Contemporary Mathematics

A NOTE ON THE PARABOLICITY OF MINIMAL GRAPHS

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ABSTRACT. We show that minimal graphs over finitely connected domains are parabolic.

1. Introduction

Let M be a noncompact Riemann surface with nonempty boundary β (cf. [AS; p. 117]). There are many equivalent definitions of parabolicity of M (cf. [CKMR], [LP], [MP], [P]); for example, M is parabolic if bounded harmonic functions on M are determined by their boundary values on β. If M were the right half plane with β the imaginary axis, then M would be parabolic, whereas if β were the positive imaginary axis it would not be.

In §1- §3 we shall confine ourselves to smooth boundaries as has been traditionally done in the study of parabolicity of minimal surfaces with boundary. We shall refer to this as the "classical sense".

In §4 we shall extend the results by giving a variant more suitable for applications to general boundary value problems.

A convenient way to define parabolicity of M is in terms of its harmonic measure $\omega(x,E)$, $x \in M$, $E \subset \beta$ (cf. [LP], [P]). For an interval $I \subset \beta$ and an open set $\Omega \subset M$ having compact closure in $M \cup \beta$, define $\omega_\Omega(x,I)$ to be the bounded solution to

$$\Delta\omega_\Omega(x,I) = 0 \quad x \in \Omega,$$

$$\omega_\Omega(x,I) = 1 \quad x \in \beta \cap I^o,$$

$$\omega_\Omega(x,I) = 0 \quad x \in \beta \backslash \overline{I}.$$

Then, if Ω ranges over an expanding sequence $\{\Omega_k\}$ such that $\cup\Omega_k = M$, the corresponding ω_{Ω_k} converge to a harmonic limit $\omega(x,I)$. If $\omega(x,\beta) \equiv 1$, then $\omega(x,I)$ serves as harmonic measure. See [P; §1] for further details. In particular M is parabolic if $\omega(x,\beta) \equiv 1$.

In [CKMR] it is proved that if M is a connected properly immersed minimal surface in the upper half space $x_3 \geq 0$ of $\mathbb{R}^3$, then M is parabolic; in particular, if M is a proper minimal graph in $\mathbb{R}^3$ which is bounded from below, it is parabolic.

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In [LP] it is proved that if M is a proper minimal graph lying above a vertical negative half catenoid then M is parabolic. As usual, proper here means that the inclusion map of the surface in $\mathbb{R}^3$ is proper, that is, the inverse image of a compact set is compact.

It has been conjectured by Meeks that proper minimal graphs are parabolic in general. We have the following

THEOREM 1.1. *Let M be a proper surface with nonempty boundary β given by the graph of a solution u to the minimal surface equation over a simply connected plane domain D. If ω is as defined above, then $\omega(x,\beta) \equiv 1$ on M.*

We then have

COROLLARY 1.1. *If M is a proper minimal graph in $\mathbb{R}^3$ over a simply connected domain D, then M is parabolic.*

2. Proof of Theorem 1.1

PROOF. We parametrize M with isothermal coordinate ζ in the unit disk Δ. Specifically, then M is given by $(f(\zeta), U(\zeta)$, where $f : \Delta \to D$ is a univalent planar harmonic mapping and $U(\zeta) = u(f(\zeta))$.

In the Weierstrass representation we may write

$$f(\zeta) = h(\zeta) + \overline{g(\zeta)}, \tag{2.1}$$

where g and h are analytic in Δ and

$$|h'(\zeta)| > |g'(\zeta)| \quad \zeta \in \Delta. \tag{2.2}$$

The height function $U(\zeta)$ can then be given by

$$U(\zeta) = \Re e 2i \int \sqrt{h'(\zeta)g'(\zeta)} d\zeta. \tag{2.3}$$

Now if $\omega(z,\beta)$ is a bounded harmonic function on M with $\omega(z,\beta) \equiv 1$ on β, and

$$\mu(\zeta) = \omega(f(\zeta), \beta), \tag{2.4}$$

then μ is harmonic in Δ. Since μ is bounded, it has radial limits a.e. on $\partial\Delta$. We wish to show that for a.e. $\theta \in [0, 2\pi)$,

$$\lim_{r \to 1^-} \mu(re^{i\theta}) = 1, \tag{2.5}$$

from which it will follow that $\mu(\zeta) \equiv 1$ in Δ.

Let T be the set of θ for which

$$\lim_{r \to 1^-} \mu(re^{i\theta}) = \alpha = \alpha(\theta) \neq 1. \tag{2.6}$$

In order to analyze the boundary behavior of μ, we must take a closer look at $f(\zeta)$ and $U(\zeta)$.

In [AL; p. 2], Abu-Muhanna and Lyzzaik showed that with h the analytic part of a univalent harmonic mapping as in (2.1),

$$\left|\frac{h''(\zeta)}{h'(\zeta)}\right| \leq \frac{c}{1 - |\zeta|} \quad \zeta \in \Delta \tag{2.7}$$

for some constant c, that is, $\log h'$ is a Bloch function. This together with (2.2) and a result of Clunie and MacGregor [CM] (cf. also [Mak]) imply that for almost every θ,

$$\int_0^1 |h'(re^{i\theta})|dr < \infty \quad \text{and} \quad \int_0^1 |g'(re^{i\theta})|dr < \infty. \tag{2.8}$$

Therefore, h and g and hence f have finite radial limits a.e. on $\partial\Delta$. By (2.3) it also follows that U also has finite radial limits a.e.

Let $r_n \to 1^-$, $\zeta_n = r_n e^{i\theta}$, and suppose that z^* is an accumulation point of $\{f(\zeta_n)\}$. Since f is a homeomorphism, $z^* \notin D$, so either $z^* \in \beta$ or $\{U(\zeta_n)\}$ is unbounded. In the first case, from (2.4) and (2.6) it follows that $\theta \notin T$, and in the latter case, the set of all such θ has measure 0.

The only other possibility is that $f(\zeta_n) \to \infty$ as $\zeta_n \to e^{i\theta}$. However, again the set of such θ has measure 0. Thus, the set T for which (2.6) holds has measure 0. This means that (2.5) holds a.e. and $\mu \equiv 1$. □

3. Finitely connected domains

It is easy to extend the results of §1 to domains which are finitely connected, that is domains having finitely generated fundamental group.

THEOREM 3.1. *Let M be a proper surface with nonempty boundary which is the graph of a solution $u : \mathbb{R}^2 \to \mathbb{R}$ of the minimal surface equation, and D is finitely connected. Then M is parabolic in the classical sense.*

PROOF. By hypothesis ∂D has a finite number of compact Jordan curve components $\gamma_1, \gamma_2, ..., \gamma_n$. Suppose first that $\partial D \setminus \cup_{k=1}^n \gamma_k \neq \phi$. Then we introduce disjoint arcs β_j, $j = 1, ..., n$ joining γ_j with $\partial D \setminus \cup_{k=1}^n \gamma_k$. We may take a neighborhood N of $(\cup_{k=1}^n \gamma_k) \cup (\cup_{k=1}^n \beta_k)$ with smooth boundary such that $\overline{N}$ is compact and $D' = D \setminus \overline{N}$ is simply connected. If M' is the portion of M above D', then by Theorem 1.1, the closure of M' in M is parabolic. Thus [P; p. 167], M is parabolic.

Otherwise, $\partial D = \cup_{k=1}^n \gamma_k$. Take R large enough so that $\cup_{k=1}^n \gamma_k$ is contained in the disk D_R of radius R centered at the origin. Let Γ_R be the circle of radius R and ω_R denote the harmonic function on the portion Δ_R of M above $D \cap D_R$ which is 1 on the portion of M corresponding to $\cup_{k=1}^n \gamma_k$, and 0 on the portion above Γ_R. Then, $|\nabla \omega_R|$ gives the extremal for the modulus of the family of curves in Δ_R joining $\cup_{k=1}^n \gamma_k$ to Γ_R [M; p. 253]. However, [M; p. 258] the moduli tend to 0 as $R \to \infty$. This implies that $|\nabla \omega_R| \to 0$ which establishes the parabolicity of M in this case as well. □

4. Extensions

As remarked in §1, the traditional setting for the study of parabolicity is very restrictive. Another definition has been proposed by Fang and Hwang for their study [FH]. Let M be the graph of a solution to the minimal surface equation over a domain D and $\{x_n\}$ a sequence of points in M which is a Cauchy sequence in the surface metric. Two Cauchy sequences $\{x_n\}$ and $\{y_n\}$ are considered equivalent if $d(x_n, y_n) \to 0$ where d is the distance on the surface given by the surface metric. Then the boundary γ will consist of those equivalence classes of Cauchy sequences which do not converge to points in M. Then the distance function d can be extended to $M \cup \gamma$ in the obvious way. With this, the definition given in [FH] is as follows.

Definition. With M and γ as above, $M \cup \gamma$ is parabolic if there are no nonconstant nonnegative bounded subharmonic functions vanishing on γ.

It is easy to see that a surface which is parabolic in the sense of §1 is parabolic in the current sense.

We next extend the Corollary of §1 to the current setting, and allow D to be finitely connected. This first requires a restating the result of Clunie and MacGregor used in §1 in slightly more generality.

THEOREM A. Let h be a locally analytic univalent function in the unit disk Δ such that for some constant C,

$$\left|\frac{h''(z)}{h'(z)}\right| \le \frac{C}{1-|z|} \quad z \in \Delta. \tag{4.1}$$

Then, for any $0 < \beta < \pi$ and $\delta > 1/2$,

$$\lim_{z \to e^{i\theta} \ z \in S_\beta(\theta)} \frac{\log|h'(z)|}{(\log(1/(1-|z|)^\delta} = 0 \tag{4.2}$$

for almost all θ, where $S_\beta(\theta)$ is the Stolz sector of opening β at $e^{i\theta}$.

We shall indicate the necessary modifications for the proof of Theorem A in §5.

THEOREM 4.1. *Let M be a minimal graph over a domain D which is finitely connected. If γ is defined as above, then $M \cup \gamma$ is parabolic.*

PROOF. We first dismiss point components in the complement of D. In fact points are removable for solutions to the minimal surface equation [B], so any bounded subharmonic function in a punctured neighborhood of the point has a removable singularity. Thus we may assume that the components of the complement of D are nondegenerate continua.

Now M is homeomorphic to the plane region D so it is schlichtartig [S; p. 219] and hence there exists a conformal mapping $F : \tilde{\Delta} \to M$ [S; p. 224] where $\tilde{\Delta}$ may be taken as $\hat{\mathbb{C}} \setminus \cup_{j=1}^n \Delta_j$ and the $\{\Delta_j\}$ are a finite collection of disjoint open disks (cf. [T; p. 424]).

Corresponding to the $\Delta_j = \{z : |z - z_j| < r_j\}$ $j = 1, ..., n$, let $\mathbb{A}_j = \{z : r_j < |z - z_j| < \rho_j\}$ be disjoint annuli, and $\mathbb{A}_j^R = \{z \in \mathbb{A}_j : -3\pi/4 < \arg(z - z_j) < 3\pi/4\}$ and $\mathbb{A}_j^L = \{z \in \mathbb{A}_j : \pi/2 < \arg(z - z_j) < 3\pi/2\}$. We can map the $\mathbb{A}_j^R$ and $\mathbb{A}_j^L$ one to one conformally onto the unit disk and apply the procedure of §1. Consider for example $\mathbb{A}_j^R$, and let ϕ denote a 1-1 conformal map of $\mathbb{A}_j^R$ onto the unit disk Δ. If we write $F(z) = (f(z), U(z))$ where f is a univalent harmonic mapping and restrict it to $\mathbb{A}_j^R$, then we may write $f(\phi^{-1}(\zeta)) = h(\zeta) + \overline{g(\zeta)}$, $\zeta \in \Delta$, and (2.7) holds as before.

By Theorem A, for a given $0 < \beta < \pi$ and $\delta > 1/2$, then (4.2) holds for h, and by (2.2) also for g. Since ϕ maps the open arc l_j of $|z - z_j| = r_j$ in $\partial\mathbb{A}_j^R$ onto an arc of $\partial\Delta$, it extends analytically across the arcs. Transferring the statement (4.2) and its counterpart for g over to l_j, and letting $T_\beta(\theta)$ denote the "exterior Stolz angle", we obtain

$$\lim_{\substack{z \to z_j + r_j e^{i\theta} \\ z \in T_\beta(\theta)}} \frac{\log|h'(z)|}{(\log(1/(|z_j + r_j e^{i\theta} - z|)^\delta} = 0, \quad \lim_{\substack{z \to z_j + r_j e^{i\theta} \\ z \in T_\beta(\theta)}} \frac{\log|g'(z)|}{(\log(1/(|z_j + r_j e^{i\theta} - z|)^\delta} = 0 \tag{4.3}$$

outside a set of one dimensional Hausdorff measure 0.

Applying this to each $\mathbb{A}_j^R$ and $\mathbb{A}_j^L$ we obtain that (4.3) holds for each $\partial\Delta_j,\ j = 1, ..., n$ outside a set of one dimensional Hausdorff measure 0.

Let ψ be a proper n to 1 mapping of $\tilde{\Delta}$ onto the unit disk Δ as in [T; p. 418]. By (4.3) and again by conformality at the boundary, if we take any fixed branch in a neighborhood of $\zeta \in \partial\Delta$, then outside a set of one dimensional Hausdorff measure 0, $h \circ \psi^{-1}$ and $g \circ \psi^{-1}$ satisfy (4.2). It follows from this that in a neighborhood of every $e^{i\theta} \in \partial\Delta$, any branches of $f \circ \psi^{-1}$ and $U \circ \psi^{-1}$ have finite radial limits a.e.

Let v be a function which is bounded and subharmonic on M, and vanishes on γ. Define the subharmonic function μ by

$$\mu(w) = \max_{\psi(\zeta)=w} v \circ F(\zeta) \quad w \in \Delta.$$

Since ψ is proper and each branch of $F \circ \psi^{-1} (= (f \circ \psi^{-1}, U \circ \psi^{-1}))$ has finite radial limit a.e., it follows that for $0 < \theta \le 2\pi$,

$$\lim_{r \to 1^-} \mu(re^{i\theta}) = 0 \quad \text{a.e.} \tag{4.4}$$

Now, in any disk $\Delta_\rho = \{|w| \le \rho < R < 1\}$,

$$\mu(w) \le \frac{1}{2\pi} \int_0^{2\pi} \frac{(R^2 - r^2)\mu(Re^{i\theta})\, d\theta}{R^2 - 2Rr\cos(\theta - \varphi) + r^2}, \qquad (w = re^{i\varphi}). \tag{4.5}$$

Fixing ρ and letting $R \to 1^-$, then (4.4) and (4.5) imply that $\mu(w) \le 0$ in Δ_ρ . Since $\rho < 1$ was arbitrary, the inequality holds in Δ. This immediately implies that $v \le 0$ on M. □

5. The Clunie-MacGregor theorem

Theorem 3 of [CM] is proved for univalent functions and radial limits. We indicate the modifications needed to carry this over to locally univalent functions satisfying (4.1) and with (4.2) holding in Stolz angles. The result in [CM] of interest is Theorem 3. One step in the proof of Theorem 3 relies on Theorem 1 of [CM], but the proof of Theorem 1 goes over without change using only (4.1). The fact that local univalence and (4.1) are only needed in these proofs was mentioned already in [Mac]. With the hypothesis of local univalence on an analytic function h with (4.1) this gives, for given $\lambda > 0$,

$$\int_0^{2\pi} |\log|h'(re^{i\theta})||^\lambda d\theta \le C(\log(1/(1-r))^{\lambda/2} \quad 0 \le r < 1. \tag{5.1}$$

Using (4.1), the proof of Theorem 3 goes through yielding (4.2) for radial limits instead of limits in Stolz angles. However, suppose that $e^{i\theta} \in \partial\Delta$ is a point for which (4.2) holds in the radial sense. Given $0 < \beta < \pi$, let $\Gamma_\beta(r)$ be the circular arc $\{re^{it} : \theta - \beta(1-r) < t < \theta + \beta(1-r)\}$. For $\zeta \in \Gamma_\beta(r)$, if $\Gamma_\beta(r, \zeta)$ denotes the subarc of $\Gamma_\beta(r)$ from $re^{i\theta}$ to ζ, then by (4.1)

$$|\log|h'(\zeta)| - \log|h'(re^{i\theta})|| \le \int_{\Gamma_\beta(r,\zeta)} \left| \frac{h''(re^{it})}{h'(re^{it})} \right| r\, dt \le \frac{C\beta(1-r)}{1-r} = C\beta,$$

so (4.2) for the radial point implies the same for all $\zeta \in \Gamma_\beta(r)$. Thus (4.2) holds in Stolz angles for a.e. θ.

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