}^{N} \sqrt{\eta_m} e_m(x) d\beta_m(t)$, and the projection operator π_N is bounded

$$\|\pi_N\|_{\mathcal{L}(\dot{H}^s, L^2)} \le 1, \quad \forall s \in \mathbb{N}.$$

3.2 Ergodicity of Spatial Semi-discrete Scheme

Theorem 3.1 *Let $u_N(t, x)$ be the solution of Eq. 3.1, then u_N possesses a unique invariant measure, denoted by μ_N. Thus, u_N is ergodic.*

Proof Following from Theorem 2.1, we need to show three properties of u_N,“strong Feller”, “irreducibility” and “Lyapunov condition”, in order to show the ergodicity of u_N. Thus the proof is divided into three parts as follows.

Part 1. Strong Feller. We transform (3.1) into its equivalent finite-dimensional SDE form. Denote $a_m(t) = \big(u_N(t, x), e_m(x)\big)$ and we have

$$u_N(t, x) = \sum_{m=1}^{N} a_m(t)e_m(x).$$

Applying the Itô's formula to $a_m(t)$ leads to

$$da_m(t) = \left[-\lambda_m a_m(t) + \left(\mathbf{i}\lambda\pi_N\left(|u_N|^2 u_N\right), e_m\right)\right]dt + \sqrt{\eta_m} d\beta_m(t), \quad 1 \le m \le N. \tag{3.2}$$

We decompose the above equation into its real and imaginary parts by denoting $a_m = a_m^1 + \mathbf{i}a_m^2$, $\lambda_m = \lambda_m^1 + \mathbf{i}\lambda_m^2$ and $\beta_m = \beta_m^1 + \mathbf{i}\beta_m^2$, where $\{\beta_m^i\}_{1\le m\le N, i=1,2}$ is a family of independent $\mathbb{R}$-valued Wiener processes and the superscripts 1 and 2 mean the real and imaginary parts of a complex number, respectively, and obtain

$$\begin{cases} da_m^1 = \Big[-\lambda_m^1 a_m^1 + \lambda_m^2 a_m^2 + Re\left(\mathbf{i}\lambda\pi_N\left(|u_N|^2 u_N\right), e_m\right)\Big]dt + \sqrt{\eta_m}d\beta_m^1(t), \\ da_m^2 = \Big[-\lambda_m^2 a_m^1 - \lambda_m^1 a_m^2 + Im\left(\mathbf{i}\lambda\pi_N\left(|u_N|^2 u_N\right), e_m\right)\Big]dt + \sqrt{\eta_m}d\beta_m^2(t). \end{cases}$$

With notations $X(t) = (a_1^1(t), a_1^2(t), \cdots, a_N^1(t), a_N^2(t))^T$, $\beta = (\beta_1^1, \beta_1^2, \cdots, \beta_N^1, \beta_N^2)^T \in \mathbb{R}^{2N}$, $F = diag\{\Lambda_1, \cdots, \Lambda_N\}$,

$$\Lambda_i = \begin{pmatrix} -\lambda_i^1 & \lambda_i^2 \\ -\lambda_i^2 & -\lambda_i^1 \end{pmatrix}, \quad G(X(t)) = \begin{pmatrix} Re\ \left(\mathbf{i}\lambda\pi_N\left(|u_N|^2 u_N\right), e_1\right) \\ Im\ \left(\mathbf{i}\lambda\pi_N\left(|u_N|^2 u_N\right), e_1\right) \\ \vdots \\ Re\ \left(\mathbf{i}\lambda\pi_N\left(|u_N|^2 u_N\right), e_N\right) \\ Im\ \left(\mathbf{i}\lambda\pi_N\left(|u_N|^2 u_N\right), e_N\right) \end{pmatrix}$$

and

$$Z = \begin{pmatrix} \sqrt{\eta_1} & & & & \\ & \sqrt{\eta_1} & & & \\ & & \ddots & & \\ & & & \sqrt{\eta_N} & \\ & & & & \sqrt{\eta_N} \end{pmatrix} := (Z_1^1, Z_1^2 \cdots, Z_N^1, Z_N^2),$$

we get an equivalent form of Eq. 3.1

$$dX(t) = \Big[FX(t) + G\big(X(t)\big)\Big]dt + \sum_{m=1}^{N}\sum_{i=1}^{2} Z_m^i d\beta_m^i := Y\left(X(t)\right)dt + \sum_{m=1}^{N}\sum_{i=1}^{2} Z_m^i d\beta_m^i.$$

It is obvious that

$$span\{Z_1^1, Z_1^2, \cdots, Z_N^1, Z_N^2\} = \mathbb{R}^{2N},$$

which means the Hörmander's condition holds. According to the Hörmander theorem [13], $X(t)$ is a strong Feller process.

Part 2. Irreducibility. By using the same notations as above, we have

$$dX = Y(X)dt + Zd\beta, \tag{3.3}$$

with $X = X(t) \in \mathbb{R}^{2N},\ X(0) = y$ and Z being invertible. Using a similar technique as [14], we consider the associated control problem

$$d\overline{X} = Y(\overline{X})dt + ZdU, \tag{3.4}$$

with $\overline{X} = \overline{X}(t)$ and a smooth control function $U \in C^1(0, T)$. For any fixed $T > 0$, $y \in \mathbb{R}^{2N}$ and $y^+ \in \mathbb{R}^{2N}$, using polynomial interpolation, we derive a continuous function $\big(\overline{X}(t),\ t \in [0, T]\big)$ such that $\overline{X}(0) = y$ and $\overline{X}(T) = y^+$. Hence,

$$dU = Z^{-1}\big(d\overline{X} - Y(\overline{X})dt\big),$$

and we get the control function U such that (3.4) is satisfied with $\overline{X}(0) = y$, $\overline{X}(T) = y^+$ and $U(0) = 0$. We subtract the resulting Eqs. 3.3 and 3.4, and achieve

$$X(t) - \overline{X}(t) = \int_0^t Y(X(s)) - Y(\overline{X}(s))ds + Z(\beta(t) - U(t)), \quad t \in [0, T].$$

According to the properties of Brownian motion,

$$P\left(\sup_{0\le t\le T} \left|\beta(t) - U(t)\right| \le \epsilon\right) > 0, \quad \forall \epsilon > 0.$$

Note that Y is locally Lipschitz because of its continuous differentiability, and the ranges of $X(t)$ and $\overline{X}(t)$ ($t \in [0, T]$) are both compact sets. Thus, it holds

$$P\left(\left|X(t) - \overline{X}(t)\right| \le \int_0^t C_1\left|X(s) - \overline{X}(s)\right|ds + C_2\epsilon, \quad \forall\, t \in [0, T]\right) > 0, \quad \forall\, \epsilon > 0$$

with C_1 and C_2 are positive constants independent of ϵ. Then the Grönwall's inequality yields

$$P\left(\left|X(t) - \overline{X}(t)\right| \le C_2(1 + e^{C_1 t})\epsilon, \quad \forall\, t \in [0, T]\right) > 0, \quad \forall\, \epsilon > 0.$$

For any $\delta > 0$, choosing $t = T$ and $\epsilon = \delta/C_2(1 + e^{C_1 T}) > 0$, we finally obtain

$$P\left(|X(T) - y^+| < \delta\right) > 0.$$

In other words, $X(T)$ hits $B(y^+, \delta)$ with positive probability. The irreducibility has been proved.

The above two conditions ensure the uniqueness of the invariant measure of $X(t)$. It suffices to show the existence of invariant measures in the following.

Part 3. Lyapunov condition. A useful tool for proving existence of invariant measures is provided by Lyapunov functions, which is introduced in Theorem 2.1. Itô's formula applied to $\|u_N(t)\|_0^2$ implies that

$$d\|u_N(t)\|_0^2 = -2\alpha\|u_N(t)\|_0^2 dt + 2Re\int_0^1 \overline{u}_N(t)\pi_N Q^{\frac{1}{2}} dx dW(t) + 2\sum_{m=1}^{N} \eta_m dt, \tag{3.5}$$

where we have used the fact that

$$Re\left[\mathbf{i}\lambda \int_0^1 \pi_N(|u_N|^2 u_N)\overline{u}_N dx\right] = Re\left[\mathbf{i}\lambda \int_0^1 \left(|u_N|^4 - (Id - \pi_N)(|u_N|^2 u_N)\overline{u}_N\right) dx\right]$$
$$= -\lambda Im\left((Id - \pi_N)(|u_N|^2 u_N), u_N\right) = 0.$$

Taking expectation on both sides of Eq. 3.5, we get

$$\frac{d}{dt}E\|u_N(t)\|_0^2 = -2\alpha E\|u_N(t)\|_0^2 + C_N,$$

where $C_N = 2\sum_{m=1}^{N} \eta_m \le 2\eta$. It is solved as

$$E\|u_N(t)\|_0^2 = e^{-2\alpha t}\left(\int_0^t C_N e^{2\alpha s} ds + E\|u_N(0)\|_0^2\right) \le e^{-2\alpha t}E\|u_N(0)\|_0^2 + C, \ \forall t > 0.$$

On the other hand,

$$\|u_N(t)\|_0^2 = \int_0^1 \left|\sum_{m=1}^{N} a_m(t)e_m(x)\right|^2 dx = \|X(t)\|_{l^2(\mathbb{R}^{2N})}^2.$$

Define $F = \|\cdot\|_{l^2(\mathbb{R}^{2N})} : \mathbb{R}^{2N} \to [0, +\infty]$. The level sets of F are tight by Heine-Borel theorem. Therefore, $X(t)$ is ergodic. We mention that the ergodicity of $X(t)$ is equivalent to the existence of a random variable $\xi = (\xi_1^1, \xi_1^2, \cdots, \xi_N^1, \xi_N^2)$ such that

$$\lim_{t\to\infty} X(t) = \xi, \text{ i.e., } \lim_{t\to\infty} a_m^i(t) = \xi_m^i, \ \forall\, m = 1, \cdots, N, \ i = 1, 2.$$

It leads to

$$\lim_{t\to\infty} u_N(t) = \sum_{m=1}^{N} \left(\xi_m^1 + \mathbf{i}\xi_m^2\right) e_m,$$

which shows the ergodicity of $u_N(t)$. □

According to the proof of Lyapunov condition, we have the following uniform boundedness for 0-norm. Moreover, 1-norm and 2-norm are also uniformly bounded, which is stated in the following proposition. Its proof is given in Appendix "The Proof of Proposition 3.1" for readers' convenience. In sequel, all the constants C are independent of the end point T of time interval and may be different from line to line.

Proposition 3.1 *Assume that $u_0 \in \dot{H}^1$, $\|Q^{\frac{1}{2}}\|_{\mathcal{HS}(L^2,\dot{H}^1)} < \infty$ and $p \geq 1$. There exists positive constants c_0 and $C = C(\alpha, p, u_0, c_0, Q)$, such that for any $t > 0$,*

$$\begin{aligned} i) \quad & E\|u_N(t)\|_0^{2p} \leq e^{-2\alpha p t} E\|u_N(0)\|_0^{2p} + C \leq C, \\ ii) \quad & E\mathcal{H}(u_N(t))^p \leq e^{-\alpha p t} E\mathcal{H}(u_N(0))^p + C \leq C, \end{aligned}$$

where $\mathcal{H}(u_N(t)) = \frac{1}{2}\|\nabla u_N(t)\|_0^2 - \frac{\lambda}{4}\|u_N(t)\|_{L^4}^4 + c_0\|u_N(t)\|_0^6$. In addition, if we assume further $u_0 \in \dot{H}^2$ and $\|Q^{\frac{1}{2}}\|_{\mathcal{HS}(L^2,\dot{H}^2)} < \infty$, we also have

$$iii) \ E\|u_N(t)\|_2^2 \leq C.$$

Remark 2 The uniform boundedness of the original solution u can also be obtained in the same procedure as Proposition 3.1 or [8]. As the $\dot{H}^2$-regularity for both the original solution and numerical solutions are essential to obtain the time-independent weak error, we need the assumption $u_0 \in \dot{H}^2$ and $\|Q^{\frac{1}{2}}\|_{\mathcal{HS}(L^2,\dot{H}^2)} < \infty$ in the error analysis.

3.3 Weak Error between Solutions u and u_N

Weak convergence is established for the spatial semi-discretization (3.1) in this section utilizing a transformation of $u_N(t)$ and the corresponding Kolmogorov equation.

Theorem 3.2 *Assume that $u_0 \in \dot{H}^2$ and $\|Q^{\frac{1}{2}}\|_{\mathcal{HS}(L^2,\dot{H}^2)} < \infty$. For any $\phi \in C_b^2(L^2)$, there exists a constant $C = C(u_0, \phi, Q)$ independent of T, such that for any $T > 0$,*

$$\left| E\left[\phi\big(u_N(T)\big)\right] - E\left[\phi\big(u(T)\big)\right] \right| \leq C N^{-2}.$$

Before the proof of Theorem 3.2, we give a useful lemma.

Lemma 1 *Assume that $S(t)$ and π_N are defined as before. We have the following estimation*

$$\|S(t) - S(t)\pi_N\|_{\mathcal{L}(\dot{H}^s, L^2)} \le Ce^{-\alpha t}N^{-s}.$$

Proof For any $u \in \dot{H}^s$, we have

$$\|S(t)u - S(t)\pi_N u\|_0 = e^{-\alpha t}\|u - \pi_N u\|_0 = e^{-\alpha t}\left(\sum_{n=N+1}^{\infty} |(u, e_n)|^2\right)^{\frac{1}{2}}$$

$$\le e^{-\alpha t}|\lambda_N|^{-\frac{s}{2}}\left(\sum_{n=N+1}^{\infty} |\lambda_n|^s |(u, e_n)|^2\right)^{\frac{1}{2}} \le Ce^{-\alpha t}N^{-s}\|u\|_s.$$

□

Proof of Theorem 3.2 We split the proof in three steps.

Step 1. Calculation of $E\,[\phi(u(T))]$.

To eliminate the unbounded Laplacian operator, we consider the modified process $Y(t) = S(T-t)u(t)$, $t \in [0, T]$, which is the solution of the following SPDE

$$\begin{aligned} dY(t) &= \mathbf{i}\lambda S(T-t)\Big[|S(t-T)Y(t)|^2 S(t-T)Y(t)\Big]dt + S(T-t)Q^{\frac{1}{2}}dW \\ &:= H(Y(t))dt + S(T-t)Q^{\frac{1}{2}}dW. \end{aligned}$$

Denote $v(T-t, y) := E[\phi(Y(T))|Y(t) = y]$ and it follows easily

$$\frac{\partial v(T-t, y)}{\partial t} = -\Big(Dv(T-t, y), H(y)\Big) - \frac{1}{2}Tr\Big[(S(T-t)Q^{\frac{1}{2}})^* D^2 v(T-t, y) S(T-t)Q^{\frac{1}{2}}\Big].$$

Note that the mild solution of u has the expression $u(T) = S(T-t)u(t) + \mathbf{i}\lambda\int_t^T S(T-s)|u|^2 u ds + \int_t^T S(T-s)Q^{\frac{1}{2}}dW$. Thus, we have

$$\begin{aligned} v(T-t, y) &= E[\phi(Y(T))|Y(t) = y] = E[\phi(u(T))|u(t) = S(t-T)y] \\ &= E\left[\phi\left(y + \mathbf{i}\lambda\int_t^T S(T-s)|u(s)|^2 u(s)ds + \int_t^T S(T-s)Q^{\frac{1}{2}}dW\right)\right]. \end{aligned}$$

For any $h \in L^2$, similar to [7] (Lemma 5.13), we have

$$(Dv(T-t, y), h) = E\left[\left(D\phi\left(y + \mathbf{i}\lambda\int_t^T S(T-s)|u(s)|^2 u(s)ds + \int_t^T S(T-s)Q^{\frac{1}{2}}dW\right), \chi^h(t)\right)\right]$$

with $\chi^h(t) = h + \mathbf{i}\lambda\int_t^T S(T-s)\left(2|u(s)|^2\chi^h(s) + u^2(s)\overline{\chi^h(s)}\right)ds$. It's easy to obtain that

$$\|\chi^h(t)\|_0 \le \|h\|_0 + C\int_t^T e^{-\alpha(T-s)}\|u(s)\|_1^2\|\chi^h(s)\|_0 ds. \tag{3.6}$$

To show the uniform boundedness of $E\|\chi^h(t)\|_0$, we define a family of subsets

$$K_m := \left\{\omega \in \Omega \;\middle|\; \sup_{t \le s \le T}\|u(s)\|_1 > m(T+1-t)^{\frac{1}{2}}\right\}, \quad m \in \mathbb{N}$$

for any $t \leq T$. We claim that $E\left(\sup_{t\leq s\leq T}\|u(s)\|_1^2\right) \leq C + C(T-t)$. In fact, we can deduce

$$d\mathcal{H}(u(t)) \leq -\frac{3}{2}\alpha\mathcal{H}(u(t)) + Cdt + dM_*(t)$$

similar to Proposition 3.1 or [8], which implies

$$\mathcal{H}(u(s)) \leq e^{-\frac{3}{2}\alpha(s-t)}\mathcal{H}(u(t)) + \int_t^s Ce^{-\frac{3}{2}\alpha(s-r)}dr + \int_t^s e^{-\frac{3}{2}\alpha(s-r)}dM_*(r)$$

with $dM_* := 6c_0\|u\|_0^4 Re\left(u, Q^{\frac{1}{2}}dW\right) - Re\left(\Delta u + \lambda|u|^2u, Q^{\frac{1}{2}}dW\right)$ and $E\mathcal{H}(u(t)) \leq C$. Taking supremum and expectation, we get

$$\begin{aligned} E\left[\sup_{t\leq s\leq T}\mathcal{H}(u(s))\right] &\leq E\mathcal{H}(u(t)) + C(T-t) + E\left[\sup_{t\leq s\leq T}\int_t^s e^{-\frac{3}{2}\alpha(s-r)}dM_*(r)\right] \\ &\leq C + C(T-t), \end{aligned}$$

where in the last step we have used the Doob's inequality for convolution integrals (see [16], Theorem 2). This complete the proof of the claim. Then the Chebyshev's inequality (see e.g. [10]) yields that

$$P\left(K_m\right) \leq \frac{E\left(\sup_{t\leq s\leq T}\|u(s)\|_1^2\right)}{m^2(T+1-t)} \leq \frac{C+C(T-t)}{m^2(T+1-t)} \leq \frac{C}{m^2}, \quad \forall\, t \leq T.$$

As $\sum\limits_{m=1}^{\infty} P(K_m) \leq \sum\limits_{m=1}^{\infty}\frac{C}{m^2} < \infty$, we get $P(\cap_{n=1}^{\infty}\cup_{m=n}^{\infty}K_m) = 0$ based on the Borel-Cantelli Lemma (see e.g. [10]). It implies that there exists a constant $M_* \in \mathbb{N}$, for any $m \geq M_*$, $\|u(t)\|_1 \leq \sup\limits_{t\leq s\leq T}\|u(s)\|_1 \leq m(T+1-t)^{\frac{1}{2}}$ almost surely. Then the backward Grönwall's inequality applied to Eq. 3.6 yields $E\|\chi^h(t)\|_0 \leq C\|h\|_0$ thanks to the exponential decay factor, and it holds

$$|(Dv(T-t,y),h)| \leq \|\phi\|_{C_b^1}E\|\chi^h(t)\|_0 \leq C\|\phi\|_{C_b^1}\|h\|_0. \tag{3.7}$$

Similarly, we also have

$$\left|\left(\left(D^2v(T-t,y),h\right),h\right)\right| \leq C\|\phi\|_{C_b^2}\|h\|_0^2. \tag{3.8}$$

The Itô's formula gives that

$$\begin{aligned} dv(T-t,Y(t)) &= \frac{\partial v}{\partial t}(T-t,Y(t))dt + \Big(Dv\left(T-t,Y(t)\right), H\left(Y(t)\right)dt \\ &\quad + S(T-t)Q^{\frac{1}{2}}dW(t)\Big) \\ &\quad + \frac{1}{2}Tr\left[(S(T-t)Q^{\frac{1}{2}})^*D^2v\left(T-t,Y(t)\right)S(T-t)Q^{\frac{1}{2}}\right]dt \\ &= \left(Dv(T-t,Y(t)), S(T-t)Q^{\frac{1}{2}}dW(t)\right). \end{aligned}$$

Therefore,

$$v(0,Y(T)) = v(T,Y(0)) + \int_0^T\left(Dv(T-s,Y(s)), S(T-s)Q^{\frac{1}{2}}dW(s)\right). \tag{3.9}$$

Noticing that $Y(0) = S(T)u_0$ and $Y(T) = u(T)$, we recall $v(T-t, y) = E[\phi(Y(T))|Y(t) = y]$ to derive

$$v(0, Y(T)) = E\left[\phi(u(T))|Y(T) = u(T)\right]$$

and

$$\begin{aligned} v(T, Y(0)) &= E\left[\phi(Y(T))|Y(0) = S(T)u_0\right] \\ &= E\left[\phi\Big(S(T)u_0 + \int_0^T H(Y(t))dt + S(T-t)Q^{\frac{1}{2}}dW(t)\Big)\Big|Y(0) = S(T)u_0\right]. \end{aligned}$$

Take expectation of both sides of Eq. 3.9 and we have

$$E[\phi(u(T))] = E\left[\phi\Big(S(T)u_0 + \int_0^T H(Y(t))dt + S(T-t)Q^{\frac{1}{2}}dW(t)\Big)\right]. \tag{3.10}$$

Step 2. Calculation of $E\left[\phi(u_N(T))\right]$.

The mild solution of Eq. 3.1 is

$$u_N(t) = S(t)\pi_N u_0 + \mathbf{i}\lambda \int_0^t S(t-s)\pi_N \left(|u_N(s)|^2 u_N(s)\right) ds + \int_0^t S(t-s)\pi_N Q^{\frac{1}{2}} dW(s).$$

Using similar argument as above, we consider the following stochastic process:

$$Y_N(t) = S(T-t)u_N(t).$$

The relevant SDE is

$$\begin{aligned} dY_N(t) &= \mathbf{i}\lambda S(T-t)\pi_N\Big[|S(t-T)Y_N(t)|^2 S(t-T)Y_N(t)\Big]dt + S(T-t)\pi_N Q^{\frac{1}{2}}dW \\ &:= H_N(Y_N(t))dt + S(T-t)\pi_N Q^{\frac{1}{2}}dW(t). \end{aligned}$$

Apply Itô's formula to $t \to v(T-t, Y_N(t))$ and we get

$$\begin{aligned} dv(T-t, Y_N(t)) &= \frac{\partial v}{\partial t}(T-t, Y_N(t))dt \\ &\quad + \Big(Dv(T-t, Y_N(t)), H_N(Y_N(t))dt + S(T-t)\pi_N Q^{\frac{1}{2}}dW(t)\Big) \\ &\quad + \frac{1}{2}Tr\Big[(S(T-t)\pi_N Q^{\frac{1}{2}})^* D^2 v(T-t, Y_N(t))S(T-t)\pi_N Q^{\frac{1}{2}}\Big]dt \\ &= \Big(Dv(T-t, Y_N(t)), S(T-t)\pi_N Q^{\frac{1}{2}}dW(t)\Big) \\ &\quad + \Big(Dv(T-t, Y_N(t)), H_N\left(Y_N(t)\right) - H\left(Y_N(t)\right)\Big)dt \\ &\quad - \frac{1}{2}Tr\Big[(S(T-t)Q^{\frac{1}{2}})^* D^2 v(T-t, Y_N(t))S(T-t)Q^{\frac{1}{2}}\Big]dt \\ &\quad + \frac{1}{2}Tr\Big[(S(T-t)\pi_N Q^{\frac{1}{2}})^* D^2 v(T-t, Y_N(t))S(T-t)\pi_N Q^{\frac{1}{2}}\Big]dt. \end{aligned}$$

Therefore,

$$
\begin{aligned}
v(0, Y_N(T)) &= v(T, Y_N(0)) + \int_0^T \Big(Dv(T-s, Y_N(s)), S(T-s)\pi_N Q^{\frac{1}{2}} dW(s) \Big) \\
&\quad + \int_0^T \Big(Dv\big(T-t, Y_N(t)\big), H_N\big(Y_N(t)\big) - H\big(Y_N(t)\big) \Big) dt \\
&\quad + \frac{1}{2} \int_0^T Tr \Big[(S(T-t)\pi_N Q^{\frac{1}{2}})^* D^2 v(T-t, Y_N(t)) S(T-t)\pi_N Q^{\frac{1}{2}} \Big] dt \\
&\quad - \frac{1}{2} \int_0^T Tr \Big[(S(T-t) Q^{\frac{1}{2}})^* D^2 v(T-t, Y_N(t)) S(T-t) Q^{\frac{1}{2}} \Big] dt. \qquad (3.11)
\end{aligned}
$$

By the construction of Y_N, we can check that

$$
Y_N(0) = S(T)\pi_N u_0 \quad \text{and} \quad Y_N(T) = u_N(T).
$$

According to the representation of v, we have

$$
v(0, Y_N(T)) = E\left[\phi(Y(T)) | Y(T) = Y_N(T)\right] = E\left[\phi(u_N(T)) | Y(T) = Y_N(T)\right]
$$

and

$$
\begin{aligned}
v(T, Y_N(0)) &= E\left[\phi(Y(T)) | Y(0) = S(T)\pi_N u_0\right] \\
&= E\Big[\phi\Big(S(T)\pi_N u_0 + \int_0^T H(Y(t)) dt \\
&\qquad + S(T-t) Q^{\frac{1}{2}} dW(t) \Big) \Big| Y(0) = S(T)\pi_N u_0 \Big].
\end{aligned}
$$

Take expectation of the two sides of Eq. 3.11 and we get

$$
\begin{aligned}
E\left[\phi(u_N(T))\right] &= E\Big[\phi\Big(S(T)\pi_N u_0 + \int_0^T H(Y(t)) dt + S(T-t) Q^{\frac{1}{2}} dW(t) \Big) \Big] \\
&\quad + E \int_0^T \Big(Dv\big(T-t, Y_N(t)\big), H_N\big(Y_N(t)\big) - H\big(Y_N(t)\big) \Big) dt \\
&\quad + \frac{1}{2} E \int_0^T \Big\{ Tr \Big[(S(T-t)\pi_N Q^{\frac{1}{2}})^* D^2 v(T-t, Y_N(t)) S(T-t)\pi_N Q^{\frac{1}{2}} \Big] \\
&\quad - Tr \Big[(S(T-t) Q^{\frac{1}{2}})^* D^2 v(T-t, Y_N(t)) S(T-t) Q^{\frac{1}{2}} \Big] \Big\} dt. \qquad (3.12)
\end{aligned}
$$

Step 3. Weak error of the solutions.

Subtracting the resulting Eqs. 3.10 and 3.12 leads to

$$
\begin{aligned}
& E\left[\phi(u_N(T))\right]-E\left[\phi(u(T))\right] \\
=\ & E\Big[\phi\Big(S(T)\pi_N u_0+\int_0^T H(Y(t))dt+S(T-t)Q^{\frac{1}{2}}dW(t)\Big) \\
& -\phi\Big(S(T)u_0+\int_0^T H(Y(t))dt+S(T-t)Q^{\frac{1}{2}}dW(t)\Big)\Big] \\
& +E\int_0^T\Big(Dv\big(T-t,Y_N(t)\big),H_N\big(Y_N(t)\big)-H\big(Y_N(t)\big)\Big)dt \\
& +\frac{1}{2}E\int_0^T\Big\{Tr\Big[(S(T-t)\pi_N Q^{\frac{1}{2}})^*D^2v(T-t,Y_N(t))S(T-t)\pi_N Q^{\frac{1}{2}}\Big] \\
& -Tr\Big[(S(T-t)Q^{\frac{1}{2}})^*D^2v(T-t,Y_N(t))S(T-t)Q^{\frac{1}{2}}\Big]\Big\}dt \\
:=\ & I+II+III.
\end{aligned}
\tag{3.13}
$$

Due to Lemma 1, terms I and II can be estimated as

$$
|I|\le C\,\|\phi\|_{C_b^1}\,E\,\|S(T)u_0-S(T)\pi_N u_0\|_0\le Ce^{-\alpha T}\|\phi\|_{C_b^1}E\|u_0\|_2N^{-2}\le Ce^{-\alpha T}N^{-2},
\tag{3.14}
$$

and

$$
\begin{aligned}
|II|\ &\le CE\int_0^T\|\phi\|_{C_b^1}\|H_N(Y_N(t))-H(Y_N(t))\|_0dt \\
&= CE\int_0^T\|\phi\|_{C_b^1}\|\mathbf{i}\lambda S(T-t)(Id-\pi_N)\big(|u_N(t)|^2u_N(t)\big)\|_0dt \\
&\le |\lambda|C\int_0^T e^{-\alpha(T-t)}\|\phi\|_{C_b^1}E\Big[\|u_N(t)\|_1^2\|u_N(t)\|_2\Big]N^{-2}dt \\
&\le |\lambda|\frac{C}{\alpha}N^{-2}
\end{aligned}
\tag{3.15}
$$

based on Lemma 1, Proposition 3.1 and the embedding $H^1\hookrightarrow L^\infty$ in $\mathbb{R}$. In the first step of Eq. 3.15, we have used the fact (3.7).

Let us now estimate term III. As $(S(T-t)\pi_N-S(T-t))Q^{\frac{1}{2}}$ is a bounded linear operator and so is D^2v shown in Eq. 3.8, we have

$$
\begin{aligned}
& \Big|Tr\Big[(S(T-t)\pi_N Q^{\frac{1}{2}})^*D^2v(T-t,Y_N(t))S(T-t)\pi_N Q^{\frac{1}{2}}\Big] \\
& -Tr\Big[(S(T-t)Q^{\frac{1}{2}})^*D^2v(T-t,Y_N(t))S(T-t)Q^{\frac{1}{2}}\Big]\Big| \\
=\ & \Big|Tr\Big[((S(T-t)\pi_N-S(T-t))Q^{\frac{1}{2}})^*D^2v(T-t,Y_N(t))(S(T-t)\pi_N+S(T-t))Q^{\frac{1}{2}}\Big]\Big| \\
\le\ & C\|S(T-t)\pi_N-S(T-t)\|_{\mathcal{L}(\dot{H}^2,L^2)}\|Q^{\frac{1}{2}}\|_{\mathcal{HS}(L^2,\dot{H}^2)}\|\phi\|_{C_b^2}\|S(T-t)\|_{\mathcal{L}(L^2,L^2)}\|Q^{\frac{1}{2}}\|_{\mathcal{HS}(L^2,L^2)} \\
\le\ & Ce^{-\alpha(T-t)}N^{-2}.
\end{aligned}
$$

Hence, integrating above equation leads to

$$
|III|\le\frac{C}{\alpha}N^{-2}.
\tag{3.16}
$$

Plugging (3.14), (3.15) and (3.16) into (3.13), we get

$$\left|E\Big[\phi\big(u_N(T)\big)\Big] - E\Big[\phi\big(u(T)\big)\Big]\right| \le C(e^{-\alpha T} + \frac{1}{\alpha})N^{-2} \le CN^{-2}, \tag{3.17}$$

in which, C is independent of time T. □

3.4 Convergence Order between Invariant Measures μ and μ_N

Based on the ergodicity of stochastic processes u and u_N, for any deterministic $u_0 \in \dot{H}^2$, we have the following two equations

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T E\phi\big(u(t)\big)dt = \int_{L^2} \phi(y)d\mu(y),$$
$$\lim_{T\to\infty} \frac{1}{T}\int_0^T E\phi\big(u_N(t)\big)dt = \int_{V_N} \phi(y)d\mu_N(y)$$

for any $\phi \in C_b^2(L^2)$. Due to the time-independence of the weak error in Theorem 3.2, it turns out for any fixed α and N,

$$\left|\int_{L^2} \phi(y)d\mu(y) - \int_{V_N} \phi(y)d\mu_N(y)\right| = \left|\lim_{T\to\infty} \frac{1}{T}\int_0^T E\phi\big(u(t)\big) - E\phi\big(u_N(t)\big)dt\right|$$
$$\le \lim_{T\to\infty} \frac{1}{T}\int_0^T \left|E\phi\big(u(t)\big) - E\phi\big(u_N(t)\big)\right| dt \le \lim_{T\to\infty} \frac{1}{T}\int_0^T C(e^{-\alpha t} + \frac{1}{\alpha})N^{-2}dt \le \frac{C}{\alpha}N^{-2},$$

which implies that μ_N is a proper approximation of μ. Thus, we give the following theorem.

Theorem 3.3 *Assume that $u_0 \in \dot{H}^2$ and $\|Q^{\frac{1}{2}}\|_{\mathcal{HS}(L^2,\dot{H}^3)} < \infty$. The error between invariant measures μ and μ_N is of order 2, i.e.,*

$$\left|\int_{L^2} \phi(y)d\mu(y) - \int_{V_N} \phi(y)d\mu_N(y)\right| < \frac{C}{\alpha}N^{-2}.$$

Remark 3 Although the time-independent weak error between u and u_N is obtained under the assumption $\|Q^{\frac{1}{2}}\|_{\mathcal{HS}(L^2,\dot{H}^2)} < \infty$, it is necessary to assume in addition $\|Q^{\frac{1}{2}}\|_{\mathcal{HS}(L^2,\dot{H}^3)} < \infty$ in order to get the unique ergodicity of u (see [8]).

4 Full Discretization

In this section, we discretize (3.1) in temporal direction by a modification of the implicit Euler scheme to get a fully discrete scheme. We prove the ergodicity of the numerical solution u_N^k of the fully discrete scheme, and get weak order $\frac{1}{2}$ of u_N^k in temporal direction. Thus, we achieve at least the same order as the weak error for the error of invariant measure, as a result of the time-independency of the weak error and the ergodicity of the solution.

4.1 Fully Discrete Scheme

We use a modified implicit Euler scheme to approximate (3.1), and obtain the following scheme

$$\begin{cases} u_N^k - e^{-\alpha\tau} u_N^{k-1} = \left(\mathbf{i}\Delta u_N^k + \mathbf{i}\lambda\pi_N \left(\dfrac{|u_N^k|^2 + |e^{-\alpha\tau} u_N^{k-1}|^2}{2} u_N^k \right) \right) \tau + \pi_N Q^{\frac{1}{2}} \delta W_k \\ u_N^0 = \pi_N u_0(x), \end{cases} \tag{4.1}$$

where u_N^k is an approximation of $u_N(t_k)$, τ represents the uniform time step, $t_k = k\tau$, and $\delta W_k = W(t_k) - W(t_{k-1})$.

The well-posedness of scheme (4.1), together with the uniform boundedness of the numerical solution, is stated in the following proposition. The time step τ is assumed to satisfy $\alpha\tau \in [0, 1]$ in sequel.

Proposition 4.1 *Assume $u_0 \in \dot{H}^0$. For sufficiently small τ, there uniquely exists a family of V_N-valued and $\{\mathcal{F}_{t_k}\}_{k\in\mathbb{N}}$-adapted solutions $\{u_N^k\}_{k\in\mathbb{N}}$ of Eq. 4.1, which satisfies that for any integer $p \geq 2$, there exists a constant $C = C(p, \alpha, u_N^0) > 0$, such that*

$$E\|u_N^k\|_0^p \leq C, \quad \forall\, k \in \mathbb{N}.$$

Proof **Step 1.** Existence and uniqueness of solution.

Similar to [6], we fix a family $\{g_k\}_{k\in\mathbb{N}}$ of deterministic functions in V_N. We also fix $\tilde{u}_N^{k-1} \in V_N$, the existence of solution $\tilde{u}_N^k \in V_N$ of

$$\tilde{u}_N^k - e^{-\alpha\tau} \tilde{u}_N^{k-1} = \mathbf{i}\tau\Delta\tilde{u}_N^k + \mathbf{i}\lambda\tau\pi_N \left(\frac{|\tilde{u}_N^k|^2 + |e^{-\alpha\tau} \tilde{u}_N^{k-1}|^2}{2} \tilde{u}_N^k \right) + \sqrt{\tau} g_k \tag{4.2}$$

can be proved by using Brouwer fixed point theorem. Indeed, multiplying (4.2) by $\bar{\tilde{u}}_N^k$, integrating with respect to x and taking the real part, we get

$$\begin{aligned} &\|\tilde{u}_N^k\|_0^2 + \|\tilde{u}_N^k - e^{-\alpha\tau}\tilde{u}_N^{k-1}\|_0^2 - e^{-2\alpha\tau}\|\tilde{u}_N^{k-1}\|_0^2 \\ &= 2\sqrt{\tau} Re \left[\int_0^1 (\bar{\tilde{u}}_N^k - e^{-\alpha\tau}\bar{\tilde{u}}_N^{k-1}) g_k dx + \int_0^1 (e^{-\alpha\tau}\bar{\tilde{u}}_N^{k-1}) g_k dx \right] \\ &\leq \|\tilde{u}_N^k - e^{-\alpha\tau}\tilde{u}_N^{k-1}\|_0^2 + e^{-2\alpha\tau}\|\tilde{u}_N^{k-1}\|_0^2 + 2\tau\|g_k\|_0^2, \end{aligned}$$

i.e.,

$$\|\tilde{u}_N^k\|_0^2 \leq 2e^{-2\alpha\tau}\|\tilde{u}_N^{k-1}\|_0^2 + 2\tau\|g_k\|_0^2. \tag{4.3}$$

Define

$$\begin{aligned} \Lambda : V_N \times V_N &\to \mathcal{P}(L^2), \\ (\tilde{u}_N^{k-1}, g_k) &\mapsto \{\tilde{u}_N^k | \tilde{u}_N^k \text{ are solutions of (42)}\}, \end{aligned}$$

where $\mathcal{P}(L^2)$ is the power set of L^2. Equation 4.3 implies that Λ is continuous, and its graph is closed by the closed graph theorem. When the spaces are endowed with their Borel σ-algebras, there is a measurable continuous function $\kappa : V_N \times V_N \to L^2$ such that

$$\kappa(u, g) \in \Lambda(u, g), \ \forall\, (u, g) \in V_N \times V_N.$$

Assume that $u_N^{k-1} \in V_N$ is a $\mathcal{F}_{t_{k-1}}$-measurable random variable, then $u_N^k = \kappa(u_N^{k-1}, \frac{\pi_N Q^{\frac{1}{2}}\delta W_k}{\sqrt{\tau}})$ is an L^2-valued solution of Eq. 4.1. Moreover,

$$(1-\mathbf{i}\Delta\tau)u_N^k = e^{-\alpha\tau}u_N^{k-1} + \mathbf{i}\lambda\tau\pi_N\left(\frac{|u_N^k|^2+|e^{-\alpha\tau}u_N^{k-1}|^2}{2}u_N^k\right) + \pi_N Q^{\frac{1}{2}}\delta W_k \in V_N.$$

Hence, u_N^k is actually a V_N-valued solution of Eq. 4.1.

For any given u_N^{k-1} and sufficiently small time step τ, the solution u_N^k is unique, which can be proved in a similar procedure as [2]. This fact will be used in proving the ergodicity of the numerical solution $\{u_N^k\}_{k\in\mathbb{N}}$, and it can be found in Appendix "The Proof of Uniqueness of the Solution for Eq. 4.1".

Step 2. Boundedness of the p-moments.

The constants C below may be different, but do not depend on time.

i) $p=2$. To show the boundedness, we multiply (4.1) by $\overline{u}_N^k$, integrate in [0,1] with respect to the space variable, take expectation and take the real part,

$$E\|u_N^k\|_0^2 + E\|u_N^k - e^{-\alpha\tau}u_N^{k-1}\|_0^2 - e^{-2\alpha\tau}E\|u_N^{k-1}\|_0^2 = 2ReE\int_0^1 \overline{u}_N^k\pi_N Q^{\frac{1}{2}}\delta W_k dx$$
$$= 2ReE\int_0^1 (\overline{u}_N^k - e^{-\alpha\tau}\overline{u}_N^{k-1})\pi_N Q^{\frac{1}{2}}\delta W_k dx \le E\|u_N^k - e^{-\alpha\tau}u_N^{k-1}\|_0^2 + E\|\pi_N Q^{\frac{1}{2}}\delta W_k\|_0^2.$$

It derives

$$\begin{aligned} E\|u_N^k\|_0^2 &\le e^{-2\alpha\tau}E\|u_N^{k-1}\|_0^2 + C\tau \le e^{-2\alpha\tau k}E\|u_N^0\|_0^2 + C\tau(1+e^{-2\alpha\tau}+\cdots+e^{-2\alpha\tau(k-1)}) \\ &\le e^{-2\alpha t_k}E\|u_N^0\|_0^2 + \frac{C\tau}{1-e^{-2\alpha\tau}} \le E\|u_N^0\|_0^2 + \frac{C}{e^{-1}2\alpha} \end{aligned}$$

for $\tau < \frac{1}{\alpha}$, where we have used $e^{-2\alpha\tau} < 1 - e^{-1}2\alpha\tau$ for $\tau < \frac{1}{\alpha}$.

ii) $p=4$. In the case when p=2, without taking expectation, we have

$$\|u_N^k\|_0^2 - e^{-2\alpha\tau}\|u_N^{k-1}\|_0^2 + \|u_N^k - e^{-\alpha\tau}u_N^{k-1}\|_0^2 = 2Re\int_0^1 \overline{u}_N^k\pi_N Q^{\frac{1}{2}}\delta W_k dx.$$

Multiply both sides by $\|u_N^k\|_0^2$, take expectation and take the real part and we get

$$\begin{aligned} (LHS) &= E\|u_N^k\|_0^4 - e^{-2\alpha\tau}E\|u_N^{k-1}\|_0^2\|u_N^k\|_0^2 + E\Big[\|u_N^k - e^{-\alpha\tau}u_N^{k-1}\|_0^2\|u_N^k\|_0^2\Big] \\ &= \frac{1}{2}\Big(E\|u_N^k\|_0^4 - e^{-4\alpha\tau}E\|u_N^{k-1}\|_0^4\Big) + \frac{1}{2}E\Big(\|u_N^k\|_0^2 - e^{-2\alpha\tau}\|u_N^{k-1}\|_0^2\Big)^2 \\ &\quad + E\Big[\|u_N^k - e^{-2\alpha\tau}u_N^{k-1}\|_0^2\|u_N^k\|_0^2\Big] \end{aligned}$$

and

$$
\begin{aligned}
(RHS) &= 2ReE\int_0^1 \|u_N^k\|_0^2 \overline{u}_N^k \pi_N Q^{\frac{1}{2}}\delta W_k dx\\
&= 2ReE\int_0^1 \left(\|u_N^k\|_0^2(\overline{u}_N^k - e^{-\alpha\tau}\overline{u}_N^{k-1})\right)\pi_N Q^{\frac{1}{2}}\delta W_k dx\\
&\quad +2ReE\int_0^1 \left((\|u_N^k\|_0^2 - e^{-2\alpha\tau}\|u_N^{k-1}\|_0^2)e^{-\alpha\tau}\overline{u}_N^{k-1}\right)\pi_N Q^{\frac{1}{2}}\delta W_k dx\\
&\le E\Big[\|u_N^k - e^{-\alpha\tau}u_N^{k-1}\|_0^2\|u_N^k\|_0^2\Big] + E\Big(\|u_N^k\|_0^2\|\pi_N Q^{\frac{1}{2}}\delta W_k\|_0^2\Big)\\
&\quad +\frac{1}{4}E\Big(\|u_N^k\|_0^2 - e^{-2\alpha\tau}\|u_N^{k-1}\|_0^2\Big)^2 + 4e^{-2\alpha\tau}E\|\overline{u}_N^{k-1}\pi_N Q^{\frac{1}{2}}\delta W_k\|_0^2\\
&\le E\Big[\|u_N^k - e^{-\alpha\tau}u_N^{k-1}\|_0^2\|u_N^k\|_0^2\Big] + \frac{1}{2}E\Big(\|u_N^k\|_0^2 - e^{-2\alpha\tau}\|u_N^{k-1}\|_0^2\Big)^2 + C\tau.
\end{aligned}
$$

Compare (LHS) with (RHS), we obtain

$$E\|u_N^k\|_0^4 \le e^{-4\alpha\tau}E\|u_N^{k-1}\|_0^4 + C\tau \le C.$$

iii) $p = 3$. Using 1) and 2), it is easy to check that the following holds true

$$E\|u_N^k\|_0^3 \le E\frac{\|u_N^k\|_0^2 + \|u_N^k\|_0^4}{2} \le C.$$

iv) $p > 4$. By repeating above procedure, we complete the proof.

□

Before showing the weak error between $u_N(t)$ and u_N^k, we need some a priori estimates on $\|u_N^k\|_1$ and $\|u_N^k\|_2$.

Proposition 4.2 *Assume that $\lambda = 0$ or -1, $u_0 \in \dot{H}^1$, $u_N^0 = \pi_N u_0$ and $\|Q^{\frac{1}{2}}\|_{\mathcal{HS}(L^2,\dot{H}^1)} < \infty$. Then for any $p \ge 1$, there exists a constant $C = C(\alpha, u_0, p)$ independent of N and t_k, such that*

$$E\mathcal{H}_k^p \le C, \ \forall\, k \in \mathbb{N},$$

where $\mathcal{H}_k := \|\nabla u_N^k\|_0^2 - \frac{\lambda}{2}\|u_N^k\|_{L^4}^4$.

Proof The proof for $\lambda = 0$ is in the same procedure as that for $\lambda = -1$ and is much easier. Here we only give the proof for $\lambda = -1$

$$u_N^k - e^{-\alpha\tau}u_N^{k-1} = \left(\mathbf{i}\Delta u_N^k - \mathbf{i}\pi_N\left(\frac{|u_N^k|^2 + |e^{-\alpha\tau}u_N^{k-1}|^2}{2}u_N^k\right)\right)\tau + \pi_N Q^{\frac{1}{2}}\delta W_k. \quad (4.4)$$

i) $p = 1$. Multiplying (4.4) by $\overline{u}_N^k - e^{-\alpha\tau}\overline{u}_N^{k-1}$, integrating with respect to x, taking the imaginary part and using the fact $\big((Id - \pi_N)v, v_N\big) = 0,\ \forall\, v \in \dot{H}^0,\ v_N \in V_N$, we have

$$\begin{aligned}&\|\nabla u_N^k\|_0^2+\|\nabla(u_N^k-e^{-\alpha\tau}u_N^{k-1})\|_0^2-e^{-2\alpha\tau}\|\nabla u_N^{k-1}\|_0^2\\=\ &-Re\int_0^1\Big(|u_N^k|^2+|e^{-\alpha\tau}u_N^{k-1}|^2\Big)u_N^k(\overline{u}_N^k-e^{-\alpha\tau}\overline{u}_N^{k-1})dx\\&+\frac{2}{\tau}Im\int_0^1\pi_NQ^{\frac12}\delta W_k(\overline{u}_N^k-e^{-\alpha\tau}\overline{u}_N^{k-1})dx\\=:\ &A+B.\end{aligned}$$

Simple computations yield

$$\begin{aligned}A\ &=\ -Re\left[\int_0^1\Big(|u_N^k|^2+|e^{-\alpha\tau}u_N^{k-1}|^2\Big)\Big(\frac{u_N^k+e^{-\alpha\tau}u_N^{k-1}}{2}+\frac{u_N^k-e^{-\alpha\tau}u_N^{k-1}}{2}\Big)(\overline{u}_N^k-e^{-\alpha\tau}\overline{u}_N^{k-1})dx\right]\\&\le\ -\frac12\|u_N^k\|_{L^4}^4+\frac12e^{-4\alpha\tau}\|u_N^{k-1}\|_{L^4}^4\le-\frac12\|u_N^k\|_{L^4}^4+\frac12e^{-2\alpha\tau}\|u_N^{k-1}\|_{L^4}^4\end{aligned}$$

and

$$\begin{aligned}B\ &=\ \frac{2}{\tau}Im\left[\int_0^1\pi_NQ^{\frac12}\delta W_k\Big[-\mathbf{i}\tau\Delta\overline{u}_N^k+\mathbf{i}\tau\frac{|u_N^k|^2+|e^{-\alpha\tau}u_N^{k-1}|^2}{2}\overline{u}_N^k+\overline{\pi_NQ^{\frac12}\delta W_k}\Big]dx\right]\\&=\ 2Re\left[\int_0^1\nabla(\pi_NQ^{\frac12}\delta W_k)\cdot\nabla\Big(\overline{u}_N^k-e^{-\alpha\tau}\overline{u}_N^{k-1}\Big)dx\right]+2Re\left[\int_0^1\nabla(\pi_NQ^{\frac12}\delta W_k)\cdot\nabla\Big(e^{-\alpha\tau}\overline{u}_N^{k-1}\Big)dx\right]\\&\quad+Re\left[\int_0^1\Big(|u_N^k|^2+|e^{-\alpha\tau}u_N^{k-1}|^2\Big)\overline{u}_N^k\cdot\pi_NQ^{\frac12}\delta W_kdx\right]\\&\le\ \frac14\|\nabla(u_N^k-e^{-\alpha\tau}u_N^{k-1})\|_0^2+C\|\nabla(\pi_NQ^{\frac12}\delta W_k)\|_0^2+2Re\left[\int_0^1\nabla(\pi_NQ^{\frac12}\delta W_k)\cdot\nabla\Big(e^{-\alpha\tau}\overline{u}_N^{k-1}\Big)dx\right]\\&\quad+Re\left[\int_0^1\Big(|u_N^k|^2+|e^{-\alpha\tau}u_N^{k-1}|^2\Big)\overline{u}_N^k\cdot\pi_NQ^{\frac12}\delta W_kdx\right].\end{aligned}$$

Denote $\mathcal{H}_k=\|\nabla u_N^k\|_0^2+\frac12\|u_N^k\|_{L^4}^4$, then

$$\begin{aligned}&E\mathcal{H}_k+\frac34E\|\nabla(u_N^k-e^{-\alpha\tau}u_N^{k-1})\|_0^2\\\le\ &e^{-2\alpha\tau}E\mathcal{H}_{k-1}+C\tau \qquad (4.5)\\&+ReE\left[\int_0^1\Big(|u_N^k|^2+|e^{-\alpha\tau}u_N^{k-1}|^2\Big)\overline{u}_N^k\cdot\pi_NQ^{\frac12}\delta W_kdx\right]. \qquad (4.6)\end{aligned}$$

Based on the formula

$$(|a|^2+|b|^2)\overline{a}=\overline{a}|a-b|^2+b(\overline{a}-\overline{b})^2+3|b|^2(\overline{a}-\overline{b})+\overline{b}|a-b|^2+(\overline{b})^2(a-b)+2|b|^2\overline{b},$$

the last term on the right hand side can be rewritten as

$$
\begin{aligned}
& ReE\left[\int_0^1 \left(|u_N^k|^2+|e^{-\alpha\tau}u_N^{k-1}|^2\right)\overline{u}_N^k\cdot\pi_N Q^{\frac12}\delta W_k dx\right]\\
= \;& ReE\int_0^1 \overline{u}_N^k\left|u_N^k-e^{-\alpha\tau}u_N^{k-1}\right|^2\pi_N Q^{\frac12}\delta W_k dx+ReE\int_0^1 e^{-\alpha\tau}u_N^{k-1}\left(\overline{u}_N^k-e^{-\alpha\tau}\overline{u}_N^{k-1}\right)^2\pi_N Q^{\frac12}\delta W_k dx\\
& +3ReE\int_0^1|e^{-\alpha\tau}u_N^{k-1}|^2\left(\overline{u}_N^k-e^{-\alpha\tau}\overline{u}_N^{k-1}\right)\pi_N Q^{\frac12}\delta W_k dx\\
& +ReE\int_0^1 e^{-\alpha\tau}\overline{u}_N^{k-1}\left|u_N^k-e^{-\alpha\tau}u_N^{k-1}\right|^2\pi_N Q^{\frac12}\delta W_k dx\\
& +ReE\int_0^1(e^{-\alpha\tau}\overline{u}_N^{k-1})^2(u_N^k-e^{-\alpha\tau}u_N^{k-1})\pi_N Q^{\frac12}\delta W_k dx+2ReE\int_0^1|e^{-\alpha\tau}u_N^{k-1}|^2e^{-\alpha\tau}\overline{u}_N^{k-1}\pi_N Q^{\frac12}\delta W_k dx\\
=:\;& a+b+c+d+e+f.
\end{aligned}
$$

Noting that $f=0$, it suffices to estimate the other five terms

$$
\begin{aligned}
a+b+d \le\;& E\Big[\|u_N^k\|_0\|u_N^k-e^{-\alpha\tau}u_N^{k-1}\|_{L^4}^2\|\pi_N Q^{\frac12}\delta W_k\|_{L^\infty}\\
&+2\|e^{-\alpha\tau}u_N^{k-1}\|_0\|u_N^k-e^{-\alpha\tau}u_N^{k-1}\|_{L^4}^2\|\pi_N Q^{\frac12}\delta W_k\|_{L^\infty}\Big]\\
\le\;& E\left[\left(\|u_N^k\|_0+2\|e^{-\alpha\tau}u_N^{k-1}\|_0\right)\|\nabla(u_N^k-e^{-\alpha\tau}u_N^{k-1})\|_0^{\frac12}\|u_N^k-e^{-\alpha\tau}u_N^{k-1}\|_0^{\frac32}\|\pi_N Q^{\frac12}\delta W_k\|_{L^\infty}\right]\\
\le\;& \frac14 E\left[\|\nabla(u_N^k-e^{-\alpha\tau}u_N^{k-1})\|_0\|u_N^k-e^{-\alpha\tau}u_N^{k-1}\|_0\right]\\
&+CE\left[\left(\|u_N^k\|_0^2+\|e^{-\alpha\tau}u_N^{k-1}\|_0^2\right)\|u_N^k-e^{-\alpha\tau}u_N^{k-1}\|_0^2\|\pi_N Q^{\frac12}\delta W_k\|_{L^\infty}^2\right]\\
\le\;& \frac14 E\|\nabla(u_N^k-e^{-\alpha\tau}u_N^{k-1})\|_0^2+CE\left(\tau^{\frac12}\left(\|u_N^k\|_0^2+\|e^{-\alpha\tau}u_N^{k-1}\|_0^2\right)\|u_N^k-e^{-\alpha\tau}u_N^{k-1}\|_0^2\right)^2\\
&+CE\left(\tau^{-\frac12}\|\pi_N Q^{\frac12}\delta W_k\|_{L^\infty}^2\right)^2\\
\le\;& \frac14 E\|\nabla(u_N^k-e^{-\alpha\tau}u_N^{k-1})\|_0^2+C\tau,
\end{aligned}
$$

where in the last step we have used Proposition 4.1,

$$
\begin{aligned}
c+e \le\;& 4E\left[\|e^{-\alpha\tau}u_N^{k-1}\|_{L^4}^2\|u_N^k-e^{-\alpha\tau}u_N^{k-1}\|_0\|\pi_N Q^{\frac12}\delta W_k\|_{L^\infty}\right]\\
\le\;& \frac12 E\|u_N^k-e^{-\alpha\tau}u_N^{k-1}\|_0^2+8\eta\tau e^{-4\alpha\tau}E\|u_N^{k-1}\|_{L^4}^4\\
\le\;& \frac12 E\|u_N^k-e^{-\alpha\tau}u_N^{k-1}\|_0^2+2E\left[\left(\sqrt{\alpha}\tau^{\frac12}e^{-\alpha\tau}\|\nabla u_N^{k-1}\|_0\right)\left(\frac{C}{2\sqrt{\alpha}}8\eta\tau^{\frac12}e^{-3\alpha\tau}\|u_N^{k-1}\|_0^3\right)\right]\\
\le\;& \frac12 E\|u_N^k-e^{-\alpha\tau}u_N^{k-1}\|_0^2+\alpha\tau e^{-2\alpha\tau}E\|\nabla u_N^{k-1}\|_0^2+C\tau.
\end{aligned}
$$

Then (4.5) turns to be

$$E\mathcal{H}_k\le(1+\alpha\tau)e^{-2\alpha\tau}E\mathcal{H}_{k-1}+C\tau\le e^{-\alpha\tau}E\mathcal{H}_{k-1}+C\tau.$$

We finally obtain that

$$E\mathcal{H}_k\le C.$$

ii) $p = 2$. From the case $p = 1$, by $\|\cdot\|_{L^4}^4 \le \|\nabla \cdot \|_0 \| \cdot \|_0^3$, we get

$$\begin{aligned}
\mathcal{H}_k - e^{-2\alpha\tau}\mathcal{H}_{k-1} \le\ & C\|\nabla(\pi_N Q^{\frac{1}{2}}\delta W_k)\|_0^2 \\
& + C Re\left[\int_0^1 \nabla(\pi_N Q^{\frac{1}{2}}\delta W_k)\cdot\nabla\left(e^{-\alpha\tau}\overline{u}_N^{k-1}\right)dx\right] \\
& + C\left(\tau^{\frac{1}{2}}\left(\|u_N^k\|_0^2 + \|e^{-\alpha\tau}u_N^{k-1}\|_0^2\right)\|u_N^k - e^{-\alpha\tau}u_N^{k-1}\|_0^2\right)^2 \\
& + C\left(\tau^{-\frac{1}{2}}\|\pi_N Q^{\frac{1}{2}}\delta W_k\|_{L^\infty}^2\right)^2 + \alpha\tau e^{-2\alpha\tau}\mathcal{H}_{k-1} \\
& + C\tau^{-1}\|u_N^{k-1}\|_0^6\|\pi_N Q^{\frac{1}{2}}\delta W_k\|_{L^\infty}^4.
\end{aligned}$$

Multiplying above formula by $\mathcal{H}_k$, we have

$$\begin{aligned}
& \mathcal{H}_k^2 + (\mathcal{H}_k - e^{-2\alpha\tau}\mathcal{H}_{k-1})^2 - e^{-4\alpha\tau}\mathcal{H}_{k-1}^2 \\
\le\ & C\mathcal{H}_k\|\nabla(\pi_N Q^{\frac{1}{2}}\delta W_k)\|_0^2 + C\mathcal{H}_k Re\left[\int_0^1 \nabla(\pi_N Q^{\frac{1}{2}}\delta W_k)\cdot\nabla\left(e^{-\alpha\tau}\overline{u}_N^{k-1}\right)dx\right] \\
& + C\tau\mathcal{H}_k\left(\|u_N^k\|_0^2 + \|e^{-\alpha\tau}u_N^{k-1}\|_0^2\right)^2\|u_N^k - e^{-\alpha\tau}u_N^{k-1}\|_0^4 \\
& + C\mathcal{H}_k\left(\tau^{-\frac{1}{2}}\|\pi_N Q^{\frac{1}{2}}\delta W_k\|_{L^\infty}^2\right)^2 + \alpha\tau e^{-2\alpha\tau}\mathcal{H}_k\mathcal{H}_{k-1} \\
& + C\tau^{-1}\mathcal{H}_k\|u_N^{k-1}\|_0^6\|\pi_N Q^{\frac{1}{2}}\delta W_k\|_{L^\infty}^4 \\
=:\ & a' + b' + c' + d' + e' + f',
\end{aligned}$$

where

$$\begin{aligned}
E[a' + b' + c' + d'] \le\ & \frac{1}{4}E(\mathcal{H}_k - e^{-2\alpha\tau}\mathcal{H}_{k-1})^2 + C\tau \\
& + C\tau e^{-2\alpha\tau}E\left[\mathcal{H}_{k-1}\left(\|u_N^k\|_0^2 + \|e^{-\alpha\tau}u_N^{k-1}\|_0^2\right)^2\|u_N^k - e^{-\alpha\tau}u_N^{k-1}\|_0^4\right] \\
\le\ & \frac{1}{4}E(\mathcal{H}_k - e^{-2\alpha\tau}\mathcal{H}_{k-1})^2 + \frac{1}{2}\tau e^{-4\alpha\tau}E\mathcal{H}_{k-1}^2 + C\tau,
\end{aligned}$$

$$\begin{aligned}
E[e'] \le\ & \frac{1}{2}E\left(\mathcal{H}_k - e^{-2\alpha\tau}\mathcal{H}_{k-1}\right)^2 + (\frac{1}{2}\alpha^2\tau^2 + \alpha\tau)e^{-4\alpha\tau}E\mathcal{H}_{k-1}^2 \\
\le\ & \frac{1}{2}E\left(\mathcal{H}_k - e^{-2\alpha\tau}\mathcal{H}_{k-1}\right)^2 + \frac{3}{2}\alpha\tau e^{-4\alpha\tau}E\mathcal{H}_{k-1}^2
\end{aligned}$$

and

$$\begin{aligned}
E[f'] \le\ & \frac{1}{4}E\left(\mathcal{H}_k - e^{-2\alpha\tau}\mathcal{H}_{k-1}\right)^2 + C\tau^{-2}E\left[\|u_N^{k-1}\|_0^{12}\|\pi_N Q^{\frac{1}{2}}\delta W_k\|_{L^\infty}^8\right] \\
& + \alpha\tau e^{-4\alpha\tau}E\mathcal{H}_{k-1}^2 + C\tau^{-3}E\left[\|u_N^{k-1}\|_0^{12}\|\pi_N Q^{\frac{1}{2}}\delta W_k\|_{L^\infty}^8\right] \\
\le\ & \frac{1}{4}E\left(\mathcal{H}_k - e^{-2\alpha\tau}\mathcal{H}_{k-1}\right)^2 + \alpha\tau e^{-4\alpha\tau}E\mathcal{H}_{k-1}^2 + C\tau.
\end{aligned}$$

Then we conclude

$$E\mathcal{H}_k^2 \le (1 + 3\alpha\tau)e^{-4\alpha\tau}E\mathcal{H}_{k-1}^2 + C\tau \le e^{-\alpha\tau}E\mathcal{H}_{k-1}^2 + C\tau \le C,$$

where we have used $(1 + 3\alpha\tau)e^{-3\alpha\tau} \le 1$ for $\alpha\tau < 1$.

iii) For $p = 2^l$, $l \in \mathbb{N}$, the result can be proved by above procedure. So it also holds for any $p \in \mathbb{N}$.

□

Corollary 1 *Under the assumptions in Proposition 4.2, we have*

$$E\|u_N^k - e^{-\alpha\tau}u_N^{k-1}\|_0^{2p} \le C\tau^p,$$

where constant C is independent of N and t_k.

Proof It is easy to check this by multiplying $\overline{u}_N^k - e^{-\alpha\tau}\overline{u}_N^{k-1}$ to both sides of Eq. 4.4, integrating with respect to x and taking expectation,

$$\begin{aligned}
& E\|u_N^k - e^{-\alpha\tau}u_N^{k-1}\|_0^{2p} \\
= \; & E\Bigg[\tau Im\int_0^1 \nabla u_N^k \nabla(\overline{u}_N^k - e^{-\alpha\tau}\overline{u}_N^{k-1})dx + Re\int_0^1 \pi_N Q^{\frac{1}{2}}\delta W_k\left(\overline{u}_N^k - e^{-\alpha\tau}\overline{u}_N^{k-1}\right)dx \\
& + \frac{\tau}{4}Im\int_0^1 \left(|u_N^k|^2 + |e^{-\alpha\tau}u_N^{k-1}|^2\right)\left(u_N^k + e^{-\alpha\tau}u_N^{k-1}\right)\left(\overline{u}_N^k - e^{-\alpha\tau}\overline{u}_N^{k-1}\right)dx\Bigg]^p \\
\le \; & CE\Bigg[\tau^p\|\nabla u_N^k\|_0^p\|\nabla\left(u_N^k - e^{-\alpha\tau}u_N^{k-1}\right)\|_0^p \\
& + \tau^p\left(\|u_N^k\|_1^{2p} + \|u_N^{k-1}\|_1^{2p}\right)\left(\|u_N^k\|_0^{2p} + \|u_N^{k-1}\|_0^{2p}\right)\Bigg] \\
& + CE\|\pi_N Q^{\frac{1}{2}}\delta W_k\|_0^{2p} + \frac{1}{2}E\|u_N^k - e^{-\alpha\tau}u_N^{k-1}\|_0^{2p} \\
\le \; & \frac{1}{2}E\|u_N^k - e^{-\alpha\tau}u_N^{k-1}\|_0^{2p} + C\tau^p.
\end{aligned}$$

Then we complete the proof by Proposition 4.2. □

Proposition 4.3 *Under the assumptions $\lambda = 0$ or -1, $u_0 \in \dot{H}^2$ and $\|Q^{\frac{1}{2}}\|_{\mathcal{HS}(L^2,\dot{H}^2)} < \infty$, we also have the uniform boundedness of 2-norm as follows*

$$E\|u_N^k\|_2^2 \le C, \ \forall\, k \in \mathbb{N},$$

where C is also independent of N and t_k.

Proof We also give the proof for $\lambda = -1$ only. Multiply (4.4) by $\Delta(\overline{u}_N^k - e^{-\alpha\tau}\overline{u}_N^{k-1})$, integrating with respect to x, and then taking the imaginary part, we obtain

$$\begin{aligned}
& \|\Delta u_N^k\|_0^2 + \|\Delta(u_N^k - e^{-\alpha\tau}u_N^{k-1})\|_0^2 - e^{-2\alpha\tau}\|\Delta u_N^{k-1}\|_0^2 \\
= \; & Re\int_0^1 \left(|u_N^k|^2 + |e^{-\alpha\tau}u_N^{k-1}|^2\right)u_N^k\Delta(\overline{u}_N^k - e^{-\alpha\tau}\overline{u}_N^{k-1})dx \\
& - \frac{2}{\tau}Im\int_0^1 \pi_N Q^{\frac{1}{2}}\delta W_k\Delta(\overline{u}_N^k - e^{-\alpha\tau}\overline{u}_N^{k-1})dx \\
=: \; & A' + B'.
\end{aligned}$$

According to the uniform boundedness of any order of 0-norm and 1-norm, we have the following estimations.

$$
\begin{aligned}
E[A'] &= ReE\int_0^1 |u_N^k|^2 u_N^k \Delta(\overline{u}_N^k - e^{-\alpha\tau}\overline{u}_N^{k-1})dx \\
&\quad + e^{-3\alpha\tau} ReE\int_0^1 |u_N^{k-1}|^2 u_N^{k-1} \Delta(\overline{u}_N^k - e^{-\alpha\tau}\overline{u}_N^{k-1})dx \\
&\quad + e^{-2\alpha\tau} ReE\int_0^1 |u_N^{k-1}|^2 (u_N^k - e^{-\alpha\tau}u_N^{k-1}) \Delta(\overline{u}_N^k - e^{-\alpha\tau}\overline{u}_N^{k-1})dx \\
&= ReE\int_0^1 |u_N^k|^2 u_N^k \Delta\overline{u}_N^k dx - e^{-4\alpha\tau} ReE\int_0^1 |u_N^{k-1}|^2 u_N^{k-1} \Delta\overline{u}_N^{k-1} dx \\
&\quad + e^{-2\alpha\tau} ReE\int_0^1 |u_N^{k-1}|^2 (u_N^k - e^{-\alpha\tau}u_N^{k-1}) \Delta(\overline{u}_N^k - e^{-\alpha\tau}\overline{u}_N^{k-1})dx \\
&\quad + ReE\int_0^1 u_N^k \Delta\overline{u}_N^k |u_N^k - e^{-\alpha\tau}u_N^{k-1}|^2 dx \\
&\quad + 2ReE\int_0^1 \overline{u}_N^k (\nabla u_N^k)^2 (\overline{u}_N^k - e^{-\alpha\tau}\overline{u}_N^{k-1})dx \\
&\quad + 4ReE\int_0^1 u_N^k |\nabla u_N^k|^2 (\overline{u}_N^k - e^{-\alpha\tau}\overline{u}_N^{k-1})dx \\
&\quad + ReE\int_0^1 (u_N^k - e^{-\alpha\tau}u_N^{k-1}) \Delta\overline{u}_N^k \left(|u_N^k|^2 - |e^{-\alpha\tau}u_N^{k-1}|^2\right) dx \\
&=: A_a^k - e^{-4\alpha\tau}A_a^{k-1} + A_b + A_c + A_d + A_e + A_f.
\end{aligned}
$$

We estimate above terms repectively and obtain

$$
\begin{aligned}
-e^{-4\alpha\tau}A_a^{k-1} &= -e^{-2\alpha\tau}A_a^{k-1} + e^{-2\alpha\tau}(1-e^{-2\alpha\tau})A_a^{k-1} \\
&\le -e^{-2\alpha\tau}A_a^{k-1} + C\tau E\|u_N^{k-1}\|_1^4 \le -e^{-2\alpha\tau}A_a^{k-1} + C\tau,
\end{aligned}
$$

$$
\begin{aligned}
A_b &\le e^{-2\alpha\tau} E\left[\|u_N^{k-1}\|_{L^\infty}^2 \|u_N^k - e^{-\alpha\tau}u_N^{k-1}\|_0 \|\Delta(u_N^k - e^{-\alpha\tau}u_N^{k-1})\|_0\right] \\
&\le \frac{1}{6}E\|\Delta(u_N^k - e^{-\alpha\tau}u_N^{k-1})\|_0^2 + C\tau E\|u_N^{k-1}\|_1^8 + C\tau^{-1}E\|u_N^k - e^{-\alpha\tau}u_N^{k-1}\|_0^4 \\
&\le \frac{1}{6}E\|\Delta(u_N^k - e^{-\alpha\tau}u_N^{k-1})\|_0^2 + C\tau,
\end{aligned}
$$

$$
\begin{aligned}
A_c &\le E\left[\|u_N^k - e^{-\alpha\tau}u_N^{k-1}\|_{L^4}^2 \|u_N^k\|_{L^\infty} \|\Delta u_N^k\|_0\right] \\
&\le C\tau^{-1}E\left[\|\nabla(u_N^k - e^{-\alpha\tau}u_N^{k-1})\|_0 \|u_N^k - e^{-\alpha\tau}u_N^{k-1}\|_0^3 \|u_N^k\|_1^2\right] + \frac{1}{8}\alpha\tau E\|\Delta u_N^k\|_0^2 \\
&\le \frac{1}{6}E\|\Delta(u_N^k - e^{-\alpha\tau}u_N^{k-1})\|_0^2 + C\tau^{-5}E\|u_N^k - e^{-\alpha\tau}u_N^{k-1}\|_0^{12} \\
&\quad + C\tau E\|u_N^k\|_1^8 + \frac{1}{8}\alpha\tau E\|\Delta u_N^k\|_0^2 \\
&\le \frac{1}{6}E\|\Delta(u_N^k - e^{-\alpha\tau}u_N^{k-1})\|_0^2 + \frac{1}{8}\alpha\tau E\|\Delta u_N^k\|_0^2 + C\tau,
\end{aligned}
$$

$$\begin{aligned}
A_d &= 2ReE\int_0^1 \overline{u}_N^k(\nabla u_N^k)^2\Big[-\mathbf{i}\tau\Delta\overline{u}_N^k+\mathbf{i}\tau\pi_N\left(\frac{|u_N^k|^2+|e^{-\alpha\tau}u_N^{k-1}|^2}{2}\overline{u}_N^k\right)\\
&\quad+\overline{\pi_N Q^{\frac12}\delta W_k}\Big]dx\\
&\le \frac{1}{16}\alpha\tau E\|\Delta u_N^k\|_0^2+C\tau+2ReE\int_0^1\overline{u}_N^k(\nabla u_N^k)^2\overline{\pi_N Q^{\frac12}\delta W_k}dx\\
&\le \frac{1}{16}\alpha\tau E\|\Delta u_N^k\|_0^2+C\tau+2ReE\int_0^1(\overline{u}_N^k-e^{-\alpha\tau}\overline{u}_N^{k-1})(\nabla u_N^k)^2\overline{\pi_N Q^{\frac12}\delta W_k}dx\\
&\quad+2ReE\int_0^1 e^{-\alpha\tau}\overline{u}_N^{k-1}\left((\nabla u_N^k)^2-(e^{-\alpha\tau}\nabla u_N^{k-1})^2\right)\overline{\pi_N Q^{\frac12}\delta W_k}dx\\
&\le \frac{1}{16}\alpha\tau E\|\Delta u_N^k\|_0^2+C\tau+CE\left[\|u_N^k-e^{-\alpha\tau}u_N^{k-1}\|_0\|\nabla u_N^k\|_{L^4}^2\|\pi_N Q^{\frac12}\delta W_k\|_{L^\infty}\right]\\
&\quad+CE\left[\|\nabla(u_N^k-e^{-\alpha\tau}u_N^{k-1})\|_0\left(\|u_N^{k-1}\|_1\|u_N^k\|_1+\|u_N^{k-1}\|_1^2\right)\|\pi_N Q^{\frac12}\delta W_k\|_{L^\infty}\right]\\
&\le \frac16 E\|\Delta(u_N^k-e^{-\alpha\tau}u_N^{k-1})\|_0^2+\frac18\alpha\tau E\|\Delta u_N^k\|_0^2+C\tau,
\end{aligned}$$

and

$$\begin{aligned}
A_f &= ReE\int_0^1(u_N^k-e^{-\alpha\tau}u_N^{k-1})\Delta\overline{u}_N^k Re\Big[\left(u_N^k-e^{-\alpha\tau}u_N^{k-1}\right)(\overline{u}_N^k+e^{-\alpha\tau}\overline{u}_N^{k-1})\Big]dx\\
&\le E\left[\|u_N^k-e^{-\alpha\tau}u_N^{k-1}\|_{L^4}^2(\|u_N^k\|_{L^\infty}+\|u_N^{k-1}\|_{L^\infty})\|\Delta u_N^k\|_0\right]\\
&\le \frac16 E\|\Delta(u_N^k-e^{-\alpha\tau}u_N^{k-1})\|_0^2+\frac18\alpha\tau E\|\Delta u_N^k\|_0^2+C\tau,
\end{aligned}$$

where A_e has an same estimation as A_d and we have used that $\|\nabla\cdot\|_0\cong\|\cdot\|_1\le\|\cdot\|_2\cong\|\Delta\cdot\|_0$. So we obtain

$$E[A']\le\frac56 E\|\Delta(u_N^k-e^{-\alpha\tau}u_N^{k-1})\|_0^2+\frac12\alpha\tau E\|\Delta u_N^k\|_0^2+C\tau.$$

For term B', we have

$$\begin{aligned}
E[B'] &= -\frac{2}{\tau}ImE\int_0^1\Delta\left(\pi_N Q^{\frac12}\delta W_k\right)\left(-\mathbf{i}\tau\Delta\overline{u}_N^k+\mathbf{i}\pi_N\left(\frac{|u_N^k|^2+|e^{-\alpha\tau}u_N^{k-1}|^2}{2}\overline{u}_N^k\right)\tau+\overline{\pi_N Q^{\frac12}\delta W_k}\right)dx\\
&= 2ReE\int_0^1\Delta\left(\pi_N Q^{\frac12}\delta W_k\right)\Delta(\overline{u}_N^k-e^{-\alpha\tau}\overline{u}_N^{k-1})dx\\
&\quad-ReE\int_0^1\Delta\left(\pi_N Q^{\frac12}\delta W_k\right)\left(|u_N^k|^2\overline{u}_N^k-|e^{-\alpha\tau}u_N^{k-1}|^2e^{-\alpha\tau}u_N^{k-1}\right)dx\\
&\quad-ReE\int_0^1\Delta\left(\pi_N Q^{\frac12}\delta W_k\right)|e^{-\alpha\tau}u_N^{k-1}|^2(\overline{u}_N^k-e^{-\alpha\tau}\overline{u}_N^{k-1})dx\\
&\le \frac16 E\|\Delta(u_N^k-e^{-\alpha\tau}u_N^{k-1})\|_0^2+C\tau.
\end{aligned}$$

Denoting $\mathcal{K}_k:=\|\Delta u_N^k\|_0^2-Re\int_0^1|u_N^k|^2u_N^k\Delta\overline{u}_N^k dx$, then $E\|\Delta u_N^k\|_0^2\le E\mathcal{K}_k+C$ and

$$E\mathcal{K}_k-e^{-2\alpha\tau}E\mathcal{K}_{k-1}\le\frac12\alpha\tau E\|\Delta u_N^k\|_0^2+C\tau\le\frac12\alpha\tau E\mathcal{K}_k+C\tau.$$

Finally,

$$E\mathcal{K}_k \le (1 - \frac{1}{2}\alpha\tau)^{-1} e^{-2\alpha\tau} E\mathcal{K}_{k-1} + C\tau \le C,$$

where we have used $(1 - \frac{1}{2}\alpha\tau)^{-1} e^{-2\alpha\tau} \le e^{-\alpha\tau}$ for $\alpha\tau < 1$. □

4.2 Ergodicity of the Fully Discrete Scheme

To prove the ergodicity of the scheme (4.1), we will use the discrete form of Theorem 2.1. We give some existing results before our theorem.

Assumption 1 (Minorization condition in [14]) *The Markov chain $(x_n)_{n\in\mathbb{N}}$ with transition kernel $P_n(x, G) = P(x_n \in G|x_0 = x)$ satisfies, for some fixed compact set $\mathcal{C} \in \mathcal{B}(\mathbb{R}^d)$, the following:*

i) *for some $y^* \in int(\mathcal{C})$ there is, for any $\delta > 0$, a $t_1 = t_1(\delta) \in \mathbb{N}$ such that*

$$P_{t_1}(x, B_\delta(y^*)) > 0 \quad \forall x \in \mathcal{C};$$

ii) *the transition kernel possesses a density $p_n(x, y)$, more precisely*

$$P_n(x, G) = \int_G p_n(x, y)dy \quad \forall x \in \mathcal{C},\ G \in \mathcal{B}(\mathbb{R}^d) \cap \mathcal{B}(\mathcal{C})$$

and $p_n(x, y)$ is jointly continuous in $(x, y) \in \mathcal{C} \times \mathcal{C}$.

Assumption 2 (Lyapunov condition in [14]) *There is a function $F : \mathbb{R}^d \to [1, \infty)$, with* $\lim_{|x|\to\infty} F(x) = \infty$*, real numbers $\theta \in (0, 1)$, and $\gamma \in [0, \infty)$ such that*

$$E[F(x_{n+1})|\mathcal{F}_n] \le \theta F(x_n) + \gamma.$$

Definition 3 We say that function F is essentially quadratic if there exist constants $C_i > 0,\ i = 1, 2, 3$, such that

$$C_1(1 + \|x\|^2) \le F(x) \le C_2(1 + \|x\|^2), \quad |\nabla F(x)| \le C_3(1 + \|x\|).$$

Theorem 4.1 ([14]) *Assume that a Markov chain $(x_n)_{n\in\mathbb{N}}$ satisfies Assumptions 1 and 2 with an essentially quadratic F, then the chain possesses a unique invariant measure.*

Based on the preliminaries above and the theory of Markov chains, we prove the following theorem.

Theorem 4.2 *For all τ sufficiently small, the solution $(u_N^k)_{k\in\mathbb{N}}$ of scheme (4.1) has a unique invariant measure μ_N^τ. Thus, it is ergodic.*

Proof i) Lyapunov condition. Based on Proposition 4.1, we can take essentially quadratic function $F(\cdot) = 1 + \|\cdot\|_0^2$ as the Lyapunov function, and the Lyapunov condition holds.

ii) Minorization condition. In scheme (4.1), it gives

$$P_N^k = e^{-\alpha\tau}P_N^{k-1} - \tau\left(\Delta Q_N^k + \frac{\lambda}{2}\pi_N\Big(\left(|P_N^k|^2 + |Q_N^k|^2 + |e^{-\alpha\tau}P_N^{k-1}|^2 + |e^{-\alpha\tau}Q_N^{k-1}|^2\right)Q_N^k\Big)\right) + \sum_{m=1}^{N}\sqrt{\eta_m}e_m\delta_k\beta_m^1, \tag{4.7}$$

$$Q_N^k = e^{-\alpha\tau}Q_N^{k-1} + \tau\left(\Delta P_N^k + \frac{\lambda}{2}\pi_N\Big(\left(|P_N^k|^2 + |Q_N^k|^2 + |e^{-\alpha\tau}P_N^{k-1}|^2 + |e^{-\alpha\tau}Q_N^{k-1}|^2\right)P_N^k\Big)\right) + \sum_{m=1}^{N}\sqrt{\eta_m}e_m\delta_k\beta_m^2, \tag{4.8}$$

where P_N^k and Q_N^k denote the real and imaginary part of u_N^k respectively, that is $u_N^k = P_N^k + \mathbf{i}Q_N^k$. Also, $\pi_N Q^{\frac{1}{2}}\delta W_k = \sum_{m=1}^N \sqrt{\eta_m}e_m\left(\delta_k\beta_m^1 + \mathbf{i}\delta_k\beta_m^2\right)$, where $\delta_k\beta_m^1$ and $\delta_k\beta_m^2$ are the real and imaginary part of δW_k respectively.

For any $y_1 = a_1 + \mathbf{i}b_1$, $y_2 = a_2 + \mathbf{i}b_2 \in V_N$ with a_i and b_i denoting the real and imaginary part of y_i $(i = 1, 2)$ respectively, as $\{e_m\}_{m=1}^N$ is a basis of V_N, $\{\delta_k\beta_m^1,\ \delta_k\beta_m^2\}_{m=1}^N$ can be uniquely determined to ensure that $(P_N^{k-1}, Q_N^{k-1}) = (a_1, b_1)$ and $(P_N^k, Q_N^k) = (a_2, b_2)$, which implies the irreducibility of u_N^k.

As stated in Proposition 4.1, the $\mathcal{F}_{t_k}$-measurable solution $\{u_N^k\}_{k\in\mathbb{N}}$ is defined through a unique continuous function: $u_N^k = \kappa(u_N^{k-1}, \frac{\pi_N Q^{\frac{1}{2}}\delta W_k}{\sqrt{\tau}})$, where δW_k has a C^∞ density. Thus, the transition kernel $P_1(x, G)$, $G \in \mathcal{B}(V_N)$ possesses a jointly continuous density $p_1(x, y)$. Furthermore, densities $p_k(x, y)$ are achieved by the time-homogeneous property of Markov chain $\{u_N^k\}_{k\in\mathbb{N}}$.

With above conditions, based on Theorem 4.1, we prove that u_N^k possesses a unique invariant measure. □

4.3 Weak Error between Solutions u_N and u_N^k

We still use modified processes to calculate the weak error of the fully discrete scheme in temporal direction. Denote $S_\tau = (Id - \mathbf{i}\tau\Delta)^{-1}e^{-\alpha\tau}$, then scheme (4.1) is rewritten as

$$\begin{aligned} u_N^k &= S_\tau u_N^{k-1} + \mathbf{i}\lambda\tau e^{\alpha\tau}S_\tau\pi_N\left(\frac{|u_N^k|^2 + |e^{-\alpha\tau}u_N^{k-1}|^2}{2}u_N^k\right) + e^{\alpha\tau}S_\tau\pi_N Q^{\frac{1}{2}}\delta W_k \\ &= S_\tau^k u_N^0 + \mathbf{i}\lambda\tau e^{\alpha\tau}\sum_{l=1}^k S_\tau^{k+1-l}\pi_N\left(\frac{|u_N^l|^2 + |e^{-\alpha\tau}u_N^{l-1}|^2}{2}u_N^l\right) + e^{\alpha\tau}\sum_{l=1}^k S_\tau^{k+1-l}\pi_N Q^{\frac{1}{2}}\delta W_l \end{aligned} \tag{4.9}$$

Lemma 2 *For any $k \in \mathbb{N}$ and sufficiently small τ, we have the following estimates,*

$$i)\ \|S_\tau^k - S(t)\|_{\mathcal{L}(\dot{H}^2, L^2)} \le C(t+\tau)^{\frac{1}{2}}e^{-\alpha t}\tau^{\frac{1}{2}}, \quad t \in [t_{k-1}, t_{k+1}],$$

$$ii)\ \|S_\tau^k - S(t)\|_{\mathcal{L}(\dot{H}^1, \dot{H}^1)} \le Ce^{-\alpha t}, \quad t \in [t_{k-1}, t_{k+1}],$$

where the constant $C = C(\alpha)$ is independent of k and τ.

Proof **Step 1.** If $t = t_k$. As $S(t)$ is the operator semigroup of equation $du(t) = (\mathbf{i}\Delta - \alpha)u(t)dt$, $u(0) = u^0 \in \dot{H}^2$, and S_τ is the corresponding discrete operator semigroup, we have

$$S_\tau^k u(0) = u^k = e^{-\alpha\tau}u^{k-1} + \mathbf{i}\tau\Delta u^k, \tag{4.10}$$

$$S(t_k)u(0) = u(t_k) = e^{-\alpha\tau}u(t_{k-1}) + \int_{t_{k-1}}^{t_k} \mathbf{i}e^{-\alpha(t_k - s)}\Delta u(s)ds. \tag{4.11}$$

Denote $e_k = u^k - u(t_k) = \left(S_\tau^k - S(t_k)\right)u(0)$ with $e_0 = 0$, then

$$e_k = e^{-\alpha\tau}e_{k-1} + \mathbf{i}\tau\Delta e_k + \mathbf{i}\int_{t_{k-1}}^{t_k}\left[\Delta u(t_k) - e^{-\alpha(t_k - s)}\Delta u(s)\right]ds.$$

Multiply $\overline{e}_k$ to above formula, integrate with respect to x, take the real part, and we get

$$\begin{aligned}
&\frac{1}{2}\left[\|e_k\|_0^2 + \|e_k - e^{-\alpha\tau}e_{k-1}\|_0^2 - e^{-2\alpha\tau}\|e_{k-1}\|_0^2\right] \\
&= Re\left[\mathbf{i}\int_0^1\int_{t_{k-1}}^{t_k}\Delta\overline{e}_k\int_s^{t_k}\mathbf{i}e^{-\alpha(t_k - r)}\Delta u(r)drdsdx\right] \\
&\leq C\int_{t_{k-1}}^{t_k}\int_s^{t_k}\|\Delta u^k - \Delta u(t_k)\|_0\|\Delta u(r)\|_0 drds \\
&\leq Ce^{-2\alpha t_k}\|\Delta u(0)\|_0^2\tau^2,
\end{aligned}$$

where we have used the fact that $\|\Delta u^k\|_0^2 \leq e^{-2\alpha t_k}\|\Delta u^0\|_0^2$ and $\|\Delta u(t)\|_0 \leq Ce^{-\alpha t}\|\Delta u(0)\|_0$. In fact, multiplying $\Delta\overline{u}^k - e^{-\alpha\tau}\Delta\overline{u}^{k-1}$ to Eq. 4.10, integrating in space and taking the imaginary part, we obtain

$$\|\Delta u^k\|_0^2 \leq e^{-2\alpha\tau}\|\Delta u^{k-1}\|_0^2 \leq e^{-2\alpha t_k}\|\Delta u^0\|_0^2.$$

Then it's easy to check that

$$\|e_k\|_0^2 \leq e^{-2\alpha\tau}\|e_{k-1}\|_0^2 + Ce^{-2\alpha t_k}\|\Delta u(0)\|_0^2\tau^2$$

leads to

$$\|e_k\|_0^2 \leq Ct_k e^{-2\alpha t_k}\|\Delta u(0)\|_0^2\tau, \tag{4.12}$$

which finally yields $\|S_\tau^k - S(t_k)\|_{\mathcal{L}(\dot{H}^2, L^2)} \leq Ct_k^{\frac{1}{2}}e^{-\alpha t_k}\tau^{\frac{1}{2}}$ in i).

For ii), we have

$$\begin{aligned}
\|\left(S_\tau^k - S(t_k)\right)u(0)\|_1^2 &= \sum_{n=1}^{\infty}\left|e^{-\alpha t_k}\left((1 + n^2\pi^2)^{-k} - e^{-n^2\pi^2 t_k}\right)(u(0), e_n)\right|^2|\lambda_n| \\
&\leq 4e^{-2\alpha t_k}\sum_{n=1}^{\infty}|(u(0), e_n)|^2|\lambda_n| = 4e^{-2\alpha t_k}\|u(0)\|_1^2.
\end{aligned}$$

In the following two steps, we only give the proof of i), and ii) can be proved in a same procedure. We use the notation $\|\cdot\| = \|\cdot\|_{\mathcal{L}(\dot{H}^2, L^2)}$, which is an operator norm defined at the beginning of this paper.

Step 2. If $t \in [t_{k-1}, t_k]$,

$$\begin{aligned}
\|S_\tau^k - S(t)\| &\leq \|S_\tau^k - S(t_k)\| + \|S(t_k) - S(t)\| \leq C t_k^{\frac{1}{2}} e^{-\alpha t_k} \tau^{\frac{1}{2}} \\
&\quad + e^{-\alpha t} |e^{-\alpha(t_k - t)} - 1| \\
&\leq C t_k^{\frac{1}{2}} e^{-\alpha t_k} \tau^{\frac{1}{2}} + e^{-\alpha t} \sum_{n=1}^{\infty} \frac{1}{n!} (\alpha\tau)^n \leq C t_k^{\frac{1}{2}} e^{-\alpha t_k} \tau^{\frac{1}{2}} \\
&\quad + e^{-\alpha t} \alpha\tau \frac{e^{\alpha\tau} - 1}{\alpha\tau} \\
&\leq C (t+\tau)^{\frac{1}{2}} e^{-\alpha t} \tau^{\frac{1}{2}}.
\end{aligned}$$

We have used the fact that $\frac{e^{\alpha\tau}-1}{\alpha\tau}$ is uniformly bounded for $\alpha\tau \in [0, 1]$.

Step 3. If $t \in [t_k, t_{k+1}]$,

$$\begin{aligned}
\|S_\tau^k - S(t)\| &\leq \|S_\tau^k - S(t_k)\| + \|S(t_k) - S(t)\| \leq C t_k^{\frac{1}{2}} e^{-\alpha t_k} \tau^{\frac{1}{2}} \\
&\quad + e^{-\alpha t} |e^{-\alpha(t_k - t)} - 1| \\
&\leq C t_k^{\frac{1}{2}} e^{-\alpha t} e^{\alpha(t - t_k)} \tau^{\frac{1}{2}} + e^{-\alpha t} \alpha\tau \frac{e^{\alpha\tau - 1}}{\alpha\tau} \leq C (t+\tau)^{\frac{1}{2}} e^{-\alpha t} \tau^{\frac{1}{2}}.
\end{aligned}$$

We have used the fact $e^{\alpha(t - t_k)} \leq e^{\alpha\tau} \leq e$.

□

Remark 4 From Eq. 4.10, we can also prove that

$$\|S_\tau^k\|_{\mathcal{L}(L^2, L^2)} \leq C e^{-\alpha t},$$

where k and t satisfying $t \in [t_{k-1}, t_{k+1}]$.

Next theorem gives the time-independent weak error of the solutions for different cases.

Theorem 4.3 *Assume that $u_0 \in \dot{H}^2$, $u_N^0 = u_N(0) = \pi_N u_0$ and $\|Q^{\frac{1}{2}}\|^2_{\mathcal{HS}(L^2, \dot{H}^2)} < \infty$. For the cases $\lambda = 0$ or -1, the weak errors are independent of time and of order $\frac{1}{2}$. That is, for any $\phi \in C_b^2(L^2)$, there exists a constant $C = C(u_0, \phi)$ independent of N, T and M, such that for any $T = M\tau$,*

$$\left| E[\phi(u_N(T))] - E[\phi(u_N^M)] \right| \leq C \tau^{\frac{1}{2}}.$$

Corollary 2 *Under above assumptions, for any $t \in [(M-1)\tau, (M+1)\tau]$, it also holds*

$$\left| E[\phi(u_N(t))] - E[\phi(u_N^M)] \right| \leq C \tau^{\frac{1}{2}}.$$

Proof Let $T = M\tau$. As

$$\left| E[\phi(u_N(t))] - E[\phi(u_N^M)] \right| = \left| E[\phi(u_N(T))] - E[\phi(u_N(t))] \right| + \left| E[\phi(u_N(T))] - E[\phi(u_N^M)] \right|$$

and

$$\left|E[\phi(u_N(T))] - E[\phi(u_N(t))]\right| \leq \|\phi\|_{C_b^1} E\|u_N(T) - u_N(t)\|_0$$
$$\leq \|\phi\|_{C_b^1}(T-t)\sup_{t\geq 0}\left[E\|u_N(t)\|_2 + E\|u_N(t)\|_0 + E\|u_N(t)\|_1^2\|u_N(t)\|_0\right]$$
$$+\|\phi\|_{C_b^1} E\|\pi_N Q^{\frac{1}{2}}\big(W(T) - W(t)\big)\|_0 \leq C\tau^{\frac{1}{2}},$$

we then complete the proof according to Theorem 4.3. □

Proof of Theorem 4.3 We split it into several steps.

Step 1. Calculation of $E[\phi(u_N(T))]$.

Recall the process we constructed in the proof of Theorem 3.2,

$$dY_N(t) = H_N(Y_N(t))dt + S(T-t)\pi_N Q^{\frac{1}{2}} dW(t).$$

Now we denote $v_N(T-t, y) = E[\phi(Y_N(T))|Y_N(t) = y]$, then

$$v_N(0, Y_N(T)) = v_N(T, Y_N(0)) + \int_0^T \Big(Dv_N(T-t, Y_N(t)), S(T-t)\pi_N Q^{\frac{1}{2}} dW(t)\Big), \tag{4.13}$$

where

$$\begin{aligned} v_N(0, Y_N(T)) &= E[\phi(u_N(T))|Y_N(T) = u_N(T)],\\ v_N(T, Y_N(0)) &= E[\phi(Y_N(T))|Y_N(0) = S(T)u_N(0)]\\ &= E\left[\phi\left(S(T)u_N(0) + \int_0^T H_N(Y_N(s))ds + \int_0^T S(T-s)\pi_N Q^{\frac{1}{2}} dW\right)\Bigg| Y_N \right.\\ &\left.= S(T)u_N(0)\right]. \end{aligned} \tag{4.14}$$

The expectation of Eq. 4.13 implies,

$$E[\phi(u_N(T))] = E\left[\phi\left(S(T)u_N(0) + \int_0^T H_N(Y_N(s))ds + \int_0^T S(T-s)\pi_N Q^{\frac{1}{2}} dW\right)\right]. \tag{4.15}$$

Step 2. Calculation of $E[\phi(u_N^M)]$.

Similar to [9], we define a discrete modified process

$$\begin{aligned} Y_N^k &:= S_\tau^{M-k} u_N^k\\ &= S_\tau^M u_N^0 + \mathbf{i}\lambda\tau e^{\alpha\tau}\sum_{l=1}^k S_\tau^{M+1-l}\pi_N\left(\frac{|u_N^l|^2 + |e^{-\alpha\tau}u_N^{l-1}|^2}{2}u_N^l\right) \quad (4.16)\\ &+ e^{\alpha\tau}\sum_{l=1}^k S_\tau^{M+1-l}\pi_N Q^{\frac{1}{2}}\delta W_l\\ &= S_\tau^M u_N^0 + \mathbf{i}\lambda\tau e^{\alpha\tau}\sum_{l=1}^k S_\tau^{M+1-l}\pi_N\left(\frac{|S_\tau^{l-M}Y_N^l|^2 + |e^{-\alpha\tau}S_\tau^{l-1-M}Y_N^{l-1}|^2}{2}S_\tau^{l-M}Y_N^l\right) \quad (4.17)\\ &+ e^{\alpha\tau}\sum_{l=1}^k S_\tau^{M+1-l}\pi_N Q^{\frac{1}{2}}\delta W_l. \end{aligned}$$

Consider the following time continuous interpolation of Y_N^k, which is also V_N-valued and $\{\mathcal{F}_t\}_{t\geq 0}$-adaped,

$$\begin{aligned}\tilde{Y}_N(t) &:= S_\tau^M u_N^0 + \mathbf{i}\lambda e^{\alpha\tau}\int_0^t \sum_{l=1}^M S_\tau^{M+1-l}\pi_N\left(\frac{|S_\tau^{l-M}Y_N^l|^2+|e^{-\alpha\tau}S_\tau^{l-1-M}Y_N^{l-1}|^2}{2}S_\tau^{l-M}Y_N^l\right)1_l(s)ds\\ &\quad + e^{\alpha\tau}\int_0^t \sum_{l=1}^M S_\tau^{M+1-l}\pi_N Q^{\frac{1}{2}}1_l(s)dW(s)\\ &=: S_\tau^M u_N^0 + \int_0^t H_\tau(Y_N^M, s)ds + e^{\alpha\tau}\int_0^t \sum_{l=1}^M S_\tau^{M+1-l}\pi_N Q^{\frac{1}{2}}1_l(s)dW(s).\end{aligned}$$

In particular for $t \in [t_{l-1}, t_l]$,

$$\begin{aligned}\tilde{Y}_N(t) &= Y_N^{l-1} + \mathbf{i}\lambda e^{\alpha\tau}S_\tau^{M+1-l}\pi_N\left(\frac{|S_\tau^{l-M}Y_N^l|^2+|e^{-\alpha\tau}S_\tau^{l-1-M}Y_N^{l-1}|^2}{2}S_\tau^{l-M}Y_N^l\right)(t-t_{l-1})\\ &\quad + e^{\alpha\tau}S_\tau^{M+1-l}\pi_N Q^{\frac{1}{2}}\Big(W(t)-W(t_{l-1})\Big),\end{aligned}\tag{4.18}$$

or equivalently,

$$\begin{aligned}\tilde{Y}_N(t) &= Y_N^{l} + \mathbf{i}\lambda e^{\alpha\tau}S_\tau^{M+1-l}\pi_N\left(\frac{|S_\tau^{l-M}Y_N^l|^2+|e^{-\alpha\tau}S_\tau^{l-1-M}Y_N^{l-1}|^2}{2}S_\tau^{l-M}Y_N^l\right)(t-t_{l})\\ &\quad + e^{\alpha\tau}S_\tau^{M+1-l}\pi_N Q^{\frac{1}{2}}\Big(W(t)-W(t_{l})\Big).\end{aligned}\tag{4.19}$$

Apply Itô's formula to $t \mapsto v_N(T-t, \tilde{Y}_N(t))$,

$$\begin{aligned}&dv_N(T-t,\tilde{Y}_N(t))\\ &= \frac{\partial v_N}{\partial t}(T-t,\tilde{Y}_N(t))dt + \left(Dv_N, H_\tau(Y_N^M,t)dt + e^{\alpha\tau}\sum_{l=1}^M S_\tau^{M+1-l}\pi_N Q^{\frac{1}{2}}1_l(t)dW(t)\right)\\ &\quad + \frac{1}{2}Tr\left[\left(e^{\alpha\tau}\sum_{l=1}^M S_\tau^{M+1-l}\pi_N Q^{\frac{1}{2}}1_l(t)\right)^* D^2v_N\left(e^{\alpha\tau}\sum_{l=1}^M S_\tau^{M+1-l}\pi_N Q^{\frac{1}{2}}1_l(t)\right)\right]dt\\ &= \Big(Dv_N, H_\tau(Y_N^M,t)-H_N(\tilde{Y}_N(t))\Big)dt + \left(Dv_N, e^{\alpha\tau}\sum_{l=1}^M S_\tau^{M+1-l}\pi_N Q^{\frac{1}{2}}1_l(t)dW(t)\right)\\ &\quad + \frac{1}{2}\sum_{l=1}^M Tr\left[\left(e^{\alpha\tau}S_\tau^{M+1-l}\pi_N Q^{\frac{1}{2}}\right)^* D^2v_N\left(e^{\alpha\tau}S_\tau^{M+1-l}\pi_N Q^{\frac{1}{2}}\right)\right]1_l(t)dt\\ &\quad - \frac{1}{2}\sum_{l=1}^M Tr\left[\left(S(T-t)\pi_N Q^{\frac{1}{2}}\right)^* D^2v_N\left(S(T-t)\pi_N Q^{\frac{1}{2}}\right)\right]1_l(t)dt,\end{aligned}$$

where Dv_N and D^2v_N are evaluated at $(T-t, \tilde{Y}_N(t))$.

The same as before, integrate the formula above from 0 to T, and take expectation based on the fact that

$$\begin{aligned} v_N(0, \tilde{Y}_N(T)) &= E[\phi(Y_N(T))|Y_N(T) = \tilde{Y}_N(T)] = E[\phi(u_N^M)|Y_N(T) = u_N^M], \\ v_N(T, \tilde{Y}_N(0)) &= E[\phi(Y_N(T))|Y_N(0) = \tilde{Y}_N(0)] \\ &= E\Bigg[\phi\Bigg(S_\tau^M u_N(0) + \int_0^T H_N(Y_N(s))ds \\ &\quad + \int_0^T S(T-s)\pi_N Q^{\frac{1}{2}} dW\Bigg)\Bigg| Y_N(0) = S_\tau^M u_N(0)\Bigg], \end{aligned}$$

we get

$$\begin{aligned} E[\phi(u_N^M)] &= E\left[\phi\left(S_\tau^M u_N(0) + \int_0^T H_N(Y_N(s))ds + \int_0^T S(T-s)\pi_N Q^{\frac{1}{2}} dW\right)\right] \\ &\quad + E\int_0^T \left(Dv_N, H_\tau(Y_N^M, t) - H_N(\tilde{Y}_N(t))\right) dt \\ &\quad + \frac{1}{2}\sum_{l=1}^M E\int_0^T Tr\Big[\left(e^{\alpha\tau} S_\tau^{M+1-l}\pi_N Q^{\frac{1}{2}}\right)^* D^2 v_N \left(e^{\alpha\tau} S_\tau^{M+1-l}\pi_N Q^{\frac{1}{2}}\right) \\ &\quad - \left(S(T-t)\pi_N Q^{\frac{1}{2}}\right)^* D^2 v_N \left(S(T-t)\pi_N Q^{\frac{1}{2}}\right)\Big] 1_l(t) dt. \end{aligned} \tag{4.20}$$

Step 3. Weak convergence order.

Subtracting (4.15) from (4.20), we derive

$$\begin{aligned} & E[\phi(u_N^M)] - E[\phi(u_N(T))] \\ =\ & E\Bigg[\phi\left(S_\tau^M u_N(0) + \int_0^T H_N(Y_N(s))ds + \int_0^T S(T-s)\pi_N Q^{\frac{1}{2}} dW\right) \\ & -\phi\left(S(T)u_N(0) + \int_0^T H_N(Y_N(s))ds + \int_0^T S(T-s)\pi_N Q^{\frac{1}{2}} dW\right)\Bigg] \\ & +E\int_0^T \left(Dv_N, H_\tau(Y_N^M, t) - H_N(\tilde{Y}_N(t))\right) dt \\ & +\frac{1}{2}\sum_{l=1}^M E\int_0^T Tr\Big[\left(e^{\alpha\tau} S_\tau^{M+1-l}\pi_N Q^{\frac{1}{2}}\right)^* D^2 v_N \left(e^{\alpha\tau} S_\tau^{M+1-l}\pi_N Q^{\frac{1}{2}}\right) \\ & - \left(S(T-t)\pi_N Q^{\frac{1}{2}}\right)^* D^2 v_N \left(S(T-t)\pi_N Q^{\frac{1}{2}}\right)\Big] 1_l(t) dt. \\ =:\ & I + II + III. \end{aligned}$$

Now we estimate I, II, and III separately. The constants C below may be different but are all independent of T and τ.

$$\begin{aligned} |I| &= \left| E\left[\phi\left(S_\tau^M u_N(0) + \int_0^T H_N(Y_N(s))ds + \int_0^T S(T-s)\pi_N Q^{\frac{1}{2}} dW\right)\right]\right. \\ &\quad \left. -E\left[\phi\left(S(T)u_N(0) + \int_0^T H_N(Y_N(s))ds + \int_0^T S(T-s)\pi_N Q^{\frac{1}{2}} dW\right)\right]\right| \\ &\le C\|\phi\|_{C_b^1}\|S_\tau^M u_N(0) - S(T)u_N(0)\|_0 \\ &\le C\|\phi\|_{C_b^1}\|S_\tau^M - S(T)\|_{\mathcal{L}(\dot{H}^2, L^2)}\|u_N(0)\|_2 \\ &\le C(T+\tau)^{\frac{1}{2}} e^{-\alpha T}\tau^{\frac{1}{2}}, \end{aligned} \tag{4.21}$$

where we have used Lemma 2 and $u_N(0) = \pi_N u_0 \in \dot{H}^2$.

Noticing $II = 0$ for $\lambda = 0$, now we consider the nonlinear term II for $\lambda = -1$. By using the notation $a_l := S_\tau^{l-M} Y_N^l = u_N^l$ and Eqs. 4.18 and 4.19, we can define b_l in two ways,

$$\begin{aligned} b_l &:= S(t-T)\tilde{Y}_N(t)1_l(t) \\ &= S(t-T)S_\tau^{M+1-l} u_N^{l-1} + e^{\alpha\tau} S(t-T) S_\tau^{M+1-l}\Bigg(\mathbf{i}\lambda\pi_N\left(\frac{|e^{-\alpha\tau}u_N^{l-1}|^2 + |u_N^l|^2}{2} u_N^l\right)(t-t_{l-1}) \\ &\quad +\pi_N Q^{\frac{1}{2}}\left(W(t) - W(t_{l-1})\right)\Bigg), \end{aligned}$$

or equivalently,

$$\begin{aligned} b_l &:= S(t-T)\tilde{Y}_N(t)1_l(t) \\ &= S(t-T)S_\tau^{M-l} u_N^{l} + e^{\alpha\tau} S(t-T) S_\tau^{M+1-l}\Bigg(\mathbf{i}\lambda\pi_N\left(\frac{|e^{-\alpha\tau}u_N^{l-1}|^2 + |u_N^l|^2}{2} u_N^l\right)(t-t_{l}) \\ &\quad +\pi_N Q^{\frac{1}{2}}\left(W(t) - W(t_{l})\right)\Bigg). \end{aligned}$$

Hence, we have

$$\begin{aligned} &a_{l-1} - b_l \\ &= \left(Id - S(t-T)S_\tau^{M+1-l}\right) u_N^{l-1} \\ &\quad -e^{\alpha\tau} S(t-T) S_\tau^{M+1-l}\left(\mathbf{i}\lambda\pi_N\left(\frac{|e^{-\alpha\tau}u_N^{l-1}|^2 + |u_N^l|^2}{2} u_N^l\right)(t-t_{l-1})\right. \\ &\quad \left. +\pi_N Q^{\frac{1}{2}}\left(W(t) - W(t_{l-1})\right)\right) \end{aligned}$$

and

$$\begin{aligned} a_l - b_l = &\left(Id - S(t-T)S_\tau^{M-l}\right) u_N^{l} \\ &-e^{\alpha\tau} S(t-T) S_\tau^{M+1-l}\left(\mathbf{i}\lambda\pi_N\left(\frac{|e^{-\alpha\tau}u_N^{l-1}|^2 + |u_N^l|^2}{2} u_N^l\right)\right. \\ &\left. +\pi_N Q^{\frac{1}{2}}\left(W(t) - W(t_{l})\right)\right), \end{aligned}$$

where $\|S(t-T)S_\tau^{M+1-l}\|_{\mathcal{L}(L^2,L^2)} \le C$ and

$$\|Id-S(t-T)S_\tau^{M-l}\|_{\mathcal{L}(\dot{H}^2,L^2)} \le \|S(t-T)\|_{\mathcal{L}(L^2,L^2)}\|S(T-t)-S_\tau^{M-l}\|_{\mathcal{L}(\dot{H}^2,L^2)} \le C(T-t+\tau)^{\frac{1}{2}}\tau^{\frac{1}{2}}$$

according to Lemma 2s. Thus, we have the following estimate

$$\|a_l-b_l\|_0 \le C\Big[(T-t+\tau)^{\frac{1}{2}}\tau^{\frac{1}{2}}\|u_N^l\|_2+\tau\Big(\|u_N^{l-1}\|_1^2+\|u_N^l\|_1^2\Big)\|u_N^l\|_0+\|\pi_N Q^{\frac{1}{2}}(W(t)-W(t_l))\|_0\Big].$$

Also, $\|a_{l-1}-b_l\|_0$ can be estimated in the same way. Thus, based on Eq. 3.7, we have

$$|II| = \left|E\int_0^T \Big(Dv_N, H_\tau(Y_N^M,t)-H_N(\tilde{Y}_N(t))\Big)\,dt\right| \le C\|\phi\|_{C_b^1}\int_0^T E\|H_\tau(Y_N^M,t)-H_N(\tilde{Y}_N(t))\|_0 dt, \tag{4.22}$$

where

$$\begin{aligned}
&H_\tau(Y_N^M,t)-H_N(\tilde{Y}_N(t))\\
&=\sum_{l=1}^M\left[e^{\alpha\tau}S_\tau^{M+1-l}\pi_N\left(\mathbf{i}\lambda\frac{|e^{-\alpha\tau}a_{l-1}|^2+|a_l|^2}{2}a_l\right)-S(T-t)\pi_N\Big(\mathbf{i}\lambda|b_l|^2b_l\Big)\right]1_l(t)\\
&=\frac{\lambda}{2}\mathbf{i}\sum_{l=1}^M\Big[e^{\alpha\tau}\Big(S_\tau^{M+1-l}-S(T-t)\Big)\pi_N\Big(|e^{-\alpha\tau}a_{l-1}|^2a_l\Big)\\
&\quad+(e^{-\alpha\tau}-1)S(T-t)\pi_N\Big(|a_{l-1}|^2a_l\Big)\\
&\quad+S(T-t)\pi_N\Big(|a_{l-1}|^2a_l-|b_l|^2b_l\Big)\Big]1_l(t)\\
&\quad+\frac{\lambda}{2}\mathbf{i}\sum_{l=1}^M\Big[e^{\alpha\tau}\Big(S_\tau^{M+1-l}-S(T-t)\Big)\pi_N\Big(|a_l|^2a_l\Big)+(e^{\alpha\tau}-1)S(T-t)\pi_N\Big(|a_l|^2a_l\Big)\\
&\quad+S(T-t)\pi_N\Big(|a_l|^2a_l-|b_l|^2b_l\Big)\Big]1_l(t)\\
&=\frac{\lambda}{2}\mathbf{i}\Big[\sum_{l=1}^M e^{\alpha\tau}\Big(S_\tau^{M+1-l}-S(T-t)\Big)\pi_N\Big(|e^{-\alpha\tau}a_{l-1}|^2a_l\Big)1_l(t)\\
&\quad+\sum_{l=1}^M S(T-t)\pi_N\Big(|a_{l-1}|^2(a_l-b_l)\Big)1_l(t)\\
&\quad+\sum_{l=1}^M S(T-t)\pi_N\Big(|b_l|^2(a_{l-1}-b_l)\Big)1_l(t)+\sum_{l=1}^M S(T-t)\pi_N\Big(a_{l-1}b_l(\overline{a}_{l-1}-\overline{b}_l)\Big)1_l(t)\\
&\quad+\sum_{l=1}^M(e^{-\alpha\tau}-1)S(T-t)\pi_N\Big(|a_{l-1}|^2a_l\Big)1_l(t)\\
&\quad+\sum_{l=1}^M e^{\alpha\tau}\Big(S_\tau^{M+1-l}-S(T-t)\Big)\pi_N\Big(|a_l|^2a_l\Big)1_l(t)
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{l=1}^{M} S(T-t)\pi_N\Big(|a_l|^2\,(a_l-b_l)\Big)1_l(t)+\sum_{l=1}^{M} S(T-t)\pi_N\Big(|b_l|^2(a_l-b_l)\Big)1_l(t)\\
&+\sum_{l=1}^{M} S(T-t)\pi_N\Big(a_l b_l(\overline{a}_l-\overline{b}_l)\Big)1_l(t)\Big]+\sum_{l=1}^{M}(e^{\alpha\tau}-1)S(T-t)\pi_N\Big(|a_l|^2 a_l\Big)1_l(t)\\
:=\;&\frac{\lambda}{2}\mathbf{i}\Big[II_1^{l-1}+II_2^{l-1}+II_3^{l-1}+II_4^{l-1}+II_5^{l-1}+II_1^{l}+II_2^{l}+II_3^{l}+II_4^{l}+II_5^{l}\Big].
\end{aligned}
$$

If $\lambda=-1$, thanks to the uniform estimations of 0-norm, 1-norm and 2-norm of u_N^k, we have the following estimates.

By the embedding $H^1\hookrightarrow L^\infty$ in $\mathbb{R}^1$, we have following exponential estimates

$$
\begin{aligned}
E\|II_1^{l-1}\|_0 &\le \frac{1}{2}\sum_{l=1}^{M}\|S_\tau^{M+1-l}-S(T-t)\|_{\mathcal{L}(\dot H^2,L^2)}E\left\|\pi_N\left(|e^{-\alpha\tau}u_N^{l-1}|^2u_N^l\right)\right\|_2 1_l(t)\\
&\le C\sum_{l=1}^{M}\|S_\tau^{M+1-l}-S(T-t)\|_{\mathcal{L}(\dot H^2,L^2)}E\left[\|u_N^{l-1}\|_1^4+\|u_N^l\|_2^2\right]1_l(t)\\
&\le C(T-t+\tau)^{\frac{1}{2}}e^{-\alpha(T-t)}\tau^{\frac{1}{2}},
\end{aligned}
$$

$$
\begin{aligned}
E\|II_2^{l-1}\|_0 &\le Ce^{-\alpha(T-t)}E\sum_{l=1}^{M}\|a_{l-1}\|_1^2\|a_l-b_l\|_0 1_l(t)\\
&\le \quad Ce^{-\alpha(T-t)}E\sum_{l=1}^{M}\|u_N^{l-1}\|_1^2\Big[C(T-t+\tau)^{\frac{1}{2}}\tau^{\frac{1}{2}}\|u_N^l\|_2\\
&\quad +C\left[\left(\|u_N^{l-1}\|_1^2+\|u_N^l\|_1^2\right)\|u_N^l\|_0\tau+\|\pi_N Q^{\frac{1}{2}}(W(t)-W(t_l))\|_0\right]\Big]1_l(t)\\
&\le C(T-t+1)^{\frac{1}{2}}e^{-\alpha(T-t)}\tau^{\frac{1}{2}},\\
E\|II_5^{l-1}\|_0 &\le e^{-\alpha(T-t)}(1-e^{-\alpha\tau})E\Big[\|u_N^{l-1}\|_1^2\|u_N^l\|_0\Big]\le Ce^{-\alpha(T-t)}\tau,
\end{aligned}
$$

and their integrals are also of order $\frac{1}{2}$. II_1^l, II_2^l and II_5^l can also be estimated in the same way, where we have used the fact that for any $T>0$, the integral $\int_0^T(T-t+\tau)^{\frac{1}{2}}e^{-\alpha(T-t)}dt$ is bounded and $\sum_{l=1}^M 1_l(t)=1$.

Other terms are proved in the same procedure by using the fact that

$$
\|b_l\|_{L^\infty}^2\le C\|S(t-T)S_\tau^{M-l}\|_{\mathcal{L}(\dot H^1,\dot H^1)}^2[\|u_N^l\|_1^4+\|u_N^{l-1}\|_1^4+\|\pi_N Q^{\frac{1}{2}}\delta W_l\|_1^2]
$$

and $\|a_lb_l\|_{L^\infty}\le\frac{1}{2}[\|a_l\|_{L^\infty}^2+\|b_l\|_{L^\infty}^2]$. Finally, we have

$$
|II|\le C\tau^{\frac{1}{2}}. \tag{4.23}
$$

Next is the estimate of III, which is similar to the same part in the proof of Theorem 3.2.

$$
\begin{aligned}
III &= \frac{1}{2}\sum_{l=1}^{M} E\int_0^T Tr\Big[\left(e^{\alpha\tau}S_\tau^{M+1-l}\pi_N Q^{\frac{1}{2}}\right)^* D^2 v_N \left(e^{\alpha\tau}S_\tau^{M+1-l}\pi_N Q^{\frac{1}{2}}\right) \\
&\quad - \left(S(T-t)\pi_N Q^{\frac{1}{2}}\right)^* D^2 v_N \left(S(T-t)\pi_N Q^{\frac{1}{2}}\right)\Big] 1_l(t)dt \\
&= \frac{1}{2}\sum_{l=1}^{M} E\int_0^T Tr\Big[\left(\left(e^{\alpha\tau}S_\tau^{M+1-l} - S(T-t)\right)\pi_N Q^{\frac{1}{2}}\right)^* D^2 v_N \left(\left(e^{\alpha\tau}S_\tau^{M+1-l} - S(T-t)\right)\pi_N Q^{\frac{1}{2}}\right)\Big] \\
&\quad + 2Tr\Big[\left(\left(e^{\alpha\tau}S_\tau^{M+1-l} - S(T-t)\right)\pi_N Q^{\frac{1}{2}}\right)^* D^2 v_N \left(S(T-t)\pi_N Q^{\frac{1}{2}}\right)\Big] 1_l(t)dt \\
&= \frac{1}{2}\sum_{l=1}^{M} E\int_0^T Tr\Big[e^{2\alpha\tau}\left(\left(S_\tau^{M+1-l} - S(T-t)\right)\pi_N Q^{\frac{1}{2}}\right)^* D^2 v_N \left(\left(S_\tau^{M+1-l} - S(T-t)\right)\pi_N Q^{\frac{1}{2}}\right) \\
&\quad + 2e^{2\alpha\tau}\left(\left(S_\tau^{M+1-l} - S(T-t)\right)\pi_N Q^{\frac{1}{2}}\right)^* D^2 v_N \left(S(T-t)\pi_N Q^{\frac{1}{2}}\right) \\
&\quad + (e^{2\alpha\tau}-1)\left(S(T-t)\pi_N Q^{\frac{1}{2}}\right)^* D^2 v_N \left(S(T-t)\pi_N Q^{\frac{1}{2}}\right)\Big] 1_l(t)dt \\
&:= \frac{1}{2}\sum_{l=1}^{M} E\int_0^T (A_l + 2B_l + C_l) 1_l(t)dt,
\end{aligned}
$$

where A_l, B_l and C_l satisfy

$$
\begin{aligned}
E|A_l| &\le C\|S_\tau^{M+1-l} - S(T-t))\|^2_{\mathcal{L}(\dot H^2, L^2)}\|\pi_N Q^{\frac{1}{2}}\|^2_{\mathcal{L}(L^2,\dot H^2)}\|\phi\|_{C_b^2} \le C(T-t+\tau)e^{-2\alpha(T-t)}\tau, \\
E|B_l| &\le C\|S_\tau^{M+1-l} - S(T-t))\|_{\mathcal{L}(\dot H^2, L^2)}\|\pi_N Q^{\frac{1}{2}}\|^2_{\mathcal{L}(L^2,\dot H^2)}\|\phi\|_{C_b^2}\|S(T-t)\|_{\mathcal{L}(L^2,L^2)} \\
&\le C(T-t+\tau)^{\frac{1}{2}} e^{-2\alpha(T-t)}\tau^{\frac{1}{2}}
\end{aligned}
$$

and

$$
E|C_l| \le C\tau\|\pi_N Q^{\frac{1}{2}}\|^2_{\mathcal{L}(L^2,L^2)}\|\phi\|_{C_b^2}\|S(T-t)\|^2_{\mathcal{L}(L^2,L^2)} \le Ce^{-2\alpha(T-t)}\tau.
$$

It follows

$$
|III| \le C\tau^{\frac{1}{2}}. \tag{4.24}
$$

We can conclude from Eqs. 4.21, 4.23 and 4.24 that,

$$
\left|E\left[\phi(u_N(T))\right] - E\left[\phi(u_N^M)\right]\right| \le C\tau^{\frac{1}{2}},
$$

where C is independent of T, M and N. □

Remark 5 For the linear case ($\lambda = 0$), as the weak convergence order depends heavily on the regularity of the solution, which depend only on the regularity of the initial value and noise, we can achieve higher order by increasing the regularity of the initial value and the noise. For example, the weak order turns out to be 1 if we assume $u_0 \in \dot H^4$ and $\|Q^{\frac{1}{2}}\|_{\mathcal{HS}(L^2,\dot H^4)} < \infty$. However, for the nonlinear case ($\lambda = \pm 1$), it is too technical to obtain the uniform higher regularity under proper assumptions, as a result, we work under the assumptions $u_0 \in \dot H^2$ and $\|Q^{\frac{1}{2}}\|_{\mathcal{HS}(L^2,\dot H^2)} < \infty$ and derive order $\frac{1}{2}$.

4.4 Convergence Order between Invariant Measures μ_N and μ_N^τ

Theorem 4.4 *For $\lambda = 0$ or -1, assume that $u_0 \in \dot{H}^2$ and $\|Q^{\frac{1}{2}}\|_{\mathcal{HS}(L^2,\dot{H}^2)} < \infty$, the error between invariant measures μ_N and μ_N^τ is of order $\frac{1}{2}$, i.e.,*

$$\left|\int_{V_N} \phi(y)d\mu_N(y) - \int_{V_N} \phi(y)d\mu_N^\tau(y)\right| < C\tau^{\frac{1}{2}}, \quad \forall\, \phi \in C_b^2(L^2).$$

Proof By the ergodicity of stochastic processes u_N and u_N^k, we have

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T E\phi\big(u_N(t)\big)dt = \int_{V_N} \phi(y)d\mu_N(y), \tag{4.25}$$

$$\lim_{M\to\infty} \frac{1}{M}\sum_{k=0}^{M-1} E\phi(u_N^k) = \int_{V_N} \phi(y)d\mu_N^\tau(y) \tag{4.26}$$

for any $\phi \in C_b^2(L^2)$. As the weak error is proved to be independent of step k and time t in Theorem 4.3, it turns out that for a fixed τ,

$$\begin{aligned}&\left|\int_{V_N} \phi(y)d\mu_N(y) - \int_{V_N} \phi(y)d\mu_N^\tau(y)\right|\\ \le\ & \lim_{\substack{M\to\infty,\\ T=M\tau\to\infty}} \frac{1}{T}\sum_{k=0}^{M-1}\int_{t_k}^{t_{k+1}} \left|E\phi\big(u_N(t)\big) - E\phi(u_N^k)\right| dt \le C\tau^{\frac{1}{2}}.\end{aligned}$$

□

Remark 6 For the case $\lambda = 1$, if the 1-norm and 2-norm of u_N^k is also uniformly bounded, we can also get order $\frac{1}{2}$ for both time-independent weak error and error between invariant measures. If not, based on the fact $\|\cdot\|_{s+1} \le N\|\cdot\|_s$, we can get the weak error depend on N

$$\left|E[\phi(u_N(T))] - E[\phi(u_N^M)]\right| \le CN^4\tau^{\frac{1}{2}},$$

as well as the error between invariant measures.

5 Numerical Experiments

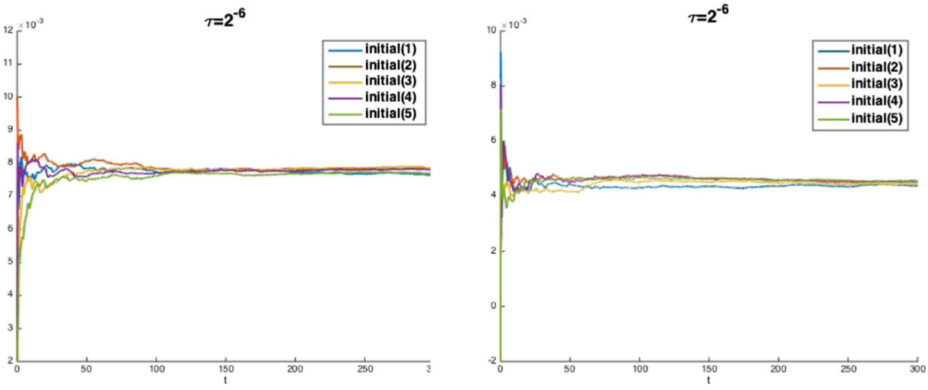
This section provides numerical experiments to test the longtime behavior of scheme (4.1) for the case $\lambda = 0$. Based on the spatial semi-discretization in stochastic ordinary differential equation form Eq. 3.2

$$da_m(t) = -\mathbf{i}(m\pi)^2 a_m(t)dt - \alpha a_m(t)dt + \sqrt{\eta_m}d\beta_m(t), \quad 1 \le m \le N,$$

we derive an equivalent form of the full discretization (4.1) as

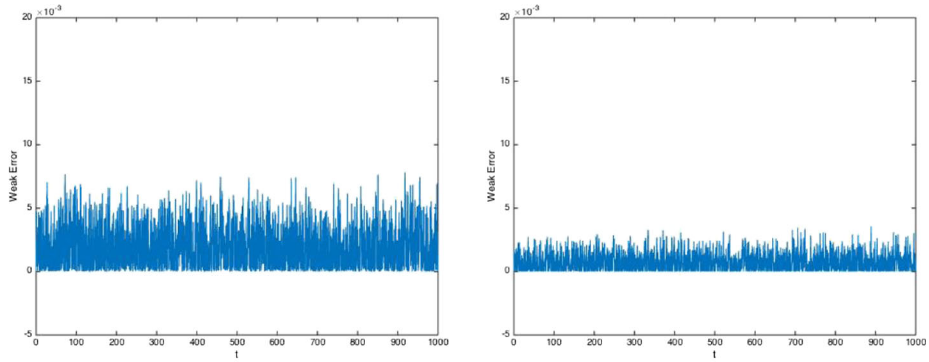
$$\vec{a}^k - e^{-\alpha\tau}\vec{a}^{k-1} = -\mathbf{i}\tau\pi^2\begin{pmatrix}1 & & \\ & \ddots & \\ & & N^2\end{pmatrix}\vec{a}^k + \begin{pmatrix}\sqrt{\eta_1}\delta_k\beta_1\\ \vdots \\ \sqrt{\eta_N}\delta_k\beta_N\end{pmatrix},$$

where $\vec{a}^k := (a_1^k, \cdots, a_N^k)^T$ is an approximation of $\vec{a}(t) := (a_1(t), \cdots, a_N(t))^T$ and $\delta_k\beta_m = \beta_m(t_k) - \beta_m(t_{k-1})$ for $1 \le m \le N$. In the sequel, we take $\alpha = 1$, $N = 100$.

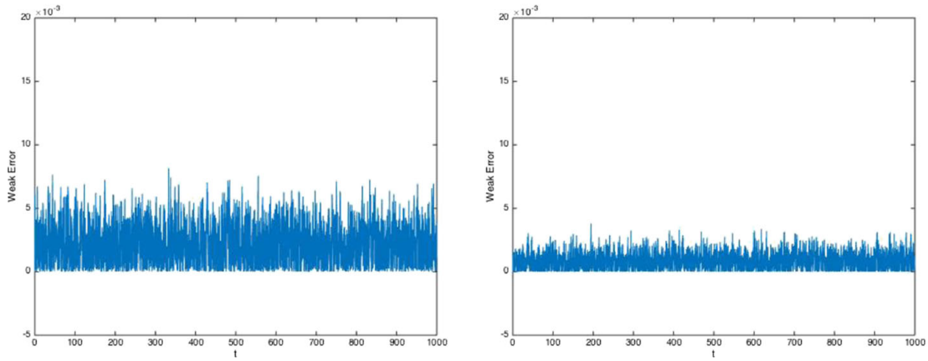


(a) $\phi(\vec{a}) = \exp(-\|\vec{a}\|_{l^2}^2)$ (b) $\phi(\vec{a}) = \sin(\|\vec{a}\|_{l^2})$

Fig. 1 The temporal averages $\frac{1}{M+1}\sum_{k=0}^{M} E[\phi(\vec{a}^k)]$ started from different initial values ($\tau = 2^{-6}$, $T = 300$)

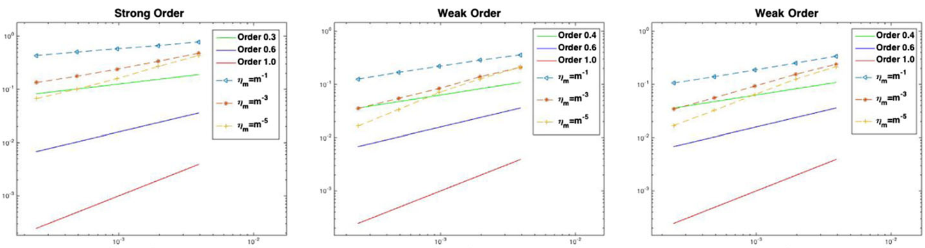


(a) $\phi(\vec{a}) = \exp(-\|\vec{a}\|_{l^2}^2), \tau = 2^{-4}$ (b) $\phi(\vec{a}) = \exp(-\|\vec{a}\|_{l^2}^2), \tau = 2^{-6}$



(c) $\phi(\vec{a}) = \sin(\|\vec{a}\|_{l^2}), \tau = 2^{-4}$ (d) $\phi(\vec{a}) = \sin(\|\vec{a}\|_{l^2}), \tau = 2^{-6}$

Fig. 2 The weak error $E[\phi(\vec{a}(t_k)) - \phi(\vec{a}^k)]$ for different ϕ and step size τ with $t_k = k\tau \in [0, T]$ and $T = 10^3$



(a) Strong Order (b) $\phi(\vec{a}) = \exp(-\|\vec{a}\|_{l^2}^2)$ (c) $\phi(\vec{a}) = \sin(\|\vec{a}\|_{l^2})$

Fig. 3 The strong and weak orders for noise in L^2, $\dot{H}^2$ and $\dot{H}^4$, i.e., $\eta_m = m^{-1}, m^{-3}, m^{-5}$. ($T = \frac{1}{2}$, $\tau \in \{2^{-i}, 5 \le i \le 9\}$)

In Fig. 1, the temporal averages $\frac{1}{M+1}\sum_{k=1}^{M} E[\phi(\vec{a}^k)]$ of the fully discrete scheme started from five different initial values initial(1) $= (1, 0, \cdots, 0)^T$, initial(2) $= (0.0003\mathbf{i}, 0, \cdots, 0)^T$, initial(3) $= \left(\sin\left(\frac{1}{101}\pi\right), \sin\left(\frac{2}{101}\pi\right), \cdots, \sin\left(\frac{100}{101}\pi\right)\right)^T$, initial(4) $= \left(\frac{2+\mathbf{i}}{20}\right)(1, 2, \cdots, 100)^T$ and initial(5) $= \left(\exp\left(-\frac{\mathbf{i}}{50}\right), \exp\left(-\frac{2\mathbf{i}}{50}\right), \cdots, \exp\left(-\frac{100\mathbf{i}}{50}\right)\right)^T$ will converge to the same value with error $\tau^{\frac{1}{2}}$ before time T, where $\tau = 2^{-6}$ and $T = 300$. This result verifies the ergodicity of the numerical solution: the temporal averages converge to the spatial average, which is a constant, for almost every initial values in the whole space. We choose 500 realizations to approximate the expectations in Figs. 1 and 2, and choose 1000 realizations in Fig. 3.

In Figs. 2 and 3, we fix the initial value $u_0(x)$ as $\sqrt{2}\sin(\pi x)$, such that $a_m(0) = (u_0, e_m)$ and $\vec{a}^0 = \vec{a}(0) = (1, 0, \cdots, 0)^T$. Figure 2 displays the weak error $E[\phi(\vec{a}(t_k)) - \phi(\vec{a}^k)]$ over long time $T = 10^3$ for different time step sizes and test functions: (a) $\tau = 2^{-4}$, $\phi(\vec{a}) = \exp(-\|\vec{a}\|_{l^2}^2)$ (b) $\tau = 2^{-6}$, $\phi(\vec{a}) = \exp(-\|\vec{a}\|_{l^2}^2)$, (c) $\tau = 2^{-4}$, $\phi(\vec{a}) = \sin(\|\vec{a}\|_{l^2})$ and (d) $\tau = 2^{-6}$, $\phi(\vec{a}) = \sin(\|\vec{a}\|_{l^2})$. The reference values are generated for the time step size $\tau = 2^{-8}$, and the noise is chosen in $\dot{H}^2$, i.e., $\eta_m = m^{-3}$. Figure 2 shows that the weak error is independent of time interval and can be controlled by $C\tau^{\frac{1}{2}}$, which coincides with our theoretical results. Figure 3 displays both (a) the strong convergence order and the rates of weak convergence for (b) $\phi(\vec{a}) = \exp(-\|\vec{a}\|_{l^2}^2)$ or (c) $\phi(\vec{a}) = \sin(\|\vec{a}\|_{l^2})$. The reference values are generated for the time step size $\tau = 2^{-14}$. As the initial value $u_0(x) = \sqrt{2}\sin(\pi x)$ is regular enough, both the strong and weak convergence order depend heavily on the regularity of the noise for the linear case. It shows in Fig. 3 that the orders slightly increase as the noise from L^2 via $\dot{H}^2$ to $\dot{H}^4$ (i.e., η_m from m^{-1} via m^{-3} to m^{-5}), which verifies Remark 5. Noticing that the orders are a little bit better than the theoretical results, because the truncation of the noise makes the noise more regular than it should be, which increases the orders slightly. Numerical tests also shows that the weak convergence order is almost the same as the strong convergence order, which is similar to the statement in [7] (Remark 5.11).

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Appendix

The Proof of Proposition 3.1

i) As it is proved in Part 3 of Theorem 3.1 that $E\|u_N(t)\|_0^2 < C$, we assume further that $E\|u_N(t)\|_0^{2n} < C,\ \forall\ n = 1, \cdots, p-1$. Denoting $dM_1 := 2Re\left(u_N, \pi_N Q^{\frac{1}{2}} dW\right)$, then Itô's formula and Eq. 3.5 yields

$$
\begin{aligned}
d\|u_N(t)\|_0^{2p} &= p\|u_N(t)\|_0^{2(p-1)} d\|u_N(t)\|_0^2 + \frac{1}{2}p(p-1)\|u_N(t)\|_0^{2(p-2)} d\langle M_1\rangle \\
&\le -2\alpha p\|u_N(t)\|_0^{2p} dt + p\|u_N(t)\|_0^{2(p-1)} dM_1(t) \\
&\quad +2p(2p-1)\sum_{m=1}^{N} \eta_m \|u_N(t)\|_0^{2(p-1)} dt,
\end{aligned}
$$

where $\langle\cdot\rangle$ denotes the quadratic variation process and in the last step we used the fact

$$
\begin{aligned}
d\langle M_1\rangle &= 4\left\langle Re\sum_{m=1}^{N}\int_0^1 \overline{u}_N(s)\sqrt{\eta_m} e_m(x)dx(d\beta_{m,1} + \mathbf{i} d\beta_{m,2})\right\rangle \\
&= 4\sum_{m=1}^{N}\left[\left(Re\int_0^1 \overline{u}_N(t,x)\sqrt{\eta_m} e_m(x)dx\right)^2 + \left(Im\int_0^1 \overline{u}_N(t,x)\sqrt{\eta_m} e_m(x)dx\right)^2\right] dt \\
&\le 8\sum_{m=1}^{N} \eta_m \|u_N(t)\|_0^2 dt.
\end{aligned}
$$

Taking expectation on both sides of above equation, we obtain

$$
\begin{aligned}
\frac{d}{dt}E\|u_N(t)\|_0^{2p} &\le -2\alpha p E\|u_N(t)\|_0^{2p} + 2p(2p-1)\sum_{m=1}^{N}\eta_m E\|u_N(t)\|_0^{2(p-1)} \\
&\le -2\alpha p E\|u_N(t)\|_0^{2p} + C
\end{aligned}
$$

by induction. Then multiplying $e^{2\alpha p t}$ to both sides of above equation yields the result.

ii) The proof in this part is similar to the proof of Lemma 2.5 in [8]. According to the Gagliardo-Nirenberg interpolation inequality, there exists a positive constant c_0, such that

$$
\frac{5}{8}\lambda\|u_N(t)\|_{L^4}^4 \le \|u_N(t)\|_{L^4}^4 \le \frac{1}{4}\|\nabla u_N(t)\|_0^2 + \frac{1}{2}c_0\|u_N(t)\|_0^6. \tag{1}
$$

Thus,

$$
\begin{aligned}
0 \le \mathcal{H}(u_N(t)) &:= \frac{1}{2}\|\nabla u_N(t)\|_0^2 - \frac{\lambda}{4}\|u_N(t)\|_{L^4}^4 + c_0\|u_N(t)\|_0^6 \\
&\le \frac{2}{3}\left(\|\nabla u_N(t)\|_0^2 - \lambda\|u_N(t)\|_{L^4}^4 + 2c_0\|u_N(t)\|_0^6\right).
\end{aligned} \tag{2}
$$

Applying Itô's formula to $\mathcal{H}(u_N(t))$, it leads to

$$\begin{aligned} d\mathcal{H}(u_N(t)) = \Bigg[& -\alpha\|\nabla u_N(t)\|_0^2 + \alpha\lambda\|u_N(t)\|_{L^4}^4 - 6\alpha c_0\|u_N(t)\|_0^6 \\ & -2\lambda\int_0^1 |u_N|^2 \sum_{m=1}^N \eta_m|e_m|^2 dx \\ & +\sum_{m=1}^N m^2\eta_m + 6c_0\|u_N(t)\|_0^4 \sum_{m=1}^N \eta_m \\ & +12c_0\|u_N(t)\|_0^2\|\pi_N Q^{\frac{1}{2}} u_N(t)\|_0^2\Bigg]dt \\ & +6c_0\|u_N(t)\|_0^4 Re\left(u_N, \pi_N Q^{\frac{1}{2}} dW\right) \\ & -Re\left(\Delta u_N(t) + \lambda|u_N(t)|^2 u_N(t), \pi_N Q^{\frac{1}{2}} dW\right), \end{aligned}$$

where we have used the fact $((Id-\pi_N)v, v_N) = 0$, $\forall\, v \in \dot{H}^0$, $v_N \in V_N$. By the following estimates

$$-2\lambda\int_0^1 |u_N|^2 \sum_{m=1}^N \eta_m|e_m|^2 dx \le 0,$$

$$6c_0\|u_N(t)\|_0^4 \sum_{m=1}^N \eta_m + 12c_0\|u_N(t)\|_0^2\|\pi_N Q^{\frac{1}{2}} u_N(t)\|_0^2 \le 4\alpha c_0\|u_N(t)\|_0^6 + C$$

and Eq. 1, we have

$$\begin{aligned} d\mathcal{H}(u_N(t)) \le \Bigg[& -\alpha\|\nabla u_N(t)\|_0^2 + \alpha\lambda\|u_N(t)\|_{L^4}^4 & (3)\\ & -2\alpha c_0\|u_N(t)\|_0^6 + \sum_{m=1}^N m^2\eta_m + C\Bigg]dt \\ & +6c_0\|u_N(t)\|_0^4 Re\left(u_N(t), \pi_N Q^{\frac{1}{2}} dW(t)\right) & (4)\\ & -Re\left(\Delta u_N(t) + \lambda|u_N(t)|^2 u_N(t), \pi_N Q^{\frac{1}{2}} dW\right) \\ \le & -\frac{3}{2}\alpha\mathcal{H}(u_N(t))dt + Cdt + dM_2, & (5) \end{aligned}$$

where

$$dM_2 := 6c_0\|u_N\|_0^4 Re\left(u_N, \pi_N Q^{\frac{1}{2}} dW\right) - Re\left(\Delta u_N + \lambda|u_N|^2 u_N, \pi_N Q^{\frac{1}{2}} dW\right).$$

Taking expectation, we derive

$$dE\mathcal{H}(u_N(t)) \le -\frac{3}{2}\alpha E\mathcal{H}(u_N(t))dt + Cdt.$$

Hence, by multiplying $e^{\frac{3}{2}\alpha t}$ to both sides of the equation above and then taking integral from 0 to t, we get the uniform boundedness for $p = 1$. By induction, we assume that the results hold for $p-1$. Then, based on the following estimates (see [8])

$$\left\langle 6\|u_N\|_0^4 Re\left(u_N, \pi_N Q^{\frac{1}{2}} dW\right)\right\rangle^2 \le C\|Q^{\frac{1}{2}}\|^2_{\mathcal{HS}(L^2,L^2)}\|u_N\|_0^{10} dt,$$

$$\left\langle Re\left(\Delta u_N + \lambda|u_N|^2 u_N, \pi_N Q^{\frac{1}{2}} dW\right)\right\rangle^2 \le C\|Q^{\frac{1}{2}}\|^2_{\mathcal{HS}(L^2,\dot{H}^1)}\left(\|\nabla u_N\|_0^2 + \|u_N\|_0^{10}\right) dt$$

and Eq. 5, we have

$$\begin{aligned} d\mathcal{H}(u_N(t))^p &= p\mathcal{H}(u_N(t))^{p-1} d\mathcal{H}(u_N(t)) + \frac{1}{2}p(p-1)\mathcal{H}(u_N(t))^{p-2} d\langle M_2\rangle \\ &\le -\frac{3}{2}\alpha p\mathcal{H}(u_N(t))^p dt + Cp\mathcal{H}(u_N(t))^{p-1} dt + p\mathcal{H}(u_N(t))^{p-1} dM_2 \\ &\quad + Cp(p-1)\mathcal{H}(u_N(t))^{p-2}\left(\|\nabla u_N(t)\|_0^2 + \|u_N(t)\|_0^{10}\right) dt. \end{aligned} \tag{6}$$

From Eq. 1, we deduce that

$$\mathcal{H}(u_N(t)) \ge \begin{cases} \frac{1}{2}\|\nabla u_N(t)\|_0^2 + c_0\|u_N(t)\|_0^6, & \lambda = 0 \text{ or } -1, \\ \frac{7}{16}\|\nabla u_N(t)\|_0^2 + \frac{7}{8}c_0\|u_N(t)\|_0^6, & \lambda = 1. \end{cases}$$

As a result, the last term in Eq. 6 can be estimated as

$$\begin{aligned} & Cp(p-1)\mathcal{H}(u_N(t))^{p-2}\left(\|\nabla u_N(t)\|_0^2 + \|u_N(t)\|_0^{10}\right) \\ \le & \left(C\mathcal{H}(u_N(t)) + C\mathcal{H}(u_N(t))^{\frac{5}{3}}\right)\mathcal{H}(u_N(t))^{p-2} \le C\mathcal{H}(u_N(t))^{p-1} + \frac{1}{2}\alpha p\mathcal{H}(u_N(t))^p, \end{aligned} \tag{7}$$

where in the last step we used the inequality of arithmetic and geometric means

$$C(\mathcal{H}(u_N(t))^2 \cdot \mathcal{H}(u_N(t))^2 \cdot \mathcal{H}(u_N(t)))^{\frac{1}{3}} \le \frac{\frac{3}{4}\alpha p\mathcal{H}(u_N(t))^2 + \frac{3}{4}\alpha p\mathcal{H}(u_N(t))^2 + C\mathcal{H}(u_N(t))}{3}.$$

Gethering Eqs. 6 and 7 and taking expectation, we obtain

$$dE\mathcal{H}(u_N(t))^p \le -\alpha p E\mathcal{H}(u_N(t))^p dt + C dt$$

by induction, which complete the proof by multiplying $e^{\alpha pt}$ on both sides of above equation.

iii) We define a functional

$$f(u) = \int_0^1 |\Delta u|^2 dx + \lambda Re\int_0^1 (\Delta\overline{u})|u|^2 u dx,$$

which satisfies

$$\|\Delta u\|_0^2 \le 2f(u) + C\|u\|_1^6 \tag{8}$$

based on the continuous embedding $H^1 \hookrightarrow L^6$ and $\left|\lambda Re\int_0^1 \Delta\overline{u}|u|^2 u dx\right| \le \frac{1}{2}\|\Delta u\|_0^2 + \frac{1}{2}\|u\|_{L^6}^6 \le \frac{1}{2}\|\Delta u\|_0^2 + C\|u\|_1^6$. The Itô's formula applied to $f(u_N)$ yields

$$\begin{aligned} df(u_N) &= Df(u_N)\left(\left(\mathbf{i}\Delta u_N + \mathbf{i}\lambda|u_N|^2 u_N - \alpha u_N\right) dt\right) + Df(u_N)\left(\pi_N Q^{\frac{1}{2}} dW\right) \\ &\quad + \frac{1}{2}D^2 f(u_N)(\pi_N Q^{\frac{1}{2}} dW, \pi_N Q^{\frac{1}{2}} dW) \\ &=: \mathcal{A} + \mathcal{B} + \mathcal{C}, \end{aligned} \tag{9}$$

where

$$
\begin{aligned}
Df(u)(\varphi) &= Re\int_0^1 \Big[2\Delta\overline{u}\Delta\varphi + 2\lambda(\Delta\overline{u})uRe(\overline{u}\varphi) + \lambda(\Delta\overline{u})|u|^2\varphi \\
&\quad +\lambda(\Delta(|u|^2u))\overline{\varphi}\Big]dx, \\
D^2f(u)(\varphi,\psi) &= Re\int_0^1 \Big[2\Delta\overline{\varphi}\Delta\psi + 2\lambda(\Delta\overline{u})uRe(\overline{\varphi}\psi) + 2\lambda(\Delta\overline{u})\varphi Re(\overline{u}\psi) \\
&\quad +2\lambda(\Delta\overline{\varphi})uRe(\overline{u}\psi) \\
&\quad +2\lambda(\Delta\overline{u})\psi Re(\overline{\varphi}u) + 2\lambda(\Delta\overline{\psi})uRe(\overline{u}\varphi) + \lambda(\Delta\overline{\varphi})|u|^2\psi \\
&\quad +\lambda(\Delta\overline{\psi})|u|^2\varphi\Big]dx
\end{aligned}
$$

and $E[\mathcal{B}] = 0$. Now we estimate $\mathcal{A}$ and $\mathcal{C}$ respectively.

$$
\begin{aligned}
E[\mathcal{A}] &= -2\alpha E[f(u_N)]dt + ReE\int_0^1 \Big[4\lambda\mathbf{i}(\Delta\overline{u}_N)u_N|\nabla u_N|^2 \\
&\quad +2\lambda\mathbf{i}(\Delta\overline{u}_N)\overline{u}_N(\nabla u_N)^2\Big]dxdt \\
&\quad +ReE\int_0^1 \Big[\lambda^2\mathbf{i}(\Delta\overline{u}_N)|u_N|^4 - 4\alpha\lambda(\Delta\overline{u}_N)u_N|u_N|^2\Big]dxdt \\
&\quad +ReE\int_0^1 \Big[-4\alpha\lambda|u_N|^2|\nabla u_N|^2 - 2\alpha\lambda(\nabla u_N)^2\overline{u}_N^2\Big]dxdt \\
&=: -2\alpha E[f(u_N)]dt + \mathcal{A}_1dt + \mathcal{A}_2dt + \mathcal{A}_3dt,
\end{aligned}
$$

where we have used the fact $\Delta(|u|^2u) = 2|u|^2\Delta u + 4u|\nabla u|^2 + 2\overline{u}(\nabla u)^2 + u^2\Delta\overline{u}$ and $\mathcal{A}_1$, $\mathcal{A}_2$ and $\mathcal{A}_3$ are estimated as follows.

$$
\begin{aligned}
|\mathcal{A}_1| &:= \left|ReE\int_0^1 \Big[4\lambda\mathbf{i}(\Delta\overline{u}_N)u_N|\nabla u_N|^2 + 2\lambda\mathbf{i}(\Delta\overline{u}_N)\overline{u}_N(\nabla u_N)^2\Big]dx\right| \\
&\le \frac{\alpha}{16}E\|\Delta u_N\|_0^2 + CE\Big[\|u_N\|_{L^\infty}^2\|\nabla u_N\|_{L^4}^2\Big] \\
&\le \frac{\alpha}{16}E\|\Delta u_N\|_0^2 + CE\Big[\|u_N\|_{L^\infty}^4 + \|\Delta u_N\|_0\|\nabla u_N\|_0^3\Big] \\
&\le \frac{\alpha}{8}E\|\Delta u_N\|_0^2 + CE\Big[\|u_N\|_1^4 + \|u_N\|_1^6\Big] \\
&\le \frac{\alpha}{8}E\|\Delta u_N\|_0^2 + C,
\end{aligned}
$$

where we have used the uniform boundedness of $\|u_N\|_1^{2p}$ for $p \ge 1$ in ii), the continuous embedding $H^1 \hookrightarrow L^\infty$ for $\mathbb{R}^1$ and the interpolation of L^4 between L^2 and H^1. Similarly, based on the continuous embedding $H^1 \hookrightarrow L^6$ and $H^1 \hookrightarrow L^8$, we have

$$
\begin{aligned}
|\mathcal{A}_2| &:= \left|ReE\int_0^1 \Big[\lambda^2\mathbf{i}(\Delta\overline{u}_N)|u_N|^4 - 4\alpha\lambda(\Delta\overline{u}_N)u_N|u_N|^2\Big]dx\right| \\
&\le \frac{\alpha}{8}E\|\Delta u_N\|_0^2 + CE[\|u_N\|_{L^8}^8 + \|u_N\|_{L^6}^6] \le \frac{\alpha}{8}E\|\Delta u_N\|_0^2 + C
\end{aligned}
$$

and

$$
|\mathcal{A}_3| := \left|ReE\int_0^1 \Big[-4\alpha\lambda|u_N|^2|\nabla u_N|^2 - 2\alpha\lambda(\nabla u_N)^2\overline{u}_N^2\Big]dx\right| \le CE\|u_N\|_1^4 \le C.
$$

Thus, we obtain

$$E[\mathcal{A}] \leq -2\alpha E[f(u_N)]dt + \frac{\alpha}{4} E\|\Delta u_N\|_0^2 + C.$$

The estimate of $\mathcal{C}$ is similar with that of $\mathcal{A}$, and we derive $E[\mathcal{C}] \leq \frac{\alpha}{4} E\|\Delta u_N\|_0^2 + C$. Taking expectation on both sides of Eq. 9 yields

$$dEf(u_N) + 2\alpha Ef(u_N)dt \leq \frac{\alpha}{2} E\|\Delta u_N\|_0^2 dt + Cdt \leq \alpha Ef(u_N)dt + Cdt.$$

Multiplying both sides of above equation by $e^{\alpha t}$ and taking integral from 0 to t, we conclude the uniform boundedness of $Ef(u_N(t))$

$$Ef(u_N(t)) \leq e^{-\alpha t} Ef(u_N(0)) + \frac{C}{\alpha}(1 - e^{-\alpha t}),$$

which yields the uniform boundedness of $E\|\Delta u_N\|_0^2$ based on Eq. 8. As the norm $\|u_N\|_2$ is equivalent to $\|\Delta u_N\|_0$ under Dirichlet boundary condition, we complete the proof.

□

The Proof of Uniqueness of the Solution for Eq. 4.1

Suppose that U and W are two solutions of the scheme, then it follows

$$U - W = \mathbf{i}\tau\Delta\big(U - W\big) + \mathbf{i}\lambda\frac{\tau}{2}\pi_N\Big[\big(|U|^2U - |W|^2W\big) + |e^{-\alpha\tau}u_N^{k-1}|^2(U - W)\Big].$$

Multiply the equation above by $\overline{U} - \overline{W}$, integrate in space and take the real and imaginary part respectively, we have

$$\|U - W\|_0^2 \leq \frac{\tau}{2}\|f(U) - f(W)\|_{L^{\frac{4}{3}}}\|U - W\|_{L^4},$$
$$\|\nabla(U - W)\|_0^2 \leq \frac{1}{2}\|f(U) - f(W)\|_{L^{\frac{4}{3}}}\|U - W\|_{L^4} + \frac{\lambda}{2}\|e^{-\alpha\tau}u_N^{k-1}\|_{L^4}^2\|U - W\|_{L^4}^2,$$

where $f(U) := |U|^2U$ and

$$\begin{aligned}
&\|f(U) - f(W)\|_{L^{\frac{4}{3}}}\\
&= \left(\int_0^1 \Big||U|^2U - |W|^2W\Big|^{\frac{4}{3}} dx\right)^{\frac{3}{4}} = \left(\int_0^1 \Big||U|^2(U - W) + |W|^2(U - W) + UW(\overline{U} - \overline{W})\Big|^{\frac{4}{3}} dx\right)^{\frac{3}{4}}\\
&\leq \left(\int_0^1 \Big||U|^2 + |W|^2 + |UW|\Big|^2 dx\right)^{\frac{1}{2}}\left(\int_0^1 |U - W|^4 dx\right)^{\frac{1}{4}} \leq \big\||U| + |W|\big\|_{L^4}^2\|U - W\|_{L^4}.
\end{aligned}$$

Since

$$
\begin{aligned}
\|U-W\|_{L^4}^4 &\le \|U-W\|_0^3\|\nabla(U-W)\|_0\\
&\le \left(\frac{\tau}{2}\|f(U)-f(W)\|_{L^{\frac{4}{3}}}\|U-W\|_{L^4}\right)^{\frac{3}{2}}\left(\frac{1}{2}\|f(U)-f(W)\|_{L^{\frac{4}{3}}}\|U-W\|_{L^4}\right.\\
&\quad\left.+\frac{|\lambda|}{2}\|e^{-\alpha\tau}u_N^{k-1}\|_{L^4}^2\|U-W\|_{L^4}^2\right)^{\frac{1}{2}}\\
&\le \frac{1}{4}\tau^{\frac{3}{2}}\big\||U|+|W|\big\|_{L^4}^3\left(\big\||U|+|W|\big\|_{L^4}^2+|\lambda|\|u_N^{k-1}\|_{L^4}^2\right)^{\frac{1}{2}}\|U-W\|_{L^4}^4\\
&\le \frac{1}{4}\tau^{\frac{3}{2}}\left(\big\||U|+|W|\big\|_{L^4}^4+|\lambda|\big\||U|+|W|\big\|_{L^4}^3\|u_N^{k-1}\|_{L^4}\right)\|U-W\|_{L^4}^4,
\end{aligned}
$$

if $U\neq W$, then

$$
\begin{aligned}
1 &\le \frac{1}{4}\tau^{\frac{3}{2}}\left(\big\||U|+|W|\big\|_{L^4}^4+|\lambda|\big\||U|+|W|\big\|_{L^4}^3\|u_N^{k-1}\|_{L^4}\right)\\
&\le C_0\tau^{\frac{3}{2}}\left(\big\||U|+|W|\big\|_{L^4}^4+|\lambda|\big\||U|+|W|\big\|_{L^4}^6+|\lambda|\|u_N^{k-1}\|_{L^4}^2\right).
\end{aligned}
$$

For cases $\lambda=0$ or -1, the L^4-norm of the solutions are uniformly bounded. So $C_0\tau^{\frac{3}{2}}>1$, which do not hold when τ is sufficiently small. For case $\lambda=1$, according to the fact that

$$
\big\||U|+|W|\big\|_{L^4}^6\le\big\||U|+|W|\big\|_0^{\frac{3}{2}}\big\|\nabla(|U|+|W|)\big\|_0^{\frac{9}{2}}\le N^{\frac{9}{2}}\big\||U|+|W|\big\|_0^6,
$$

we have $C_0N^{\frac{9}{2}}\tau^{\frac{3}{2}}>1$, which is also a contradiction when τ is sufficiently small.

Thus, the numerical solution for Eq. 4.1 is unique. □

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