A SHARP BOUND FOR THE GROWTH OF MINIMAL GRAPHS

ALLEN WEITSMAN

Abstract. We consider minimal graphs $u = u(x, y) > 0$ over unbounded domains $D \subset R^2$ bounded by a Jordan arc γ on which $u = 0$. We prove a sort of reverse Phragmén-Lindelöf theorem by showing that if D contains a sector

$$S_\lambda = \{(r, \theta) := \{-\lambda/2 < \theta < \lambda/2\} \quad (\pi < \lambda \leq 2\pi),$$

then the order of growth is at most $r^{\pi/\lambda}$.

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1. Introduction

Let D be an unbounded plane domain. In this paper we consider the boundary value problem for the minimal surface equation

$$\begin{cases} \operatorname{div} \dfrac{\nabla u}{\sqrt{1 + |\nabla u|^2}} = 0 \quad \text{and } u > 0 \quad \text{in } D \\ u = 0 \quad \text{on } \partial D \end{cases} \tag{1.1}$$

We shall study the constraints on growth of nontrivial solutions to (1.1) as determined by the maximum

$$M(r) = \max u(x, y),$$

where the max is taken over the values $r = \sqrt{x^2 + y^2}$ and $(x, y) \in D$.

The methods of this paper extend the results of [6], where the following is proved.

Theorem A. *Let D be a simply connected domain whose boundary is a Jordan arc, and D contains a sector $S_\lambda := \{z : |\arg z| \leq \lambda/2\}$, with $\pi < \lambda \leq 2\pi$. With $M(r)$ defined as above, if u satisfies (1.1) in D, then there exist positive constants K and R such that*

$$M(r) \leq Kr, \quad |z| > R. \tag{1.2}$$

As in Theorem A above, throughout this paper we shall use complex notation for convenience.

The study of upper and lower bounds for the growth of solutions to (1.1) are rather scarce and fragmented. To begin with, the first relevant theorem in this direction was proved by Nitsche [9, p. 256] who observed that if D is contained in a sector of opening strictly less than π, then (1.1) hs no nontrivial solutions.

For domains contained in a half plane, but not contained in any such sector, there are solutions to (1.1) with differing growth rates given in [6].

For angles $\lambda \geq \pi$, in terms of the order ρ of u defined by

$$\rho = \lim_{r\to\infty} \sup \frac{\log M(r)}{\log r},$$

it follows by using the module estimates of Miklyukov [7] (see also chapter 9 in [8]) as in [12] that if D omits a sector of opening $2\pi - \alpha$, $(\pi \leq \alpha \leq 2\pi)$, the omitted set in the case $\alpha = 2\pi$ being a line, then the order ρ of any nontrivial solution to (1.1) is at least π/α, More precisely, the results in [12] are phrased in terms of the asymptotic angle β defined as follows.

Let $\Theta(r)$ be the angular measure of the set $D \cap \{|z| = r\}$, and $\Theta^*(r) = \Theta(r)$ if D does not contain the circle $|z| = r$, and $+\infty$ otherwise. Then

$$\beta = \lim \sup_{r\to\infty} \Theta^*(r).$$

With this definition, the lower bound is given by

Theorem B. *Let D be an unbounded domain whose boundary ∂D is a piecewise differentiable arc, and u satisfies (1.1). If $\beta \geq \pi$, then $\rho \geq \pi/\beta$.*

Regarding upper bounds, it has been conjectured [13] that solutions to (1.1) in general have at most exponential growth, and this is achieved by the horizontal catenoid. In [13] the following is proved.

Theorem C. *If D is an unbounded domain contained in a half plane and bounded by a Jordan arc, then*

$$Cr \leq M(r) \leq e^{Cr} \quad (r > r_0)$$

for some positive constants C and r_0.

The main result of this paper is the following bound for the order ρ of solutions when D contains a large sector.

Theorem 1.1. *Let D be a simply connected domain whose boundary is a Jordan arc, and D contains a sector $S_\lambda := \{z : |\arg z| < \lambda/2\}$, with $\lambda > \pi$. If u satisfies (1.1) in D, then $\rho \leq \pi/\lambda$.*

The examples given in [6] show that the theorem is sharp. Further details regarding those prototypes can be found in [14].

2. PRELIMINARIES

Let u be a solution to the minimal surface equation over a simply connected domain D. We shall make use of the parametrization of the surface given by u in isothermal coordinates using Weierstrass functions $(x(\zeta), y(\zeta), U(\zeta))$ with ζ in the right half plane H, $U(\zeta) = u(x(\zeta), y(\zeta))$. Our notation will then be given by

$$f(\zeta) = x(\zeta) + iy(\zeta) \quad \zeta \in H. \tag{2.1}$$

Then $f(\zeta)$ is univalent and harmonic, and since D is simply connected it can be written in the form

$$f(\zeta) = h(\zeta) + \overline{g(\zeta)} \tag{2.2}$$

where $h(\zeta)$ and $g(\zeta)$ are analytic in H,

$$|h'(\zeta)| > |g'(\zeta)|, \tag{2.3}$$

and

$$U(\zeta) = 2\Re e\, i \int \sqrt{h'(\zeta)g'(\zeta)}\, d\zeta. \tag{2.4}$$

(cf. [3]).

Now, $U(\zeta)$ is harmonic and in (2.4) can be taken as positive in H and vanishing on ∂H. Thus, (cf. [11, p. 151]),

$$U(\zeta) = K\, \Re e\, \zeta, \tag{2.5}$$

where K is a positive constant. This with (2.4) gives

$$g'(\zeta) = -\frac{C}{h'(\zeta)} \tag{2.6}$$

where C is a positive constant.

In order to estimate the function $f(\zeta)$ in (2.2), we shall use the following lemma on quasiconformal mappings from [2] (see [1, Lemma 5.8]).

Lemma A. *Let φ be quasiconformal in the plane such that $\varphi(0) = 0$, $\varphi(1) = 1$, $\varphi(\infty) = \infty$, and the dilatation*

$$\mu(z) = \varphi_{\overline{z}}(z)/\varphi_z(z)$$

satisfies

$$\int_0^{2\pi} |\mu(re^{i\theta})|\, d\theta \to 0 \qquad r \to \infty.$$

Then, in any fixed annulus $A_R = \{R^{-1} \le |z| \le R\}$ $(R > 1)$,

$$\frac{\varphi(tz)}{\varphi(t)} \to z$$

uniformly in $A(R)$ *as* $0 < t \to \infty$. *In particular,*

$$|\varphi(z)| = |z|^{1+o(1))} \quad z \to \infty.$$

At the last stage we shall need a barrier argument based on the following [4, p.827].

Lemma B. Let $u(z)$ be a solution to the minimal surface equation over a domain Ω of the form $S_\lambda \backslash E$ with $\lambda < \pi$ and $u(z) \le ax^m + b$ $(0 < m < 1, a,\ b \ge 0)$ for $z \in \partial S_\lambda$ and $u(z) = 0$ on ∂E. Then $u(z) \le ax^m + b$ in Ω.

Proof. Let $T_1 = S_\lambda \cap \{z : \Re e\, z < 1\}$. Then, there exists [5, p.322] a solution $v_1(z)$ to the minimal surface equation over T_1 with values $ax + b$ on ∂S_λ, and $v_1(z) \to +\infty$ if $\Re e\, z \to 1$ and $|\arg z| \le \lambda' < \lambda$ in T_1. The dilations $v_R(z) = Rv_1(z/R)$ have corresponding properties for $T_R = S_\lambda \cap \{z : \Re e\, z < R\}$. Now $\{v_R\}$ is a decreasing sequence on compact subsets of S_λ, so $V_R \to v$ on S_λ, where v is a solution to the minimal surface equation with planar boundary values in a sector of opening less than π. Thus [9, p.256] $v(z) \equiv ax + b$ on S_λ. If Ω is as in the hypothesis, and $U(z)$ is a solution to the minimal surface equation over Ω with $U(z) \le ax + b$ on ∂S_λ and $U(z) = 0$ on ∂E, then for any $R > 1$, $v_R(z)$ dominates $U(z)$ inside $\Omega \cap \{z : \Re e\, z < R\}$.. Thus letting $R \to \infty$, it follows that $U(z) \le ax + b$ in Ω.

To apply this to $u(z)$ as in the statement of the lemma, take $x_0 > 0$ and note that

$$u(z) \le a\left(x_0^m + mx_0^{m-1}(x - x_0)\right) + b \quad z \in \partial S_\lambda. \tag{2.7}$$

By the above linear case, it follows that the inequality in (2.7) holds in Ω. Applying this in particular a point $z = x_0 + iy$ in Ω, we get $u(x_0 + iy) \le ax_0^m + b$. Since x_0 was arbitrary, the lemma is proved. □

As a final preliminary lemma, we need the following qualitative growth estimate.

Lemma 2.1. *Let* $u(z)$ *be a solution to (1.1) over a domain* D *containing a sector* S_λ *with* $\lambda > \pi$, *and* $f(\zeta)$, $h(\zeta)$, $g(\zeta)$, *and* $U(\zeta)$ *be as in (2.2), (2.4), and (2.6) corresponding to* $u(z)$. *Then, for any proper subsector* $S_{\lambda'}$ *with* $\pi < \lambda' < \lambda$ *and* $D_{\lambda'} = f^{-1}(S_{\lambda'})$,

$$h'(\zeta) \to \infty \text{ as } \zeta \to \infty$$

uniformly for $\zeta \in D_{\lambda'}$.

Proof. Let $f(\zeta)$, $U(\zeta)$ be as above. So, $u(f(\zeta)) = U(\zeta) = K\Re e\,\zeta$.

Let $P_\alpha := \{\zeta : \Re e\; e^{i\alpha} f(\zeta) > 0\}$ $\;(|\alpha| < \lambda/2 - \pi/2)$ and introduce a new variable $\tilde\zeta$ and let $\psi(\tilde\zeta)$ be a conformal map from the right half plane $H := \{\tilde\zeta : \Re e\,\tilde\zeta > 0\}$ onto P_0.

Define

$$\begin{cases} \tilde f(\tilde\zeta) := f(\psi(\tilde\zeta)) \\ \tilde g(\tilde\zeta) := g(\psi(\tilde\zeta)) \\ \tilde h(\tilde\zeta) := h(\psi(\tilde\zeta)) \end{cases}$$

Then $\tilde f$ is a harmonic map, and

$$\tilde f(\tilde\zeta) = \tilde h(\tilde\zeta) + \overline{\tilde g(\tilde\zeta)}.$$

Let $\tilde F(\tilde\zeta) = \tilde h(\tilde\zeta) + \tilde g(\tilde\zeta)$ be the analytic function with the same real part as $\tilde f$. Then $\Re e\tilde F$ is positive in H and vanishes on ∂H, and therefore, after renormalizing we may write (see [11, p. 151])

$$\tilde F(\tilde\zeta) = \tilde\zeta \implies \tilde F'(\tilde\zeta) = 1. \tag{2.8}$$

In particular we have

$$\Re e\{\tilde h(\tilde\zeta) + \overline{\tilde g(\tilde\zeta)}\} = \Re e\{\tilde h(\tilde\zeta) + \tilde g(\tilde\zeta)\} = \Re e\tilde\zeta.$$

Now,

$$\tilde h'(\tilde\zeta) = h'(\psi(\tilde\zeta)) \cdot \psi'(\tilde\zeta), \tag{2.9}$$

and

$$\tilde g'(\tilde\zeta) = -\frac{\psi'(\tilde\zeta)}{h'(\psi(\tilde\zeta))} = -\frac{\psi'(\tilde\zeta)^2}{\tilde h'(\tilde\zeta)}. \tag{2.10}$$

Combining this with (2.8)) we have

$$1 = \tilde F'(\tilde\zeta) = \tilde h'(\tilde\zeta) - \frac{\psi'(\tilde\zeta)^2}{\tilde h'(\tilde\zeta)}$$

which implies

$$\tilde h'(\tilde\zeta)^2 - \tilde h'(\tilde\zeta) - \psi'(\tilde\zeta)^2 = 0.$$

Thus,

$$\tilde h'(\tilde\zeta) = \frac{1 + \sqrt{1 + 4\psi'(\tilde\zeta)^2}}{2}. \tag{2.11}$$

Since $\psi(\tilde\zeta)$ is a conformal map with $\Re e\,\psi(\tilde\zeta) > 0$ in H, there exists a real constant $0 \le c < \infty$ such that in any sector $S_\beta := \{\tilde\zeta : |\arg\tilde\zeta| \le \beta < \pi/2\}$ the limit $\psi'(\tilde\zeta) \to c$ exists as $\tilde\zeta \to \infty$ in S_β (see [11, p. 152]).

Suppose that $c > 0$ so that $\zeta = \psi(\tilde{\zeta}) = c\,\tilde{\zeta}(1 + o(1))$ as $\tilde{\zeta} \to \infty$ in S_β. This means that P_0 is asymptotically the half plane H, that is, for any sector $S_{\beta'}$ $(0 < \beta' < \pi/2)$, $S_{\beta'} \cap \{|\zeta| > R\} \subseteq P_0$ for some $R = R(\beta')$.

Furthermore, by (2.10) and (2.11),

$$\tilde{h}'(\tilde{\zeta}) \to \frac{1 + \sqrt{1+4c^2}}{2}, \quad \tilde{g}'(\tilde{\zeta}) \to \frac{-2c^2}{1 + \sqrt{1+4c^2}}, \qquad \tilde{\zeta}/\zeta \to c. \tag{2.12}$$

which implies that

$$\tilde{h}(\tilde{\zeta}) + \overline{\tilde{g}(\tilde{\zeta})} = \left[\Re e\,\tilde{\zeta} + i\left(1 + \frac{4c^2}{1+\sqrt{1+4c^2}}\right)\Im m\,\tilde{\zeta}\right](1 + o(1))$$

as $\tilde{\zeta} \to \infty$ uniformly in S_β. From this it follows that

$$h(\zeta) + \overline{g(\zeta)} = \left[\Re e\,\zeta/c + i\left(1 + \frac{4c^2}{1+\sqrt{1+4c^2}}\right)\Im m\,\zeta/c\right](1 + o(1))) \tag{2.13}$$

uniformly as $\zeta \to \infty$ in proper subsectors of H.

By (2.1) and (2.5), the graph of the minimal surface is given parametrically by $(\Re e\,f(\zeta), \Im m\,f(\zeta), K\Re e\,\zeta)$. Using (2.13) we then have that the suface is asymptotic to a plane, that is, its parametrization has the form

$$\left((1 + o(1))\Re e\,\zeta,\ (1 + o(1))(1 + \frac{4c^2}{1+\sqrt{1+4c^2}})\Im m\,\zeta,\ K\Re e\,\zeta\right)$$

as $\zeta \to \infty$ in proper subsectors of H.

Now, if we consider P_α and $P_{-\alpha}$ for $\alpha = (\lambda - \pi)/2$ along with P_0, then $f(P_\alpha)$, $f(P_{-\alpha})$, and $f(P_0)$ are an overlapping cover of S_λ. The transformation $e^{i\alpha}f(\zeta) = e^{i\alpha}h(\zeta) + \overline{e^{-i\alpha}g(\zeta)}$ rotates P_α onto the right half plane, and an analysis corresponding to this mapping gives a conformal mapping which we again denote by $\psi(\tilde{\zeta})$, and as before $\psi'(\tilde{\zeta})$ tends to a limit in proper subsectors. We claim that if this limit for P_0 was not 0, then the same must be true for P_α. In fact, in P_0, it follows from (2.9) and (2.12) that $h'(\zeta)$ remains bounded as $\zeta \to \infty$ in proper subsectors of P_0. However, for P_α if the corresponding $\psi'(\tilde{\zeta}) \to 0$, then by (2.9) and (2.11), $h'(\zeta)$ becomes unbounded. This creates a contradiction in the overlapping regions. The same analysis applies to $P_{-\lambda}$.

Thus, the graphs of the minmal surface over proper subsectors of $P_{-\lambda}$, P_0, and P_λ are all asymptotic to planes. However, since these subsectors overlap, they must be asymptotic to the same plane over compact subsets of S_λ, But since $\lambda > \pi$ this contradicts the fact that $u(z) > 0$ over D.

Since this shows that in $D_{\lambda'}$ the above conformal maps $\psi(\tilde{\zeta})$ all satisfy $\psi'(\tilde{\zeta}) \to 0$, this again by (2.9), implies that $h'(\zeta) \to \infty$. $\square$

3. Proof of Theorem 1.1

Proof. For fixed λ, let $f_1(\zeta)$ denote the function in (2.1) corresponding to a solution to (1.1) over a domain D containing S_λ. Then for λ' such that $\pi < \lambda' < \lambda$ we define $f_2(\zeta) = \zeta^{\lambda'/\pi}+1$. Let $\tilde{S}_{\lambda'} = f_2(H)$ and $\tilde{H} = f_1^{-1}(\tilde{S}_{\lambda'})$. Then if $\psi(\zeta)$ is a $1-1$ conformal mapping of H onto $\tilde{H}$ with $\psi(\infty) = \infty$, it follows that $f_1(\psi(H)) = f_2(H)$ and there exists an orientation preserving homeomorphism $\varphi : H \to H$ with $\varphi(\infty) = \infty$ such that

$$f_1(\psi(\zeta)) = f_2(\varphi(\zeta)) \quad \zeta \in H. \tag{3.1}$$

Differentiating (3.1) with respect to ζ and $\overline{\zeta}$, and using (2.6) we obtain

$$\psi'(\zeta)h_1'(\psi(\zeta)) = \varphi_\zeta(\zeta)f_2'(\varphi(\zeta)) \tag{3.2}$$

and

$$-C\overline{\frac{\psi'(\zeta)}{h_1'(\psi(\zeta))}} = \varphi_{\overline{\zeta}}(\zeta)f_2'(\varphi(\zeta)) \tag{3.3}$$

Dividing (3.3) by (3.2) we have

$$\frac{C}{|h_1'(\psi(\zeta))|^2} = \left|\frac{\varphi_{\overline{\zeta}}(\zeta)}{\varphi_\zeta(\zeta)}\right|. \tag{3.4}$$

Now, $\psi(\zeta) \to \infty$ as $\zeta \to \infty$ in H, so by Lemma 2.1 it follows that the left side of (3.4) tends to 0.

It therefore follows from (3.4) and the fact that φ is a sense preserving differentiable homeomorphism, that φ is quasiconformal in H and that the dilatation of φ satisfies

$$\left|\frac{\varphi_{\overline{\zeta}}(\zeta)}{\varphi_\zeta(\zeta)}\right| \to 0 \qquad \zeta \to \infty \ \ \zeta \in H. \tag{3.5}$$

The mapping φ can then be extended by reflection to a quasiconformal mapping of the complex plane onto the complex plane with (3.5) still in force. Thus, Lemma A can be applied to φ. In fact, since φ maps the vertical axis to itself, the conclusion in Lemma A can be improved to

$$\varphi(re^{i\theta}) = r^{(1+o(1))}e^{i(\theta+o(1))}$$

so that

$$f_1(\psi(re^{i\theta})) = f_2(\varphi(re^{i\theta})) = r^{(\lambda'/\pi+o(1))}e^{i(\lambda'\theta/\pi+o(1))} \quad \zeta = re^{i\theta} \to \infty \quad \zeta \in H.$$

From this we see that, given any λ'' such that $\pi < \lambda'' < \lambda'$, there is a proper sector $\Sigma_{\lambda''}$ in H such that $f(\psi(\Sigma_{\lambda'''}))$ covers $S_{\lambda''}$. But $\psi(\zeta)$ is a conformal mapping of H into H, so $\psi'(\zeta) \to C$ as $\zeta \to \infty$ in $\Sigma_{\lambda''}$ for some $C \geq 0$ (cf. [11, p. 151]). Combining this with (2.5) we conclude that for some $k > 0$ and sufficiently large z,

$$u(z) < |z|^{(\pi/\lambda'+o(1))} \quad z \in S_{\lambda''}. \tag{3.6}$$

The boundary of the sector $\Sigma_{\lambda''}$ on which (3.6) holds forms an angle in the left half plane of opening less than π. On the remaining portion of the boundary of D in the left half plane $u(z) = 0$. Therefore Lemma B implies that (3.6) holds in the portion in the left half plane as well. Thus (3.6) holds throughout D. Since λ' can be taken arbitrarily close to λ in (3.6), the proof is complete. □

References

1. D .Drasin, *On Nevanlinna's inverse problem*, Complex Variables **37** (1999), 123-143
2. D. Drasin, A. Weitsman, *Meromorphic functions with large sums of deficiencies*, Advances in Math. **15** (1974), 93-126.
3. P. Duren, *Harmonic mappings in the plane*, Cambridge Tracts in Mathematics, 2004.
4. J-F Hwang, *Phragmén Lindelöf theorem for the minimal surface equation*, Proc. Amer. Math. Soc. **104** (1988), 825-828.
5. H. Jenkins, j. Serrin, *Variational problems of minimal surface type II. Boundary value problems for the minimal surface equation*, Arch. Rat. Mech. Anal., **21** (1965/66), 321-342.
6. E. Lundberg, A. Weitsman, *On the growth of solutions to the minimal surface equation over domains containing a half plane*, Calc. Var. Partial Differential Equtions **54** (2015), 3385-3395.
7. V. Miklyukov, *Some singularities in the behavior of solutions of equations of minimal surface type in unbounded domains*, Math. USSR Sbornik **44** (1983), 61-73.
8. V. Miklyukov, *Conformal maps of nonsmooth surfaces and their applications*, Exlibris Corp. (2008).
9. J.C.C. Nitsche,*On new results in the theory of minimal surfaces*, Bull. Amer. Mat. Soc. **71** (1965), 195-270.
10. R. Osserman, *A survey of minimal surfaces.* Dover Publications Inc. (1986).
11. M. Tsuji, *Potential Theory in Modern Function Theory*, Maruzen Co., Ltd., Tokyo (1959).
12. A. Weitsman, *On the growth of minimal graphs*, Indiana Univ. Math. J. **54** (2005), 617-625.
13. A. Weitsman, *Growth of solutions to the minimal surface equation over domains in a half plane*, Communications in Analysis and Geometry **13** (2005), 1077-1087.
14. A. Weitsman, *Level curves of minimal graphs.* Arxiv:2008.10197, To appear in Communications in Analysis and Geometry

Email: weitsman@purdue.edu