ON THE GAUSS MAP OF MINIMAL GRAPHS

DAOUD BSHOUTY, DEPT. OF MATHEMATICS, TECHNION INST. OF TECHNOLOGY, 3200 HAIFA, ISRAEL, AND ALLEN WEITSMAN, DEPT. OF MATHEMATICS, PURDUE UNIVERSITY, W. LAFAYETTE, IN 47907

Abstract. We consider graphs of solutions to the minimal surface equation which are unbounded over subarcs of the domain boundary. An extensive study of such surfaces was made by Jenkins and Serrin. In this note, properties of the Gauss map are studied.

Subject Classification: 30C62, 31A05, 31A20, 49Q05

Keywords: harmonic mappings, dilatation, minimal surfaces

I. Introduction. Let S be the graph of a minimal surface over a Jordan domain D, and given by $\varphi = \varphi(x, y)$, $(x, y) \in D$. Thus φ satisfies the minimal surface equation

$$\operatorname{div}\left(\frac{\nabla\varphi}{\sqrt{1+|\nabla\varphi|^2}}\right) = 0$$

in D. We are interested in studying those surfaces S which are $\pm\infty$ over all but a finite number of points on ∂D. More precisely, we consider ∂D as the union of finitely many arcs $\{A_k\}$ and $\{B_k\}$ having only endpoints in common such that for each j

$$\text{(1.1)} \qquad \begin{cases} \varphi(x, y) \to +\infty \text{ as } (x, y) \to \text{Int } A_k,\ (x, y) \in D \\ \varphi(x, y) \to -\infty \text{ as } (x, y) \to \text{ Int } B_k,\ (x, y) \in D. \end{cases}$$

Since if $\varphi \to \infty$ or $\varphi \to -\infty$ over a boundary arc, then the arc must be a line segment [7; p. 102], the domains we consider are polygons having $\{A_k\}$, $\{B_k\}$ on their sides. Also, no two segments of $\{A_j\}$ nor two segments of $\{B_j\}$ meet at a convex corner [5; pp.329-330]. The prototype is Scherk's surface which plays a central role for extremal problems, and is given by

$$\varphi(x, y) = \log(\cos x / \cos y)$$

with D the square $\{(x, y) : -\pi/2 < x < \pi/2, -\pi/2 < y < \pi/2\}$.

In [5], Jenkins and Serrin made a detailed study of minimal surfaces satisfying (1.1). We shall henceforth refer to these as JS surfaces. They characterized those polygons for which there is a minimal surface satisfying (1.1). To describe this, let P be a connected polygonal domain in D whose boundary is the union of some segments A_i and B_j, along with some cross cuts in D connecting endpoints of some A_i and B_j. Let α, β denote, respectively, the total length of segments A_i and the total length of segments B_j on the boundary of P, and let γ be the perimeter of P. Then [5; pp.338-339] we have

Theorem A. *There exists a minimal surface over D given by $\varphi = \varphi(x, y)$ satisfying (1.1) if and only if*

$$2\alpha < \gamma \qquad\qquad 2\beta < \gamma$$

for each such P as above, properly contained in D, and

$$\alpha = \beta$$

when $P = D$.

The purpose of this note is to supplement the work of Jenkins and Serrin with some recent advances in the study of planar harmonic mappings. Just as the JS surfaces arise naturally from extremal problems in nonparametric theory, the corresponding harmonic mappings arise in the parametric theory from Poisson integrals of step functions.

We shall refer to a univalent harmonic mapping f in the unit disk $U = \{\zeta : |\zeta| < 1\}$ in the complex plane as

$$f(\zeta) = u(\zeta) + iv(\zeta) \qquad\qquad \zeta \in U$$

where u and v are real harmonic, and f is orientation preserving.

Then f can also be written $f = h + \overline{g}$ with h, g analytic in U, and

$$g'(\zeta) = \overline{f_{\overline{\zeta}}}(\zeta) = a(\zeta) f_\zeta(\zeta) = a(\zeta)h'(\zeta). \tag{1.2}$$

Here $a(\zeta)$ is analytic in U with $|a(\zeta)| < 1$ for $\zeta \in U$. We shall refer to $a(\zeta)$ as the analytic dilatation. Many of the function theoretic properties of univalent harmonic mappings are given in [2].

The connection between a minimal surface S and a univalent harmonic mapping comes from the conformal representation of S via the Weierstrass formulas

$$\begin{cases} x = \frac{1}{2}\ \Re e \int p(\zeta)(1 - G(\zeta)^2)d\zeta \\ y = \frac{1}{2}\ \Re e\, i \int p(\zeta)(1 + G(\zeta)^2)d\zeta \\ \varphi = \Re e \int p(\zeta)G(\zeta)\, d\zeta \end{cases} \tag{1.3}$$

where p is analytic, G is meromorphic, and p has zero at the poles of G of twice the order. Now, the first two coordinate functions $x(\zeta)$, $y(\zeta)$ are harmonic, and since S is given as a graph, then

$$f(\zeta) = x(\zeta) + i\, y(\zeta) \tag{1.4}$$

is univalent. The meromorphic function $G(\zeta)$ is the (stereographic projection of the) Gauss map of the surface, where we take the surface normal to be upward so that $|G(\zeta)| > 1$. A useful relation then results from the fact that the Gauss map $G(\zeta)$ and analytic dilatation $a(\zeta)$ are related by

$$a(\zeta) = \frac{-1}{G(\zeta)^2}. \tag{1.5}$$

The formulas (1.3) can be written

$$\begin{cases} x = \Re e \int h'(\zeta)(a(\zeta)+1))d\zeta \\ y = \Re e\, i \int h'(\zeta)(a(\zeta)-1)d\zeta \\ \varphi = \Re e\, 2i \int h'(\zeta)\sqrt{a(\zeta)}\, d\zeta \end{cases} \tag{1.6}$$

See [4] for a discussion. Now, (1.5) implies that $|a(\zeta)| \to 1$ at points when the surface normal is tending to the horizontal direction. This observation can be used to prove [9; p.449], [1; p.145]

Theorem B. *Let S be a minimal surface given by $\varphi(x, y)$ for (x, y) in a domain Ω, with $\gamma \subset \partial\Omega$ a C^1 curve, convex with respect to the interior of Ω. If the Gauss map G of S satisfies $|G(x, y)| \to 1$ as $(x, y) \to \gamma$, then γ is a line segment.*

We shall pursue the connection (1.5) between the Gauss map and the parametric representation (1.3), (1.4).

In what follows, we shall refer to the endpoints of the segments $\{A_k\}$ and $\{B_k\}$ as vertices, even though it is possible for the angle at such a point to be π.

Throughout we shall use ζ to denote points in the parameter disk as in (1.3), and $z = x + iy$ to denote points in the domain D of φ. For convenience, we shall use $G(z)$ as $G(f^{-1}(z))$ with f as in (1.4) when working with the nonparametric representation.

II. The Gauss map for JS surfaces.

Theorem 1. *Let S be a JS surface over a polygonal domain D having vertices z_j, $j = 1, \dots, n$, in positive cyclic order for the endpoints of $\{A_k\}$ and $\{B_k\}$. Then, in the conformal representation (1.3), the Gauss map $G(\zeta)$ is of the form $c/B(\zeta)$ where c is a constant of modulus 1 and $B(\zeta)$ is a Blaschke product of degree at most $(n-2)/2$ and precisely $(n-2)/2$ if D is convex.*

In terms of the univalent harmonic function $f(\zeta)$ given by (1.4), this is the Poisson integral of a step function $f^*(e^{i\theta})$ whose values are contained in the set $\{z_k\}$; the segments $\{A_k\}$ and $\{B_k\}$ arise as the cluster sets of f at the endpoints of the intervals of constancy of f^*.

Proof. Let p' be an interior point of a side of D, $p \in D$, and $p^* \in S$ be the point over p. By [5; p.325, Lemma 1] if r is the distance on S from p^* to ∂S, d the length of the vector $\vec{\nu}$ from p to p', and $d \le r/8$, then

$$\frac{|\nu \cdot \nabla\varphi|}{\sqrt{1 + |\nabla\varphi|^2}} \ge 1 - 4d^2/r^2. \tag{2.1}$$

If we represent S parametrically by (1.3), then the Gauss map G in terms of $z = x + iy$ can be taken to satisfy [8; p.105]

$$|G(z)|^2 = \frac{W+1}{W-1} \qquad W = \sqrt{1 + |\nabla\varphi|^2}. \tag{2.2}$$

Then (2.1) implies that $|\nabla\varphi| \to \infty$ as $p \to p'$ and thus (2.2) implies that $|G(z)| \to 1$ at interior points of the sides of D.

If $f = x(\zeta) + iy(\zeta)$ is as in (1.4), then f is the Poisson integral of its radial limit function f^* whose values are a.e. $[0, 2\pi)$ contained in ∂D.

Let L be one of the line segments A_j so that $\varphi \to \infty$ over the interior of L. By a translation and rotation we may assume that L is an interval $\{0\} \times [y_1, y_2]$ in the y axis, and D is to the left. Now let $\epsilon > 0$ and $\delta > 0$ be small enough so that the segment $L_{\epsilon\delta} = \{-\delta\} \times [y_1 + \epsilon, y_2 - \epsilon]$ is contained in D. Then, as in [5; p.328], revisiting (2.1) we may write for small $\delta > 0$,

$$\frac{\partial\varphi/\partial x}{\sqrt{1 + |\nabla\varphi|^2}} \geq 1 - 4d^2/r^2$$

over $L_{\epsilon\delta}$. It follows that the unit upward normals on S tend to $\vec{i}$ over L.

We now have that the Gauss map $G(\zeta)$ defined on U as in (1.3) has constant value for those points on ∂U corresponding to the interiors of A_j and B_j. Thus, these lines corresponds to the cluster sets of points of ∂U by f. This implies that f^* is simply the step function with values corresponding to endpoints of $\{A_l\}$ and $\{B_m\}$. Thus, f being the Poisson integral of a step function with n values, its analytic dilatation is a multiple of a Blaschke product of degree at most $n - 2$, and equal $n - 2$ if D is convex [7; p.469], [3; p.203], [6; p.995]. The result follows from (1.5). $\square$

III. Convexity and resting points. Let S be a JS surface given by $\varphi(x, y)$ over D, and let $\{A_k\}$, $\{B_k\}$ be as in (1.1). Let $\{z_j\}_{j=1}^{n}$ be the end points of the $\{A_k\}$ and $\{B_k\}$ arranged in positive cyclic order.

Following [1] for the current setting, we call z_j a *point of convexity* if it is a vertex of D having an interior angle less than π. From the straight line lemma [5; p.329] we have that z_j is a common endpoint of sides A_i and B_j, that is, $\varphi(x, y)$ alternates signs on each side of z_j. Although the straight line lemma and maximum principle are useful in describing the boundary behavior, a more detailed understanding can be obtained by following the Gauss map by the methods of [1].

Let f be a univalent harmonic mapping as in (1.3) and (1.4) corresponding to S, and having analytic dilatation $a(z)$. Now, f is the Poisson integral of a step function having values $z_1, z_2, \ldots, z_n$. The sides $\{A_k\}$, $\{B_k\}$ occur as the cluster sets as ζ approaches the endpoints $\{\zeta_k\}$ of the intervals where f has the constant values $\{z_k\}$.

Let ζ_j, ζ_{j+1} be the (positively oriented) endpoints of the arc C_j on ∂U corresponding to z_j. By [1; p.150] if $0 < \alpha_j < 2\pi$ is the interior angle at z_j and we take a continuous branch of $\arg a(\zeta)$ on $\overline{C}_j$, then

$$\frac{1}{2}\left(\arg a(\zeta_{j+1}) - \arg a(\zeta_j)\right) = \alpha_j, \tag{3.1}$$

or

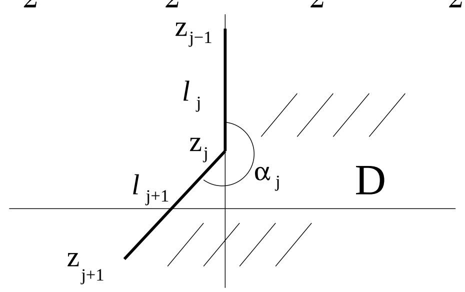
$$\frac{1}{2}\left(\arg a(\zeta_{j+1}) - \arg a(\zeta_j)\right) = \alpha_j - \pi. \tag{3.2}$$

In case $0 < \alpha_j < \pi$, then z_j is a point of convexity and (3.1) must hold. However, if $\alpha_j \geq \pi$ and (3.1) still holds, we call z_j a *full resting point*. Over a full resting point, the Gauss map has a behavior similar to that of a point of convexity and we have

Theorem 2. *If z_j is a full resting point for the JS surface S given by $\varphi(x,y)$, then φ changes signs on the sides adjacent to z_j. If z_j is neither a point of convexity nor full resting point then φ does not change sign at z_j.*

Proof. Let ℓ_j, ℓ_{j+1} be adjacent sides to z_j on ∂D, ordered positively. We may represent ℓ_j and ℓ_{j+1} with the notation $\ell_j = [z_{j-1}, z_j]$, $\ell_{j+1} = [z_j, z_{j+1}]$. We take the Gauss map G as upward pointing, and assume that $\varphi_j \to +\infty$ over ℓ_j. We may further take ℓ_j to be in the y axis with D to the right, so that the normals are tending to the vector $\vec{i}$, and $G = 1$ over ℓ_j. If ζ_j and ζ_{j+1} are as above, then ℓ_j is the cluster set of f at ζ_j and ℓ_{j+1} is the cluster set at ζ_{j+1}. By (1.5) we then have $a(\zeta_j) = -1$, and we are taking the branch of G such that arg $G(\zeta_j) = 0$.

Assume first that z_j is a full resting point so that (3.1) holds. Again, using (1.5) we find that over ℓ_{j+1} $\arg(G) = \dfrac{\pi}{2} - \dfrac{\arg(a(\zeta_{j+1}))}{2} = \dfrac{\pi}{2} - \alpha_j - \dfrac{\pi}{2} = -\alpha_j$



Now, the angle from z_j to z_{j+1} is $\dfrac{\pi}{2} - \alpha_j$. Thus, G over ℓ_{j+1} corresponds to the direction parallel to the outward normal to D. Since G is the stereographic projection of the upward normal, it follows that $\varphi \to -\infty$ on ℓ_{j+1}.

If z_i were not a point of convexity nor a full resting point then we could use a similar argument with (3.2). □

The introduction of the notion of resting points enables us to more closely link the Gauss map with a JS surface as our next result illustrates. We shall refer to the number of sign changes of S as the number of endpoints z_j of the $\{A_k\}$ and $\{B_k\}$ which are simultaneously an endpoint of an A_i and a B_j.

Theorem 3. *If a JS surface S has n sign changes, then its Gauss map G as in (1.3) is of the form $c/B(z)$ where $B(z)$ is a Blaschke product of order $(n-2)/2$.*

Proof. Let f be a univalent harmonic mapping corresponding to S by (1.4), Then its analytic dilatation $a(z)$ is a multiple of a Blaschke product by (1.5) and Theorem 1.

Suppose $a(z)$ has order d. According to [1, p.156], the total number of points of convexity and full resting points is precisely $d+2$. The result then follows from Theorem 2 and (1.5). □

It is interesting to note that φ generally has a similar behavior at a vertex whether or not there is a sign change.

Theorem 4. *Let z_j be a vertex for a JS surface over D given by φ and suppose the boundary angle at a vertex z_j is not π. Then there are paths γ leading to z_j in D such that $\varphi(x,y)$ remains bounded as $z \to z_j$ along γ.*

Proof. With the parametric representation (1.3) and (1.4), the function f is a Poisson integral of a step function with z_j as a value on some interval I of ∂U. At interior points of I, then f extends to a $\mathcal{C}^\infty$ function across the interval, and in particular the partials of f are bounded. Also, $a(z)$ being a Blaschke product also extends across I and has continuous partials.

Using (1.3), (1.4), and (1.5) we find that $\varphi(\zeta)$ is a branch

$$\varphi(\zeta) = \int f_\zeta(\zeta)\sqrt{a(\zeta)}\,d\zeta$$

and thus will remain bounded on curves Γ leading to any interior point of I. Its image $f(\Gamma)$ is then a curve γ in D which leads to z_j, and over which φ remains bounded. □

The previous result describes the boundary behavior for endpoints z_j of $\{A_k\}$ and $\{B_k\}$ which do not have interior angle π. However, in any case we have

Theorem 5. *Let z_j be a vertex for a JS surface S whose boundary angle is π. If φ remains bounded over some curve γ in D leading to z_j, then φ must change sign at z_j.*

Proof. Let f be the function corresponding to S as in (1.4). Then f is a Poisson integral of a step function. Now, z_j is a point on the line segment joining adjacent endpoints z_{j-1} and z_{j+1}. If z_j does not correspond to an interval on ∂U for f, then $\varphi = +\infty$ or $\varphi = -\infty$ over the entire segment joining z_{j-1} to z_j, and in particular also at z_j. (See Theorem C below.)

Otherwise there is an interval corresponding to z_j on ∂U. By [1; Theorem 2.13(ii)] z_j is then a full resting point, and by Theorem 2, the function φ therefore changes sign at z_j. □

In [10] a relation is given for the height function corresponding to a univalent harmonic mapping coming from the Poisson integral f of a step function. In the current setting, f corresponds to a minimal surface by (1.4). This gives the following

Theorem C. *Let P be a polygon having vertices $c_1, \dots, c_n$ given cyclically and ordered by a positive orientation on ∂P. Let f be a univalent harmonic mapping of U such that $f(\zeta)$ is the Poisson integral of a step function having the ordered sequence $c_1, \dots, c_n$ as its values. Then the analytic dilatation $a(\zeta)$ of $f(\zeta)$ is of the form $cB(\zeta)$ where $B(\zeta)$ is a finite Blaschke product of order at most $n-2$ and $f(U) = P$. Suppose that f corresponds to a minimal graph S with height function φ given as in (1.6). Then φ tends to $+\infty$ or $-\infty$ at points over the open segments between the vertices; if P is convex then $+\infty$ and $-\infty$ alternate on adjacent sides.*

Remark. In the present paper we have made no assumptions that the "vertices" must be at true corners (of angles $\neq \pi$) whereas in Theorem C the $c_1, \dots, c_n$ are true corners of P. However, the proof of the result in [10] goes through without change except for the last statement concerning the convex case which is done with a barrier argument assuming the angles at the c_j are strictly less than π. However, if the angle were π, this would be a full resting point as mentioned in §3, and by Theorem 2 the signs of φ would again change. Thus, Theorem C is valid even in the present case where the "vertices" may occur even with interior angle π.

IV. A converse to Theorem 1. We know that if S is a JS surface over a polygonal domain D, then its analytic dilatation is of the form $c/B(\zeta)$ where $B(\zeta)$ is a finite Blaschke product. The converse of Theorem 1 is then

Theorem 6. *Let S be a minimal graph over a polygonal domain D having k sides. If the Gauss map for S in the parametrization (1.3) has the form $c/B(\zeta)$ where c is a constant of modulus 1 and $B(\zeta)$ is a Blaschke product of order n, then S is a JS surface, and $k \geq 2n+2$.*

Proof. Let f be a univalent harmonic mapping of U onto D as in (1.4) and f^* its radial limit function. Collecting facts from [1; p.145], we have that at each point $e^{it} \in \partial U$ either f^* is continuous or has a jump discontinuity, and each situation can be analyzed.

Suppose first that f^* is continuous on an interval containing a point e^{it}. Then, f^* must be constant on the interval. To see this, suppose it were not. Then by [1; (2.13)],

$$\lim_{h\to 0^+} \arg[f^*(e^{i(t+h)}) - f^*(e^{i(t)})] = -(1/2)\arg a(e^{it}) \bmod \pi. \tag{4.1}$$

Since arg $a(^{i\theta})$ is increasing, if we apply (4.1) to each point of the interval we get a contradiction. Thus, f^* is constant on intervals where it is continuous.

Now let e^{it} be a point at which f^* jumps. Then, by [1; (2.11)]

$$\arg[f^*(e^{i(t+0)}) - f^*(e^{i(t-0)})] = -(1/2)\arg a(e^{it}) \bmod \pi. \tag{4.2}$$

In (4.2), we again note that arg $a(e^{i\theta)})$ is increasing. Also, the total variation of the argument is $4n\pi$. For jump points e^{it} whose right and left limits of f^* lie on an open segment L of ∂D, the left hand side of (4.2) remains constant, and thus, arg a must undergo an increment of at least 2π. Thus, there are fewer than $2n$ such points corresponding to L. Since ∂D has only a finite number of sides and a finite number of corners, it follows that f is the Poisson integral of a step function f^* having a finite number of steps. By Theorem C and the Remark which follows it, we have that S is a JS surface and $k \geq 2n+2$. □

References

[1] D. Bshouty and W. Hengartner, *Boundary values versus dilatations of harmonic mappings*, J. Analyse Math. **72** (1997), 141–164.

[2] J. Clunie and T. Sheil-Small, *Harmonic univalent functions*, Ann. Acad. Sci. Fenn. Ser. AI Math **9** (1984), 3–25.

[3] W. Hengartner and G. Schober, *On the boundary behavior of orientation-preserving harmonic mappings*, Complex Variables Theory Appl. **5** (1986), 197–208.

DAOUD BSHOUTY, DEPT. OF MATHEMATICS, TECHNION INST. OF TECHNOLOGY, 3200 HAIFA, ISRAEL,

[4] ———, *Curvature estimates for some minimal surfaces*, Complex Analysis: Articles dedicated to Albert Pfluger on the occasion of his 80th birthday (J. Hersch and A. Huber, editors; Birkhäuser Verlag, Basel, 1988), 87-100.

[5] H. Jenkins and J. Serrin, *Variational problems of minimal surface type II. Boundary value problems for the minimal surface equation*, Arch. Rational Mech. Anal. **21** (1965/66), 321–342.

[6] R. Langevin, G. Levitt, and H. Rosenberg, *Complete minimal surfaces with long line boundaries*, Duke Math. J. **55** (1987), 985-995.

[7] R. Osserman, *A Survey of Minimal Surfaces*, Dover, 1986.

[8] T. Sheil-Small, *On the Fourier series of a step function*, Michigan Math. J., **36** (1989), 459–464.

[9] A. Weitsman, *On the dilatation of univalent planar harmonic mappings*, Proc. Amer. Math. Soc. **126** (1998), 447–452.

[10] ———, *On univalent harmonic harmonic mappings and minimal surfaces*, Pacific J. Math. **192** (2000), 191-200.