Valency of Harmonic Mappings onto Bounded Convex Domains

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Abstract

We give two harmonic mappings of the open unit disc onto bounded convex domains that extend continuously to the unit circle as 2-valent local homeomorphisms onto the boundary of the image domains such that the valency of one mapping is 6 and of the other is 8. The paper concludes with the following conjecture: For every positive integer N there exists a harmonic mapping that satisfies the above boundary properties and whose valency is at least N.

1. Introduction A *harmonic* map in a region $\mathcal{D}$ is a function of the form

$$f = \overline{g} + h \tag{1}$$

where g and h are analytic functions in $\mathcal{D}$ that are single-valued if $\mathcal{D}$ is simply-connected and multiple-valued which can be determined up to additive constants otherwise. The *Jacobian* of f is

$$J_f = |h'|^2 - |g'|^2 = |h'|^2(1 - |a|^2)$$

where $a = g'/h'$ is the *second dilatation* function of f. It is known that f is locally one-to-one in $\mathcal{D}$ if and only if J_f is nonzero there.

Let $\mathbf{D}$ be the unit disc $\{z : |z| < 1\}$ and let $\mathbf{K}$ be a bounded convex domain. In 1926, T. Radó [10] asked whether the harmonic extension to $\mathbf{D}$ of a sense-preserving homeomorphism of $\partial\mathbf{D}$ to $\partial\mathbf{K}$ is itself a homeomorphism. Shortly afterwards, H. Kneser [4] gave a geometric proof for a positive answer and observed that the proof also holds if the convexity of $\mathbf{K}$ is exchanged for the set-inclusion $f(\mathbf{D}) \subset \mathbf{K}$. G. Choquet [2], apparently unaware of Kneser's work, also gave yet another but analytic proof. Recently, the theorem of Radó-Kneser-Choquet was extended to multiply connected domains by P. Duren and W. Hengartner and, independently by A. Lyzzaik; see [3] and [6]. We say that a complex-valued function f of $\mathbf{D}$ is $p-$*valent*, or has valency p, if it admits each value at most p times and some value exactly p times. In 1985, T. Sheil-Small [11] considered harmonic mappings of $\mathbf{D}$ to $\mathbf{K}$ that extend continuously to n-valent sense-preserving local homeomorphisms between $\partial\mathbf{D}$ and $\partial\mathbf{K}$ and asked whether these functions are $(3n-2)-$valent in $\mathbf{D}$. A negative answer was given for all $n = 3, 6, 9, \cdots$ by A. Lyzzaik [5]. Subsequently, in a slightly different direction, Sheil-Small [12] conjectured that a harmonic polynomial of degree n is at most n^2-valent in the complex plane, $\mathbb{C}$; a *harmonic polynomial* of degree n is of the form $p + \overline{q}$ where p and q are analytic polynomials with degrees n and m, $m < n$, respectively. R. Peretz and J. Schmidt [9] supplied the answer. Further, A. Wilmshurst [13] and, independently, D. Bshouty, W. Hengartner and T. Suez [1] showed that n^2 is indeed the best possible bound by constructing harmonic polynomials of degree n and valency n^2. These results disprove once again Sheil-Small's former question, this time however for all values n. This led T. Sheil-Small (personal communication) about a decade ago to make the following conjecture.

Sheil-Small's Conjecture. If f is a harmonic mapping of $\mathbf{D}$ onto a convex Jordan domain $\mathbf{K}$ that extends continuously to an n-valent sense-preserving local homeomorphism between $\partial\mathbf{D}$ and $\partial\mathbf{K}$ and assumes every point in $\partial\mathbf{K}$ exactly n times, then the valency of f is at most n^2.

If the convexity of $\mathbf{K}$ replaces the set-inclusion $f(\mathbf{D}) \subset \mathbf{K}$, then the conjecture is false as was recently shown by A. Lyzzaik [7].

The aim of this paper is to show that the latter Sheil-Small's conjecture is false. We do this by constructing two harmonic mappings of $\mathbf{D}$ onto convex Jordan domains $\mathbf{K}$ that extend continuously to 2-valent sense-preserving local homeomorphisms between $\partial\mathbf{D}$ and $\partial\mathbf{K}$ such that the valency of one mapping is 6 and of the other is 8. These mappings, though shall be studied independently, are related in the sense that each could be obtained from the previous one by a specific variation. We state the result of this paper as follows.

Theorem *There exist* $6-$ *and* $8-$*valent harmonic mappings* f *of the unit disc* $\mathbf{D}$ *onto a convex domain* $\mathbf{K}$ *that extend continuously to 2-valent sense-preserving homeomorphisms between* $\partial\mathbf{D}$ *and* $\partial\mathbf{K}$.

The paper is based on the following three harmonic mappings.

$$f_1(z) = \overline{z} + z^2/2; \tag{2}$$

$$\begin{aligned} f_2(z) \;&=\; \overline{z - 2.5/(z-3)^2} \\ &\quad + z^2/2 - 5/(z-3) - 7.5/(z-3)^2; \end{aligned} \tag{3}$$

$$\begin{aligned} f_3(z) \;&=\; \overline{z - 2.5/(z-3)^2 + 0.0015/(z-1.4)^2} \\ &\quad + z^2/2 - 5/(z-3) - 7.5/(z-3)^2 \\ &\quad + 0.003/(z-1.4) + 0.0021/(z-1.4)^2. \end{aligned} \tag{4}$$

A requisite for the study of the geometry of these maps is the following notation. Let f be a complex-valued function of a Jordan domain G of the sphere, and let $z_0 \in \overline{G}$. We write

$$f_{z_0} \sim z^r(\overline{G}) \qquad (r = 1, 2, \cdots)$$

if there exist open neighborhoods U and V of z_0 and $f(z_0)$, respectively, and sense-preserving homeomorphisms h_1 and h_2 that map U and V to (i) the discs $|\zeta| < 1$ and $|\eta| < 1$ if $z_0 \in G$, and to (ii) the half-closed semi-discs $|\zeta| < 1$, $\Im\zeta \geq 0$ and $|\eta| < 1$, $\Im\eta \geq 0$ otherwise, respectively, such that

$$\eta = h_2 \circ f \circ h_1^{-1}(\zeta) = \zeta^r \qquad (\zeta \in h_1(U \cap \overline{G})).$$

Similarly, we write

$$f_{z_0} \sim \overline{z}^r(\overline{G}) \qquad (r = 1, 2, \cdots),$$

if

$$\eta = h_2 \circ f \circ h_1^{-1}(\zeta) = \overline{\zeta}^r \qquad (\zeta \in h_1(U \cap \overline{G})).$$

Note that if $z_0 \in \partial G$, then r is odd if and only if $f_{|\partial G}$ is locally one-to-one at z_0, and $r = 1$ if and only if $f_{|G}$ is locally one-to-one at z_0. Also, we write

$$f_{z_0} \sim z^r, \overline{z}^s(G) \qquad (r, s = 1, 2, \cdots)$$

if $z_0 \in G$ and there exists a cross-cut of G through z_0 that divides G into two Jordan domains G_1 and G_2 such that $f_{z_0} \sim z^r(\overline{G}_1)$ and $f_{z_0} \sim \overline{z}^s(\overline{G}_2)$.

Now note that the dilatation of each f_j is $a(z) = 1/z$ and the Jacobian is $J(f_j) = (|z|^2 - 1)|g_j'|^2$. Thus each f_j is locally one-to-one everywhere in $\mathbb{C}$ except on $\partial\mathbf{D}$ and at the points where g_j' either vanishes or is infinite (in case of f_2 and f_3), and is sense-reversing in $\mathbf{D}$ and sense-preserving in $\mathbb{C} \setminus \overline{\mathbf{D}}$; further, each f_j admits ∞ as a critical point of order 1 [8]. To study the behavior of each f_j on $\partial\mathbf{D}$, we write

$$f_j(e^{it} = \overline{g}_j(e^{it}) + h_j(e^{it})$$

where g_j and h_j are given by (2), (3) and (4). Then

$$\begin{aligned} \frac{d}{dt} f_j(e^{it}) &= -ie^{-it}\overline{g}_j'(e^{it}) + ie^{it}h_j'(e^{it}) \\ &= -ie^{-it}\overline{g}j'(e^{it}) + ie^{2it}g_j'(e^{it}) \\ &= -2e^{it/2}\Im[e^{3it/2}g_j'(e^{it})]. \end{aligned} \tag{5}$$

Observe that $\arg df_j(e^{it})/dt$ makes a jump of size an odd multiple of π at each value t where $\Im[e^{3it/2}g_j'(e^{it})]$ changes sign and is otherwise continuously increasing by π. Thus $f_j(\partial\mathbf{D})$ is everywhere locally convex except for a cusp of angle size zero associated with each of the jumps. We shall see that each h_j' (and g_j') are never zero on $\partial\mathbf{D}$. This implies that for any $z_0 \in \partial\mathbf{D}$, $f_{z_0} \sim z, \overline{z}\,(\mathbf{D})$ unless z_0 is a point where a jump of $\arg df_j(e^{it})/dt$ occurs in which case $f_{z_0} \sim z^3, \overline{z}$ or $f_{z_0} \sim \overline{z}^3, z\,(\mathbf{D})$ and the size of the associated jump is $-\pi$ or π respectively

[8]. If the number of jumps is κ, then the total variation of $\arg df_j(e^{it})/dt$ is $(\kappa+1)\pi$. Further, since f_j is locally one-to-one and sense-reversing in $\mathbf{D}$, the net variation of $\arg df_j(e^{it})/dt$ is exactly -2π which yields κ odd.

The paper is organized as follows. In Section 2, we study the geometry of f_1 in order to establish a 4−valent harmonic mapping F_1 of the unit disc $\mathbf{D}$ onto a convex domain $\mathbf{K}$ that extends continuously to 2−valent sense-preserving homeomorphisms between $\partial\mathbf{D}$ and $\partial\mathbf{K}$. The same exercise is carried out in Sections 3 and 4 with f_2 and f_3 in order to establish 6− and 8−valent harmonic mappings F_2 and F_3, respectively, which otherwise satisfy the same properties of F_1. This disproves Sheil-Small's conjecture for $n=2$.

2. The Geometry of f_1 and the Construction of F_1

In this section, we construct a 4-valent harmonic mapping F_1 of $\mathbf{D}$ onto a convex Jordan domain $\mathbf{K}$ that extends continuously to a 2-valent sense-preserving local homeomorphism between $\partial\mathbf{D}$ and $\partial\mathbf{K}$.

Using (2) and (5), we have

$$\frac{d}{dt}f_1(e^{it}) = -e^{it/2}\sin(3t/2). \tag{6}$$

Then there exist branches of $\arg df_1(e^{it})/dt$ that increase steadily and continuously by π on $[0,2\pi]$ without account of a jump of size an odd multiple of π at each $t_k = 2(k-1)\pi/3$, $k=1,2,3$. It follows that f_1 maps $\partial\mathbf{D}$ homeomorphically to a deltoid whose three cusps have vertices at the points $f(e^{it_k})$; a deltoid is a triangle with concave sides and angles of size zero; see Figure 1(a). Let Δ be the Jordan domain determined by the deltoid. Since $J_{f_1}(z) = |z|^2-1$ vanishes if and only if $z\in\partial\mathbf{D}$, f_1 is locally one-to-one in $\mathbb{C}\setminus\partial\mathbf{D}$. Also, since J_{f_1} is positive in $\mathbb{C}\setminus\overline{\mathbf{D}}$ and negative in $\mathbf{D}$, it is sense-preserving in the former region and reversing in the latter. It follows at once that f_1 is a sense-reversing homeomorphism from $\overline{\mathbf{D}}$ to $\overline{\Delta}$. Further, in view of the fact that the analytic and co-analytic parts of f_1 are non-vanishing on $\partial\mathbf{D}$ [8], $f_{z_0} = z, \overline{z}\,(\overline{\mathbf{D}})$ for every $z_0\in\partial\mathbf{D}$ and $z_0\notin e^{it_k}$, and $f_{z_0} = z^3, \overline{z}\,(\overline{\mathbf{D}})$ for every $z_0 = e^{it_k}$.

Let γ_k be the positively-directed subarc of $\partial\mathbf{D}$ starting and ending at e^{it_k} and $e^{it_{k+1}}$ respectively, with $e^{it_4} = e^{it_1}$, $A_k = f_1(e^{it_k})$ and $\Gamma_k = f_1(\gamma_k)$. Evidently, each Γ_k is a directed arc starting and ending at A_k and A_{k+1} respectively, with $A_4 = A_1$, and the points A_k and the arcs Γ_k are the vertices and sides of the deltoid respectively. Let L_k be the ray from A_k in the direction opposite to the tangent vector to Γ_k at A_k. Because of the local behavior of f_1 on $\partial\mathbf{D}$, in particular at the points e^{it_k} [8], there exists an unbounded Jordan arc ℓ_k from e^{it_k} that lies otherwise in $\mathbb{C}\setminus\overline{\mathbf{D}}$ and maps under f_1 homeomorphically to L_k. Obviously, the arcs ℓ_k are mutually disjoint and together with $\partial\mathbf{D}$ divide $\mathbb{C}$ into $\mathbf{D}$ and three Jordan domains D_k, $k=1,2,3$, where D_k is bounded by ℓ_k, γ_k

and ℓ_{k+1}, with $\ell_4 = \ell_1$. Let Δ_k be the Jordan domain bounded by Ł_k, Γ_k and Ł_{k+1} and lying on the right-hand side of Γ_k. Observe that f_1 (i) as restricted to each $\overline{D_k}$ is a locally one-to-one map, (ii) maps each ∂D_k homeomorphically to $\partial\Delta_k$, and (iii) is an open sense-preserving map in D_k. Then, by the Monodromy theorem, we conclude that f_1 maps $\overline{D_k}$ homeomorphically to $\overline{\Delta_k}$.

View $\overline{\Delta}$ and each $\overline{\Delta_k}$ as a bordered covering surface of $\mathbb{C}$ with the identity as the projection map; see Figure 1(b). Adjoin these surfaces by identifying crosswise any two boundary arcs with the same projection. This yields the folded image surface of f_1 [8]. It is immediate that this surface covers every point of Δ, $\partial\Delta$, and $\mathbb{C} \setminus \mathbf{D}$ exactly 4, 3, and 2 times respectively, and that it spreads over $\mathbb{C} \setminus \overline{\Delta}$ as a 2-sheeted smooth covering. Let $\mathbf{K}$ be a convex Jordan domain containing $\partial\boldsymbol{\Delta}$, and let $\mathcal{D} = f^{-1}(\mathbf{K})$. It follows at once that $\mathcal{D}$ is a Jordan domain containing $\mathbf{D}$, f is a 4-valent function from $\mathcal{D}$ onto $\mathbf{K}$, and f extends continuously to a 2-valent local homeomorphism from $\partial\mathbf{D}$ onto $\mathbf{K}$. Precomposing f_1 with a univalent function from the $\mathbf{D}$ onto $\mathcal{D}$ yields the desired harmonic mapping F_1 from $\mathbf{D}$ onto $\mathbf{K}$.

3. The Construction of F_2

Here we construct a 6-valent harmonic mapping F_2 of $\mathbf{D}$ onto a convex Jordan domain $\mathbf{K}$ that extends continuously to a 2-valent sense-preserving local homeomorphism between $\partial\mathbf{D}$ and $\partial\mathbf{K}$.

Using (3) and (5), we obtain

$$\frac{d}{dt} f_2(e^{it}) = -\varphi_2(t) e^{it/2},$$

where

$$\varphi_2(t) = \Im[e^{i3t/2}(1 + \frac{5}{(e^{it} - 3)^3})].$$

We can write

$$\varphi_2(t) = -16 \frac{\sin(t/2)}{|e^{it} - 3|^6} \psi_2(u),$$

where $u = \sin^2 t/2$ and

$$\psi_2(u) = 3 - 127u - 180u^2 + 108u^3 + 432u^4.$$

Clearly, $\psi_2(0)$ is nonzero and φ_2 changes sign at $t_1 = 0$. Also, φ_2 changes sign at $t \in (0, 2\pi)$ if and only if ψ_2 changes sign at $u \in (0,1)$. The number of the latter values is found by adhering to the variations of ${\psi_2}'$ and ${\psi_2}''$ which imply that ψ_2 decreases from $\psi_2(0) = 3$ to $\psi_2(1/2) = -65$ and then increases to $\psi_2(1) = 236$. Thus ψ_2 changes sign exactly twice in $(0,1)$, namely at the approximate values 0.02289 and 0.77059 (these values and others appearing henceforth are found by the software *Mathematica*). Solving $\sin^2 t/2 \approx 0.02289$ and $\sin^2 t/2 \approx 0.77059$ yield the values $t_2 \approx 0.30375$, $t_3 \approx 2.14264$, $t_4 \approx 2\pi - 2.14264 \approx 4.13954$ and $t_5 \approx 2\pi - 0.303754 \approx 5.97843$ at which φ_2 changes sign in $(0, 2\pi)$. Hence, φ_2 changes sign exactly five times in an interval $(-\epsilon, 2\pi)$ for a sufficiently small positive ϵ, namely at the values t_k, $1 \leq k \leq 5$.

We conclude that there exist branches of $\arg df_2(e^{it})/dt$ that increase steadily and continuously by π on $[0, 2\pi]$ without account of a jump of size an odd multiple of π at each value t_k. Note that

$$J_{f_2}(z) = (|z|^2 - 1)|g_2'|^2 = (|z|^2 - 1)|1 + \frac{5}{(z-3)^3}|^2$$

which vanishes on $\partial\mathbf{D}$ and at the zeros of g_2' which are $3 + \sqrt[3]{-5}$, or $z_1 \approx 3.85499 + 1.48088i$, $z_2 = \overline{z}_1$ and $z_3 \approx 1.29002$. Then f_2 is locally one-to-one in $\mathbb{C} \setminus \partial\mathbf{D}$ except at 3 where g_2 is infinite, or at the points z_k. Since each $|z_k| > 1$, $(f_2)_\xi \sim z^3, \overline{z}\,(\mathbf{D})$ or $(f_2)_\xi \sim \overline{z}^3, z\,(\mathbf{D})$ for any $\xi = e^{it_k}$ and the size of each jump of $\arg df_2(e^{it})/dt$ is $-\pi$ or π respectively. Further, f_2 is locally one-to-one in $\mathbf{D}$ and consequently the net variation of $\arg df_2(e^{it})/dt$ on $[0, 2\pi]$ is -2π. It follows that if m is the number of the values t_k at which $\arg df_2(e^{it})/dt$ makes a jump of size $-\pi$, then $-2\pi = \pi - m\pi + (5-m)\pi$ and $m = 4$; that is, $\arg df_2(e^{it})/dt$ makes a jump of size π at only one t_k and size $-\pi$ at each of the others.

Now we describe $f_2(\partial\mathbf{D})$. Let $A_k = f(e^{it_k})$, $1 \leq k \leq 5$. Then $A_1 = 1.5$, $A_2 \approx 1.51622 + 0.00129i$, $A_3 \approx -0.08600 - 1.14829i$, and, since $f_2(z) = \overline{f_2(\overline{z})}$, $A_4 = \overline{A}_3$ and $A_5 = \overline{A}_1$. Let γ_k, $1 \leq k \leq 5$, be the subarc of $\partial\mathbf{D}$ determined by $z(t) = e^{it}$, $t_k \leq t_{k+1}$, with $t_6 = t_1$, and let $\Gamma_k = f_2(\gamma_k)$; see Figure 2(a). We conclude the following:

(i) Each Γ_k is a locally convex Jordan arc;

(ii) Any two arcs Γ_k and Γ_{k+1}, with $\Gamma_6 \equiv \Gamma_1$, form a cusp whose vertex is A_{k+1}, with $A_6 \equiv A_1$;

(iii) $\Gamma_1 \setminus \{A_1\}$ and $\Gamma_5 \setminus \{A_1\}$ lie in the upper and lower half-planes, respectively;

(iv) Γ_k and Γ_{k+3}, $k = 1, 2$, cross at exactly one point;

(v) Γ_k and Γ_{6-k}, $k = 1, 2$, are symmetric to each other about the real axis, and Γ_3 is also symmetric about the real axis;

(vi) The arc-products $\Gamma_{k+2}\Gamma_{k+1}\Gamma_k$, $k = 1, 3$, are Jordan arcs;

(vii) Except for endpoints, Γ_3 does not meet any other Γ_k;

(viii) Γ_1, Γ_2, Γ_4 and Γ_5 bound a Jordan region, R, which lies on the right-hand side of every Γ_k;

(ix) The restriction of f_2 to $\mathbf{D}$ assumes each value in R exactly twice and elsewhere at most once;

(x) $f_2(\overline{\mathbf{D}})$ lies in the convex-hull of the points A_k, $1 \le k \le 4$.

In view of (x), $\arg df_2(e^{it})/dt$ makes a jump of size $-\pi$ at each t_k, $2 \le k \le 5$, and of size π at t_1.

We describe now the image surface Ω of f_2 in $\mathbf{D}$; see Figure 2(b). Recall that f_2 is locally on-to-one in $\mathbf{D}$ ($z_k \notin \mathbf{D}$, $0 \le k \le 3$.) Let ω be the point of intersection of Γ_3 with the real axis, and let Γ_{31} and Γ_{32} be the subarcs of Γ_3 from A_3 to ω and from ω to A_4, respectively. Also, let ωA_1 be the straight-line segment from A_1 to ω. Denote by $\overline{\Omega_1}$ the closure of the Jordan domain bounded by Γ_1, Γ_2, Γ_{31} and ωA_1, and by $\overline{\Omega_2}$ the closure of the Jordan region bounded by Γ_5, Γ_4, Γ_{32} and ωA_1. Observe that $\overline{\Omega_1}$ is symmetric to $\overline{\Omega_2}$ with respect to the real axis. View $\overline{\Omega_1}$ and $\overline{\Omega_2}$ along with the identity maps as bordered covering surfaces of the complex plane, then identify these surfaces crosswise along the boundary arcs associated with ωA_1. This yields the bordered image surface $\overline{\Omega}$ of f_2 in $\overline{\mathbf{D}}$ whose interior Ω is the image surface of f_2 in $\mathbf{D}$.

As for f_2 in the complementary set of $\overline{\mathbf{D}}$, it is locally one-to-one there except at each z_k, 3 and ∞. Since each z_k is a zero of h' and g' of order 1 and f_2 is sense-preserving in some deleted neighborhood of z_k, z_k is a critical point of order 1 of f_2 [8]. Similar considerations also yield 3 and ∞ critical points of f_2 of order 1. If $w_k = f_2(z_k)$, then $w_1 \approx 10.43690 + 8.24106i$, $w_2 = \overline{w}_1$ by symmetry, and $w_3 \approx 1.62617$; further, $f(3) = f(\infty) = \infty$. Hence, there exists an open neighborhood of every $w \ne w_k, \infty$ in which every continuous branch of the inverse function of f_2 is one-to-one.

Let L_k, $2 \le k \le 5$, be the ray whose initial point is A_k and direction is opposite to the tangent vector to Γ_k at A_k; see Figure 2(a). Let Δ_1 be the Jordan domain bounded by L_2, Γ_1, Γ_5 and L_5, and let Δ_k, $2 \le k \le 4$, be the Jordan domain bounded by L_k, Γ_k and L_{k+1}, and lying on the right-hand side of Γ_k. In view of the local behavior of f_2 at each e^{it_k}, $2 \le k \le 5$, there exists a Jordan arc ℓ_k from e^{it_k} to infinity that maps under f_2 homeomorphically to L_k. Now let D_1 be the Jordan domain bounded by ℓ_2, γ_1, γ_5 and ℓ_5, and let D_k, $2 \le k \le 4$, be the Jordan domain bounded by ℓ_k, γ_k and ℓ_{k+1} and lying on the right-hand side of γ_k. Since f_2 is sense-preserving and open in $\mathbb{C} \setminus \overline{\mathbf{D}}$, $\overline{\Delta_k} \subset f_2(\overline{D}_k)$ for each k.

It is immediate that there exists a Jordan convex domain K that contains $f_2(\overline{\mathbf{D}})$ and avoids the points w_k $1 \le k \le 3$; in fact, K can be an open disk; see Figure 2(a). Let B_k be the point of intersection of L_k with ∂K. Considering the argument of the tangent vector to each Γ_k at A_k, $2 \le k \le 5$, and the locations of the points A_k, we conclude that the points B_k appear on the positively-directed ∂K in the order B_2, B_4, B_3, B_5. Let $\Delta'_k = K \cap \Delta_k$, $1 \le k \le 4$. Also, let β_k be the common boundary arc of Δ'_k and K starting from B_k and ending at B_{k+1} if $2 \le k \le 4$, and the common boundary arc of Δ'_1 and K starting from B_5 and ending at B_2 otherwise. Evidently, each Δ'_k is a Jordan domain that contains none of the points w_k and satisfies $\overline{\Delta'_k} \subset f_2(\overline{D}_k)$; see Figure 2(c). Hence, by the Monodromy theorem, there exists a Jordan domain

D'_k in D_k that maps under f_2 homeomorphically to Δ'_k. Observe that each D'_k can be chosen so that its boundary contains e^{it_k}. A homotopy argument then implies that D'_k, $2 \leq k \leq 4$, is a Jordan domain bounded by ℓ_k, γ_k, ℓ_{k+1} and a Jordan arc α_k starting from an interior point b_k of ℓ_k and ending in an interior point b_{k+1} of ℓ_{k+1} and lying otherwise in D_k, and D'_1 by ℓ_2, γ_1, γ_5, ℓ_5 and a Jordan arc α_1 starting from b_5 and ending at b_2 and lying otherwise in D_1. It is immediate that each $f_2 : \overline{D'_k} \to \overline{\Delta'_k}$ is a homeomorphism that maps α_k to β_k in a sense-preserving manner, and that $\mathcal{D} = \mathbf{D} \bigcup (\cup_{k=1}^4 \overline{D'_k})$ is a Jordan domain bounded by $\alpha = \alpha_1\alpha_2\alpha_3\alpha_4$. Let $\beta = \beta_1\beta_2\beta_3\beta_4$, and observe that β is a $2-$fold $1-$dimensional covering of ∂K with the identity as the projection map. It follows that as z traverses α positively once starting from b_1, $f_2(z)$ traverses ∂K positively twice starting from B_1. Hence f_2 is a $2-$valent sense-preserving local homeomorphism from $\partial\mathcal{D}$ to ∂K. Since each Δ'_k contains the region R, f_2 of $\mathcal{D} \setminus \overline{\mathbf{D}}$ assumes each value in R exactly 4 times, a conclusion that could also be obtained by the Argument principle. Recall that f_2 of $\mathbf{D}$ assumes each value in R twice. Hence, f_2 of $\mathcal{D}$ assumes each value in R exactly 6 times. Precomposing f_2 with a conformal map from the open unit disc to $\mathcal{D}$ yields at once the desired function F_2. The image surface of F_2 can be obtained by considering the bordered surface $\overline{\Omega}$ along with the closed Jordan domains $\overline{\Delta'_k}$, $1 \leq k \leq 4$, with the identity map viewed also as bordered covering surfaces of $\mathbb{C}$, then identifying these surfaces crosswise along the boundary arcs with the same projection; see Figure 2(c).

Remark 1. We show here that a local surgery on the image surface of F_1 yields essentially the image surface of F_2. We start off with the former image surface and its associated notation. Let $\mathbf{B}$ be a closed disc in the interior of K centered at A_1 and not meeting Γ_2. see Figure 3(a). Let $c_1 = \Gamma_1 \cap \mathbf{B}$, $c_2 = \Omega \cap \partial\mathbf{B}$ and $c_3 = \Gamma_2 \cap \mathbf{B}$. With $A_1 < B \in \partial\mathbf{B}$, let c_4 and c_5 be the major subarcs of $\partial\mathbf{B}$ starting from B and ending in c_3 and c_1 respectively. Consider the Jordan domains: W_1 bounded by the line segment $[A_1, B]$, c_4 and c_3, W_2 bounded by $[A_1, B]$, c_1 and c_5, and W_3 bounded by c_1, c_2 and c_3; see Figure 3(b). By viewing each of these domains as a covering surface of $\mathbb{C}$ with the identity as the projection map, the image surface of F_1 over the interior of $\mathbf{B}$ is obtained by identifying these surfaces crosswise along the boundary arcs with the same projection. Denote this surface by $\mathcal{W}$.

Let $e = [A_1, v]$ be a real line segment starting from A_1 and lying otherwise in W_3. Cut W_3 along e, and denote by e_1 and e_2 the lower and upper edges of the cut respectively. With fixed c_2 and v, bend e_1 and e_2 slightly into convex arcs tangent to the real axis at the common endpoint v and lie otherwise in the upper and lower half-planes in the manner depicted in Figure 3(c); denote the other endpoints of the new e_1 and e_2 by A_{11} and A_{12} respectively. This converts c_1 and c_3 to non-adjacent convex arcs ending in A_{12} and A_{11}, respectively, instead of A_1, W_1 to a Jordan domain bounded by the line segment $[A_{11}, B]$, c_4 and c_3,

W_2 to a Jordan domain bounded by $[A_{12}, B]$, c_1 and c_5, and W_3 to a covering surface with the same covering properties of the image surface of f_2 in $\mathbf{D}$. Let W_4 be the Jordan domain bounded by the segments $[A_{11}, B]$, $[A_{12}, B]$, e_1 and e_2; see Figure 3(d). Now view each of the new domains W_k, $1 \leq k \leq 4$, as a covering surface of $\mathbb{C}$ with the identity as the projection map, and identify these surfaces crosswise along the boundary arcs with the same projection. Denote the resulting surface by $\mathcal{W}'$.

Note that the borders of the image surfaces $\overline{\mathcal{W}}$ and $\overline{\mathcal{W}'}$ are "identical" when viewed as $1-$dimensional coverings of $\partial\mathbf{B}$. Thus $\overline{\mathcal{W}}$ can replace $\overline{\mathcal{W}'}$ in the image surface of F_1 in the appropriate manner which results in a covering surface of $\mathbb{C}$ that has the same covering properties of the image surface of F_2 in $\mathbf{D}$.

We conclude that the image surfaces of F_1 and F_2 in $\mathbf{D}$ differ only in a local neighborhood in which a cusp in the former surface splits into three cusps in the latter in a manner that contributes to a higher valency by 2 for F_2.

4. The Construction of F_3

In this section, we construct an 8-valent harmonic mapping F_3 of $\mathbf{D}$ onto a convex Jordan domain $\mathbf{K}$ that extends continuously to a 2-valent sense-preserving local homeomorphism between $\partial\mathbf{D}$ and $\partial\mathbf{K}$.

Using (3) and (5), we obtain

$$\frac{d}{dt} f_3(e^{it}) = -\varphi_3(t)e^{it/2},$$

where

$$\varphi_3(t) = \Im[e^{i3t/2}(1 + \frac{5}{(e^{it}-3)^3} - \frac{0.003}{(e^{it}-1.4)^3})].$$

We can write

$$\varphi_3(t) = -\frac{2 \times 0.4^6 \sin(t/2)}{|e^{it}-3|^6|e^{it}-1.4|^6}\psi_3(u),$$

where $u = \sin^2(t/2)$ and

$$\psi_3(u) = 8(1+35u)^3\psi_2(u) + 27(13u-1)(1+3u)^3.$$

Note that $\psi_3(0) = -3 \neq 0$; so φ_3 changes sign at $t_1 = 0$. Also, note that φ_3 changes sign at $t \in (0, 2\pi)$ if and only if ψ_3 changes sign at $u \in (0,1)$. To find the number of the latter values, observe that for each of the derivatives $\psi_3^{(n)}$, $2 \leq n \leq 5$, there exists a value $0 \leq u \leq 1$ such that the derivative is negative in $[0, u)$ and positive in $(u, 1]$, and that $\psi_3'(0.5) < 0 < \psi_3'(0)$ and $\psi_3'(1) > 0$. Thus there exist two values $0 < u_1 < u_2 < 1$ such that ψ_3' is positive in the intervals $[0, u_1)$ and $(u_2, 1]$ and negative in the interval (u_1, u_2). Further, Since $\psi_3(0) < 0$, $\psi_3(0.01) = 8.03135 > 0$, $\psi_3(0.02) = -8.50987 < 0$

and $\psi_3(0.8) = 3.60846 \times 10^6 > 0$, ψ_3 changes sign exactly three times in $(0, 1)$, namely at the approximate values 0.00191, 0.01747 and 0.77051. The equations $\sin^2 t/2 \approx 0.00191$, $\sin^2 t/2 \approx 0.01747$ and $\sin^2 t/2 \approx 0.77051$ yield the values $t_2 \approx 0.08751$, $t_3 \approx 0.26516$, $t_4 \approx 2.14244$, $t_5 \approx 2\pi - 2.14244 \approx 4.14074$, $t_6 \approx 2\pi - 0.26516 \approx 6.01802$ and $t_7 \approx 2\pi - 0.08751 \approx 6.19567$ in $(0, 2\pi)$ at which φ_3 changes sign. Hence, φ_3 changes sign exactly seven times in an interval $(-\epsilon, \pi)$ for a sufficiently small positive ϵ, namely at the values t_k, $1 \le k \le 7$.

We conclude that there exist branches of $\arg df_3(e^{it})/dt$ that increase steadily and continuously by π on $[0, 2\pi]$ without account of a jump of size an odd multiple of π at each value t_k. The Jacobian of f_3 is given by

$$J_{f_3} = (|z|^2 - 1)|g_3'(z)|^2 = (|z|^2 - 1)||1 + \frac{5}{(z-3)^3} - \frac{0.003}{z-1.4}^{3}|^2.$$

Clearly, $J_{f_3} < 0$ in $\mathbf{D}$ and $J_{f_3} > 0$ in $\mathbb{C} \setminus \overline{\mathbf{D}}$. Hence f_3 is sense-reversing in $\mathbf{D}$, sense-preserving in $\mathbb{C} \setminus \overline{\mathbf{D}}$, and locally one-to-one in $\mathbb{C} \setminus \partial\mathbf{D}$ except at 3, 1.4 and the zeros of

$$1 + \frac{5}{(z-3)^3} - \frac{0.003}{(z-1.4)^3}$$

which are: $z_1 \approx 3.85505 + 1.48084i$, $z_2 = \overline{z_3}$, $z_3 \approx 1.51652 + 0.124996i$, $z_4 = \overline{z_5}$, $z_5 \approx 1.22843 + 0.147502i$ and $z_6 = \overline{z_7}$. Since 3, 1.4 and $|z_k|$ are larger than 1, the following hold: (a) $(f_3)_\xi \sim z^3, \overline{z}\,(\mathbf{D})$ or $(f_3)_\xi \sim \overline{z}^3, z\,(\mathbf{D})$ for any $\xi = e^{it_k}$, (b) the size of each jump of $\arg df_3(e^{it})/dt$ is $-\pi$ or π respectively, and (c) the net variation of $\arg df_3(e^{it})/dt$ on $[0, 2\pi]$, with account of the jumps at the values t_k, is -2π; see [8]. It follows that if m is the number of the values t_k at which $\arg df_3(e^{it})/dt$ makes a jump of size $-\pi$, then $-2\pi = \pi - m\pi + (7 - m)\pi$ and $m = 5$; that is, $\arg df_3(e^{it})/dt$ makes a jump of size $-\pi$ at 5 values t_k and of size π at two.

Now we describe $f_3(\partial\mathbf{D})$. Let $A_k = f(e^{it_k})$, $1 \le k \le 7$. Then $A_1 = 1.515$, $A_2 \approx 1.51483 - 3.77285 \times 10^{-6}i$, $A_3 \approx 1.51686 + 0.00018i$, $A_4 \approx -0.08675 - 1.14875i$, and, since $f_3(z) = \overline{f_3(\overline{z})}$, $A_5 = \overline{A_4}$, $A_6 = \overline{A_3}$ and $A_7 = \overline{A_2}$. For $1 \le k \le 7$, let γ_k be the subarc of $\partial\mathbf{D}$ determined by $z = e^{it}$, $t_k \le t \le t_{k+1}$, with $t_8 = t_1$, and let $\Gamma_k = f_3(\gamma_k)$; see Figure 4(a). We conclude the following:

(i) Each Γ_k is a locally convex Jordan arc;

(ii) Any two arcs Γ_k and Γ_{k+1}, with $\Gamma_8 \equiv \Gamma_1$, form a cusp whose vertex is A_{k+1}, with $A_8 \equiv A_1$;

(iii) $\Gamma_1 \setminus \{A_1\}$ and $\Gamma_7 \setminus \{A_1\}$ lie in the lower and upper half-planes, respectively;

(iv) Γ_k, $k = 2, 3$, crosses each of arc Γ_{8-k} and Γ_{9-k} at exactly one point; also, Γ_k, $k = 5, 6$, crosses each arc Γ_{7-k} and Γ_{8-k} at exactly one point.

(v) Γ_k and Γ_{8-k}, $1 \le k \le 3$, are symmetric about the real axis, and Γ_4 is also symmetric about the real axis;

(vi) Γ_3 does not intersect Γ_1 and Γ_7, and it meets the real axis at exactly one point in (A_1, ∞). To see this, observe that, by (5), the tangent line to Γ_3 through A_3 is given by $A_3 + te^{it_3/2}$, $t \in (\infty, \infty)$, and it meets the real axis at $\approx 1.51599 > A_1$ and separates Γ_3 from the arcs Γ_1, $\Gamma_2 \setminus \{A_3\}$ and Γ_7. Likewise, because of symmetry, Γ_5 does not intersect Γ_1 and Γ_7;

(vii) Γ_4 lies in the left half-plane bounded by the vertical line passing through A_2 and A_7. To see this, observe that, by (5), the tangent line to Γ_4 through A_4 is given by $A_4 + te^{it_4/2}$, $t \in (\infty, \infty)$, and it meets the real axis at $\approx 0.71368 < \Re A_1$;

(viii) The arc-products $\Gamma_4\Gamma_3\Gamma_2\Gamma_1$ and $\Gamma_7\Gamma_6\Gamma_5\Gamma_4$ are Jordan arcs symmetric to each other about the real axis.

(ix) Γ_1, Γ_2, Γ_6 and Γ_7 bound a Jordan domain, Q, which lies on the right-hand side of every Γ_k;

(x) The restriction of f_3 to $\mathbf{D}$ assumes each value in Q exactly three times and elsewhere at most twice;

(xi) $f_3(\overline{\mathbf{D}})$ lies in the convex-hull of the points A_k, $3 \leq k \leq 6$.

Because of (xi) and symmetry, we conclude that $\arg df_3(e^{it})/dt$ makes a jump of size π at t_2 and t_7 and of size $-\pi$ at the remaining values t_k.

We describe now the image surface Ω of f_3 in $\mathbf{D}$; see Figure 4(b). Recall that f_3 is locally one-to-one in $\mathbf{D}$. Let ω be the point of intersection of Γ_4 with the real axis, and let Γ_{41} and Γ_{42} be the subarcs of Γ_4 from A_4 to ω and from ω to A_5 respectively. Also, let ωA_2 and ωA_7 be the straight-line segments from ω to A_2 and to A_7 respectively. Denote by $\overline{\Omega_1}$ the closed Jordan domain bounded by Γ_2, Γ_3, Γ_{41} and ωA_2, by $\overline{\Omega_2}$ the closed Jordan domain bounded by ωA_7, Γ_{42}, Γ_5 and Γ_6, and by $\overline{\Omega_3}$ the closed Jordan domain bounded by Γ_1, ωA_2, ωA_7 and Γ_7. Observe that $\overline{\Omega_1}$ is symmetric to $\overline{\Omega_2}$ about the real axis, and $\overline{\Omega_1}$ is symmetric about the real axis. View each $\overline{\Omega_j}$ with the identity map as a bordered covering surface of the complex plane, then identify the surface $\overline{\Omega_3}$ with the surfaces $\overline{\Omega_1}$ and $\overline{\Omega_2}$ crosswise along the boundary arcs associated with ωA_2 and ωA_7, respectively. This yields the bordered image surface $\overline{\Omega}$ of f_3 in $\overline{\mathbf{D}}$ whose interior Ω is the image surface of f_3 in $\mathbf{D}$.

As for f_3 in the complementary set of $\overline{\mathbf{D}}$, it is locally one-to-one there except at the points z_k, $1 \leq k \leq 6$, 1.4, 3 and ∞. Since each z_k is a zero of h' and g' of order 1 and f_3 is sense-preserving in some deleted neighborhood of z_k, z_k is a critical point of order 1 of f_3 [8]. Similar considerations also yield 1.4, 3 and ∞ as critical points of f_2 of order 1. If $w_k = f_3(z_k)$, then $w_1 \approx 10.438 + 8.24045i$, $w_2 = \overline{w_1}$, and $w_3 \approx 1.56031 - 0.06427i$; also, because of symmetry, $w_4 = \overline{w_3}$, $w_5 \approx 1.65472 + 0.00835i$, and , $w_6 = \overline{w_5}$. Further, $f_3(1.4) = f_3(1.3) = f_3(\infty) = \infty$. Hence, there exists an open neighborhood of every $w \neq w_k, \infty$ in which every continuous branch of the inverse function of f_3 is one-to-one.

Let L_k, $k=1,3,4,5,6$, be the ray whose initial point is A_k and whose direction is opposite to the tangent vector to Γ_k at A_k; see Figure 4(a). Let Δ_2 be the Jordan domain bounded by L_1, Γ_1, Γ_2 and L_3 and lying on the right-hand side of Γ_1 and Γ_2, Δ_k, $3\leq k\leq 5$, be the Jordan domain bounded by L_k, Γ_k and L_{k+1} and lying on the right-hand side of Γ_k, and Δ_6 be the Jordan domain bounded by L_6, Γ_6, Γ_7 and L_1 and lying on the right-hand side Γ_6 and Γ_7. In view of the local behavior of f_3 at each e^{it_k}, $k=1,3,4,5,6$, there exists a Jordan arc ℓ_k from e^{it_k} to infinity that maps under f_3 homeomorphically to L_k. Now let D_2 be the Jordan domain bounded by ℓ_1, γ_1, γ_2 and ℓ_3 and lying on the right-hand side of γ_1 and γ_2, D_k, $3\leq k\leq 5$, be the Jordan domain bounded by ℓ_k, γ_k and ℓ_{k+1} and lying on the right-hand side of γ_k, and let D_6 be the Jordan domain bounded by ℓ_6, γ_6, γ_7 and ℓ_1 and lying on the right-hand side of γ_6 and γ_7. Since f_3 is sense-preserving in $\mathbb{C}\setminus\mathbf{D}$, $\overline{\Delta_k}\subset f_3(\overline{D}_k)$ for each k.

It is immediate that there exists a Jordan convex domain K that contains $f_3(\overline{\mathbf{D}})$ and avoids the points w_k $1\leq k\leq 6$; in fact, K can be an open disk. Let B_k be the point of intersection of L_k with ∂K. Considering the argument of the tangent vector to each Γ_k at A_k, $k=1,3,4,5,6$, and the locations of the points A_k, we conclude that the points B_k appear on the positively-directed ∂K in the order B_1, B_3, B_5, B_4, B_6. Let $\Delta'_k=K\cap\Delta_k$, $2\leq k\leq 6$. Also, let β_k be the common boundary arc of Δ'_k and K; β_k has endpoints B_1 and B_3 if $k=2$, B_k and B_{k+1} if $k=3,4,5$, and B_6 and B_1 if $k=6$. Evidently, each Δ'_k is a Jordan domain that contains none of the points w_k, and satisfies $\overline{\Delta_k}\subset f_3(\overline{D}_k)$. Hence, by the Monodromy theorem, there exists a Jordan domain D'_k in D_k that maps under f_3 homeomorphically to Δ'_k. Observe that each D'_k can be chosen so that its boundary contains e^{it_k}. A homotopy argument then implies that D'_k is a Jordan domain bounded in case $k=2$ by ℓ_1, γ_1, γ_2, ℓ_3 and a Jordan arc α_2 starting from an interior point b_1 of ℓ_1 and ending in an interior point b_3 of ℓ_3 and lying otherwise in D_2, in case $k=3,4,5$, by ℓ_k, γ_k, ℓ_{k+1} and a Jordan arc α_k starting from an interior point b_k of ℓ_k and ending in an interior point b_{k+1} of ℓ_{k+1} and lying otherwise in D_k, and in case $k=6$ by ℓ_6, γ_6, γ_7, ℓ_1, and a Jordan arc α_6 starting from an interior point b_6 of ℓ_6 and ending at b_1 and lying otherwise in D_6. It is immediate that each $f_3:\overline{D'_k}\to\overline{\Delta'_k}$ is a homeomorphism that maps α_k to β_k in a sense-preserving manner, and that $\mathcal{D}=\mathbf{D}\bigcup(\cup_{k=2}^{6}\overline{D'_k})$ is the Jordan domain bounded by the Jordan arc-product $\alpha=\alpha_2\alpha_3\alpha_4\alpha_5\alpha_6$. Let $\beta=\beta_2\beta_3\beta_4\beta_5\beta_6$, and observe that β is a $2-$fold $1-$dimensional covering of ∂K with the identity as the projection map. It follows that as z traverses α positively once starting from b_1, $f_3(z)$ traverses ∂K positively twice starting from B_1. It follows that f_3 is a $2-$valent sense-preserving local homeomorphism from $\partial\mathcal{D}$ to ∂K. Since each Δ'_k, $2\leq k\leq 6$, contains the domain Q, f_3 of $\mathcal{D}\setminus\overline{\mathbf{D}}$ assumes each value in Q exactly 5 times, a conclusion that could also be drawn by the Argument principle. Recall that f_3 of $\mathbf{D}$ assumes each value in Q three times. Hence, f_3 of $\mathcal{D}$ assumes each value in Q exactly 8 times. Precomposing f_3 with a conformal map from the open unit disc to $\mathcal{D}$ yields at once the desired function

F_3. The image surface of F_3 can be obtained by considering the bordered surface $\overline{\Omega}$ along with the closed Jordan domains $\overline{\Delta'_k}$, $2 \leq k \leq 6$, with the identity map viewed also as bordered covering surfaces of $\mathbb{C}$, then identifying these surfaces crosswise along the boundary arcs with the same projection; see Figure 4(c).

Remark 2. A surgery as in Remark 1 on the image surface of F_2 over a local neighborhood of the vertex A_1 (associated with F_2) of the cusp yields essentially the image surface of F_3.

The geometric procedure described in Remarks 1 and 2 can be carried out indefinitely, however f_2 and f_3 pose some analytical concern in view of their poles and branch points. despite of this, it is believed that these points can always be dealt with suitably.

We conclude the paper with the following conjecture.

Conjecture. For every positive integer N there exists a $p-$valent, $p \geq N$, harmonic map f of the unit disc $\mathbf{D}$ onto a bounded convex domain $\mathbf{K}$ that extends to the unit circle as a $2-$valent local homeomorphism onto ∂K.

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References

[1] D. Bshouty, W. Hengartner and T. Suez, *The exact bound on the number of zeros of harmonic polynomials*, J. Analyse. Math. **67** (1995), 207-218.

[2] G. Choquet, *Sur un type de transformation analytique généralisant la représentation conforme et définie au moyen de fonctions harmoniques*, Bull. Sci. Math **69** (1945), 156-165.

[3] P. Duren, W. Hengartner, *Harmonic mappings of multiply connected domains*, Pacific J. Math., **180** (1997), 201-220.

[4] H. Kneser, *Lösung der Aufgabe 41*, Jahresber. Deutsch. Math.-Verein **35** (1926), 123-124.

[5] A. Lyzzaik, *On the valence of some classes of harmonic maps*, Math. Proc. Cambridge Phil. Soc. **110** (1990), 313-325.

[6] A. Lyzzaik, *Univalence criteria for harmonic mappings in multiply-connected domains*, J. London Math. Soc., (2) 58 (1998), 163-171.

[7] A. Lyzzaik, *A note on the valency of harmonic maps*, J. Math. Anal. Appl. **218** (1998), 611-620.

[8] A. Lyzzaik, *Local properties of light harmonic mappings*, Can. J. Math. Vol. **44** (1992), 135-153.

[9] R. Peretz, J. Schmid, *The zero set of certain complex polynomials*, Proceedings of the Ashkelon Workshop on Complex Function Theory (1996), 203-208.

[10] T. Radó, *Aufgabe 41*, Jahresber. Deutsch. Math. Verein. **35** (1926), 49.

[11] T. Sheil-Small, *On the Fourier series of a finitely described curve and a conjecture of H. S. Shapiro*, Math. Proc. Cambridge Phil. Soc. **98** (1985), 513-527.

[12] T. Sheil-Small, *Tagesberichte*, Mathematisches Forschungsinstitut Oberwolfach, Funktionentheorie, 16 bis 22.2.1992 (1992), 19.

[13] A. Wilmshurst, *The valence of harmonic polynomials*, Proc. Amer. Math. Soc., **126** (1998), 2077-2081.

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