Computing Slowly Advancing Features in Fast-Slow Systems without Scale Separation - A Young Measure Approach

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Overview

Background

- Classical Theory of Levinson–Tikhonov
- Motivating Example for Limit Cycle
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1 Background

- Classical Theory of Levinson–Tikhonov
- Motivating Example for Limit Cycle

2 The use of Young measures

- Young measures and description of the limit of fast dynamics
- Fast-slow systems without separation of state variables
- Computing slow observables
We consider a classical singular perturbed system

\[
\frac{dx}{dt} = f(x, y) \\
\varepsilon \frac{dy}{dt} = g(x, y)
\]

with \( x \in \mathbb{R}^n \) and \( y \in \mathbb{R}^m \), and initial conditions

\[ x(0) = x_0, \quad y(0) = y_0. \]
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Moreover, what is the equation of motion that governs this limit behavior?

Can we develop an efficient numerical algorithm for computing the above limit behavior?
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Thus, AFP stands for Asymptotically stable Fixed Point.
Consider the four dimensional system with planar slow and fast equations given by

\[
\begin{align*}
\frac{d\xi_1}{dt} &= \xi_2 \\
\frac{d\xi_2}{dt} &= -2\xi_1 - \xi_2 - \eta_1 + F(\eta_2) \\
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We compare now the behavior as $\varepsilon \to 0$ of solutions to the above system on a finite time interval, for two different potentials $F(\cdot)$:
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When the slow variable \( x = (\xi_1, \xi_2) \) is kept fixed, the fast solution \( y(t) = (\eta_1(t), \eta_2(t)) \) converges as \( t \to \infty \) to the solution of the corresponding algebraic equations, namely, to the point \( (-\xi_1 - \xi_2, 0) \).
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For the case of the stable potential:

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Summary of Asymptotically Fixed Point Case

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The Limit Cycle Case

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The stationary point determined by the algebraic equation is unstable with respect to the fast dynamics (hence the subscript \( u \), standing for unstable, is placed); in particular, the solution to the full equation is not attracted to the mentioned manifold.
However, for a fixed \((\xi_1, \xi_2)\) and with initial condition \((\eta_1, \eta_2)\) different from the fixed point, the fast solution converges to a limit cycle, thus, in this case the following Assumption LC holds.

**Assumption LC.** In the region where the analysis is carried out, when \(x\) is held fixed the solutions of the fast equation, namely of \[
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converge to a limit cycle which we denote by \(\Gamma(x)\). Thus, LC stands for Limit Cycle.
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This kind of work is in Bogoliubov-Mitropolsky and in Pontryagin-Rodygin.
Z. Artstein, J. Linshiz and E.S. Titi, *SIAM, Multiscale Modeling and Simulation, 6(4) (2007), 1085–1097.*

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<tr>
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$$\frac{dy}{ds} = g(x, y), \quad y(s_0) = \hat{y},$$

(1)

converges, as $s \to \infty$, to the limit cycle $\Gamma(x)$.

One way in which the limit cycle can be represented is as a periodic function

$$\gamma_x(\cdot) : [0, T_x] \to \mathbb{R}^m$$

(2)

with a period $T_x$ which depends on the fixed slow state $x$. 
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The measure $\mu_x$ is called in literature as Young Measure.
Young Measure

probability distribution
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for any measurable subset $C$ of $\mathbb{R}^m$, and where $\lambda$ is the Lebesgue measure on the real line. Notice that the measure $\mu_x$ is an invariant measure of fast dynamics.
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Denote by \((x_\varepsilon(t), y_\varepsilon(t))\) the solution to our system, under Assumption LC. The goal is to describe the structure of the limit of \((x_\varepsilon(\cdot), y_\varepsilon(\cdot))\), as \(\varepsilon \to 0\), on a fix time interval, say \([0, \tau_0]\).
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We also need a convergence notion for the probability measures $\mu_x$. To this end we adopt the weak convergence of measures.
Theorem Under Assumption LC the slow parts $x_\varepsilon(\cdot)$ of the solutions converge as $\varepsilon \rightarrow 0$, uniformly on the time interval $[0, \tau_0]$, to a solution $x_0(\cdot)$ of
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The present result follows from standard averaging techniques; e.g., Sanders and Verhulst.
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The topological limit is given as follows:

**Proposition** Under Assumption LC, the couple $(x_\varepsilon(t), y_\varepsilon(t))$ of slow and fast trajectories approaches, as $\varepsilon \to 0$, the tube located in $\mathbb{R}^n \times \mathbb{R}^m$, having a circular $m$-cross section, and given by

$$\{(x_0(t), \Gamma_{x_0(t)}): t \in [0, \tau_0]\}.$$  

where $x_0(t)$ is the uniform limit of $x_\varepsilon(t)$ as $\varepsilon \to 0$.  

{(5)}
A quantitative form of the convergence of the fast part can be formed when resorting to the distributions $\mu_x$. 
Quantitative description of the limit of the fast dynamics and Young measures

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Indeed, the family $\mu_{x_0}(t)$, parameterized by the time variable $t$ over the interval of integration $[0, \tau_0]$, can be viewed as a probability measure-valued map $\mu_{x_0}(\cdot)$, called in the literature a Young measure.
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The mappings \( y_{\varepsilon}(\cdot) \) can also be viewed as (degenerate) Young measures, namely, with values being Dirac measures. It can be represented as a measure

\[
\delta_{y_{\varepsilon}(t)}(dy).
\]
The Limit of the Graphs of Fast Oscillating Functions

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Fast-Slow Multi-scale Systems
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Suppose that a small interval, say $I$, is given near, say, a point $\tau$ in the interval. Then for $\varepsilon$ small enough the distribution of the fast dynamics $y_\varepsilon(\cdot)$ over the interval $I$ is very similar to the distribution $\mu_{x_0}(\tau)$ on the corresponding limit cycle.
The following computations are reported in:

Z. Artstein, J. Linshiz and E.S. Titi, *SIAM, Multiscale Modeling and Simulation*, 6(4) (2007), 1085–1097.

System I

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\end{align*}
\]
Figure 1

Transient to Limit Cycle

- transient trajectory
- limit cycle at $t=0$
- initial condition

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Fast-Slow Multi-scale Systems
The drift of the slow dynamics

Figure 2
Limit Solution tube generated by the fast dynamics

Figure 3
The second example is given by the following set of equations.

\[
\begin{align*}
\frac{dx_1}{dt} &= x_2 - 1 \\
\frac{dx_2}{dt} &= 1 - \sqrt{y_1^2 + y_2^2} \\
\varepsilon \frac{dy_1}{dt} &= x_1 \frac{y_1}{\sqrt{y_1^2 + y_2^2}} - y_1 - y_2 x_2 \\
\varepsilon \frac{dy_2}{dt} &= x_1 \frac{y_2}{\sqrt{y_1^2 + y_2^2}} - y_2 + y_1 x_2.
\end{align*}
\]

This system also has a limit solution which can be computed analytically. This can be seen when the variables in the latter two equations are written in polar coordinates as follows.
In fact, it is easy to see that the fast dynamics converges toward a limit cycle parameterized by the slow dynamics, which, in turn, oscillates.
Again, the numerics of the explicit expression for the slow dynamics cannot be distinguished from the solution obtained by our algorithm.
the limit tube of the fast dynamics (here the lines depict the limit cycles while the dots reflect the approximation to the corresponding invariant measures).
General fast dynamics with compact attractor

\[
\begin{align*}
\frac{dx}{dt} &= f(x, y) \\
\varepsilon \frac{dy}{dt} &= g(x, y)
\end{align*}
\]

with \(x \in \mathbb{R}^n\) and \(y \in \mathbb{R}^m\), and initial conditions

\[
x(0) = x_0, \quad y(0) = y_0.
\]

Suppose the fast dynamics \(\frac{dy}{ds} = g(x, y)\) for each fixed \(x\) in the domain of interest has a compact attractor with unique invariant measure \(\mu_x\).
Theorem Under assumption that for each fixed $x$, in the relevant domain of interest, the fast dynamics, $\frac{dy}{ds} = g(x, y)$, has a compact attractor with unique invariant measure $\mu_x$, then the slow parts $x_\varepsilon(\cdot)$ of the solutions converge as $\varepsilon \to 0$, uniformly on the time interval $[0, \tau_0]$, to a solution $x_0(\cdot)$ of
Theorem Under assumption that for each fixed \( x \), in the relevant domain of interest, the fast dynamics, \( \frac{dy}{ds} = g(x, y) \), has a compact attractor with unique invariant measure \( \mu_x \), then the slow parts \( x_\varepsilon(\cdot) \) of the solutions converge as \( \varepsilon \to 0 \), uniformly on the time interval \([0, \tau_0]\), to a solution \( x_0(\cdot) \) of

\[
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**Theorem** Under assumption that for each fixed $x$, in the relevant domain of interest, the fast dynamics, $\frac{dy}{ds} = g(x,y)$, has a compact attractor with unique invariant measure $\mu_x$, then the slow parts $x_\varepsilon(\cdot)$ of the solutions converge as $\varepsilon \to 0$, uniformly on the time interval $[0, \tau_0]$, to a solution $x_0(\cdot)$ of

$$\frac{dx}{dt} = \int_{\mathbb{R}^m} f(x, y) \mu_x(dy).$$

Moreover, the fast dynamics $y_\varepsilon(\cdot)$, viewed as a delta Dirac measure, converges to $\mu_{x_0}(\cdot)(dy)$ weakly in the sense of Young measures.
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In many interesting applications one does not have separation of scales in the state variables?

Therefore, one might have to identify slow functionals/observables of the state variables and find an efficient algorithm to compute them.
Consider a system of the form

\[ \frac{dU}{d\tau} = \frac{1}{\epsilon} F(U) + G(U), \]
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**Theorem** [Artstein-Kevrekidids-Slemrod-T.] Let the initial condition \( U^0 \) for the above equation be given. Let \( T > 0 \) be given, and fixed. Denote by \( U_\varepsilon(\tau) \) the solution of the above system, for a given \( \varepsilon \), over the interval \([0, T]\). Then a subsequence \( \varepsilon_k \to 0 \) exists such that \( U_{\varepsilon_k}(\tau) \) converge as \( k \to \infty \), in the sense of Young measures on \([0, T]\), to a Young measure, say \( \mu_0(\tau) \), \( \tau \in [0, T] \). Moreover, for almost every \( \tau \in [0, T] \) the measure \( \mu_0(\tau) \) is an invariant measure of the fast equation \( \frac{dU}{ds} = F(U) \).
The constants of motion of the fast dynamics, \( \frac{dU}{ds} = F(U) \), to the above system are candidates for slow observables. And the idea is to find an equation of motion for the way they are drifted by the slow flow.
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That is, we consider the functionals \( v(U) \), which for every solution \( U(s) \) for \( \frac{dU}{ds} = G(U) \) they satisfy

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\frac{dv(U(s))}{ds} = 0.
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Observe that whenever \( \mu \) is an invariant measure for \( \frac{dU}{ds} = F(U) \), and \( v(U) \) is a constant of motion for it, then \( v(U) \) is constant on the support of \( \mu \). We denote this value by \( \hat{v}(\mu) \).
Theorem Let $U_{\varepsilon_k}(\tau)$ be as in the statement of last Theorem which converge, as $k \to \infty$, in the Young measures sense, to the Young measure $\mu_0(\tau)$, for $\tau \in [0, T]$. Let $v(U)$ be an integral/constant of motion for the fast dynamics. Denote $\hat{v}(\tau) = \hat{v}(\mu_0(\tau))$, namely, the measurement on the limit invariant measures. Then the function $\hat{v}_j(\mu_0(\tau))$ satisfies the differential equation

$$\frac{d\hat{v}}{d\tau}(\tau) = \int_{\mathbb{R}^N} (\nabla v)(U) \cdot G(U) \mu_0(\tau)(dU), \quad \hat{v}(0) = v(U(0)),$$

where $G(U) = G(U_1, \ldots, U_N)$, and the $\nabla$ operator is with respect to the vector $U$. Furthermore, the sequence of measurements $v(U_{\varepsilon_k}(\tau))$ converge weakly to $\hat{v}(\mu_0(\tau))$, as $k \to \infty$. 

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Fast-Slow Multi-scale Systems
Application to discretized Burgers with small diffusion

\[
\frac{dU_k}{d\tau} + \frac{U_k(U_{k+1} - U_{k-1})}{2h\varepsilon} = \frac{U_{k+1} - 2U_k + U_{k-1}}{h^2}.
\]

Denote by \( U \) the vector \((U_1, \cdots, U_N)\); the above system can be rewritten as:

\[
\frac{dU}{d\tau} = \frac{1}{\varepsilon} F(U) + G(U),
\]

where

\[
F(U) = \frac{-1}{2h} \begin{pmatrix}
U_1(U_2 - U_N) \\
U_2(U_3 - U_1) \\
\vdots \\
U_N(U_1 - U_{N-1})
\end{pmatrix},
G(U) = \frac{1}{h^2} \begin{pmatrix}
U_2 - 2U_1 + U_N \\
U_3 - 2U_2 + U_1 \\
\vdots \\
U_1 - 2U_N + U_{N-1}
\end{pmatrix}.
\]
This system was investigated by Goodman and Lax on the whole line. Here we deal with the periodic case and find that the fast dynamics is integrable and has at least \( N/2 \) constants of motion.
This system was investigated by Goodman and Lax on the whole line. Here we deal with the periodic case and find that the fast dynamics is integrable and has at least $N/2$ constants of motion.

In joint work with Artstein, Gear, Kevrekidis, Slemrod and E.S.T. we investigate this computationally and find a great saving by using the Young measure approach.
Figure: Torus for the case $N = 6$ of the fast system. Initial values were $[1 \ 1 \ 1 \ 3 \ 2 \ 1]$. 

Invariant Torus for the fast dynamics
Evidence of fast dynamics

Figure: $U_1(\sigma)$ for the case in the first figure.
Fast evolution of the integrand

Figure: Evolution of $\nabla v_3(U(\sigma)) \cdot G(U(\sigma))$ along the trajectory $U(\sigma)$ of the first figure.
Figure: Averaged $\nabla v_3(U(\sigma)) \cdot G(U(\sigma))$ along the trajectory $U(\sigma)$ of the first figure.
The motion of slow observables

Figure: Behavior of the slow observables $v_2, v_3$ and $v_4$ as they are drifted by the slow diffusion.
Figure: Evolution of Tori of system (22) for initial condition [1 1 1 1.4]