SPIRALING MINIMAL GRAPHS

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ABSTRACT. We consider minimal graphs $u = u(x, y) > 0$ over unbounded spiraling domains D with $u = 0$ on ∂D. We show that such surfaces do exist, but only if the rate of spiraling is restricted. Restrictions are obtained through the method of extremal length of path families, and constructions are achieved by means of quasiconformal mappings.

1. Introduction. Let D be an unbounded domain and $u(x, y)$ a positive solution to the minimal surface equation with vanishing boundary values

$$\text{div}\frac{\nabla u}{\sqrt{1+|\nabla u|^2}} = 0, \qquad u > 0 \qquad \text{in } D, \tag{1.1}$$
$$u = 0 \quad \text{on } \partial D.$$

It follows from the maximum principle that D must be unbounded, but even here there are further obstructions. In fact, as observed by Nitsche [N; p.256], if D is contained in a sector of opening less than π, then (1.1) has no solution.

In this paper we shall examine the obstructions due to spiraling of D. To be precise, we consider D unbounded and simply connected with ∂D a piecewise differentiable Jordan arc. Then D will be a *spiraling domain* and its graph F given by (1.1) a *spiraling minimal graph*, if ∂D contains a subarc β tending to ∞ on which, for a branch of $\arg z$ on β, we have

$$\arg_{z\in\beta} z \to +\infty \quad \text{as } z \to \infty. \tag{1.2}$$

Here we are using complex notation $z = x + iy$ for points in D and β for convenience. If $z(t)$ $-\infty < t < \infty$ is a parametrization of ∂D, then there exists a branch of $\arg z(t)$ which is unbounded in at least one direction, that is, as $t \to -\infty$ or $t \to \infty$. Of course $+\infty$ could be replaced in (1.2) by $-\infty$.

Spiraling minimal graphs will be constructed in §4. However, an interesting question is whether there are obstructions due to the rate of spiraling of D. In order to quantify this we shall define the *order of a spiral* β satisfying (1.2) by

$$\sigma(\beta) = \lim_{z\to\infty\ z\in\beta} \frac{\arg z}{\log|z|}. \tag{1.3}$$

We shall prove

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Theorem 1. *There exists a universal constant σ_0 such that if D is a spiraling domain with β as in (1.2) and u satisfying (1.1), then $\sigma(\beta) \leq \sigma_0$*

It would be interesting to know the best possible constant σ_0 and in particular if $\sigma(\beta)$ in (1.3) can be positive.

2. Modulus of a path family. Let D be a simply connected unbounded domain in $\mathbb{R}^2$. Let F denote the surface given by $u(z)$, $z = x_1 + ix_2 \in D$ with

$$ds_F^2 = (1 + u_{x_1}^2)dx_1^2 + 2u_{x_1}u_{x_2}dx_1dx_2 + (1 + u_{x_2}^2)dx_2^2$$

and

$$dS_F = \sqrt{1 + |\nabla u|^2}\, dx_1dx_2$$

the respective length and area elements for F.

For a family Γ of curves in D we define the *modulus* of Γ in the metric of F by

$$\mathrm{mod}_F\Gamma = \inf \iint_D \rho^2(z)dS_F,$$

the inf being taken over all nonnegative measurable functions ρ on D satisfying

$$\inf_{\gamma\in\Gamma} \int_\gamma \rho(z)ds_F \geq 1.$$

The utility of the modulus comes from the elementary observation that it is a conformal invariant (cf. [M2; p.65]).

We shall use estimates on the modulus for path families of curves on a surface F given by solutions $u(z)$ to the minimal surface equation over domains D as in (1.1).

With D as in Theorem 1, we introduce a complex isothermal coordinate ζ for the surface F given by u over D so that the map $\zeta \to (x_1(\zeta), x_2(\zeta), u(x_1(\zeta), x_2(\zeta)))$ is a conformal mapping onto F. We take the parameter space as the upper half plane $\mathcal{H} = \{\zeta : \Im m\, \zeta > 0\}$ with specified positively oriented points $a, b \in \partial D$ corresponding by $(x_1(\zeta), x_2(\zeta))$ to $(0,0)$, $(0,1)$ respectively, and $\infty \to \infty$. The mapping $f(\zeta) = x_1(\zeta) + ix_2(\zeta)$ is then a univalent harmonic mapping of $\mathcal{H}$ onto D.

Path families Γ in $\mathcal{H}$ correspond to path families on F which project to path families $f(\Gamma)$ in D. By conformal invariance, the modulus may be computed either in $\mathcal{H}$ with the plane metric, or with the surface metric in D.

When expressed in the coordinates of $\mathcal{H}$, then $u = u(\zeta)$ is harmonic. With the special conditions here that $u = 0$ on $\partial\mathcal{H}$, u reflects to a harmonic function in the entire plane; since $u > 0$ in $\mathcal{H}$, it must be that u is of the form $c\,\Im m\,\zeta$ for some real constant $c > 0$.

In §3 we shall use an ingenious method developed by V. Mikljukov in [M1] and [M2]. (See also [M3; Chapter 9]).

3. Proof of Theorem 1. We first need the following elementary lemma.

Lemma 1. *Suppose that u and D are as in Theorem 1 and $\sigma(\beta) \geq 1$. Then there exists an absolute constant A such that*

$$\limsup_{z\to\infty \;\; z\in D} u(z)/|z| \leq A \tag{3.1}$$

Proof. Let $\tau = e^{4\pi}$. By [JS; Theorem 4] there exists a function $V_0(z)$ which has the value $V_0(z) = 0$ for $z = \tau e^{i\theta}$ $(0 < \theta < 2\pi)$, the value $+\infty$ on the portion $I = (1, \tau)$ of the real axis, and satisfies the minimal surface equation in $\Delta = \{|z| < \tau\}\backslash \overline{I}$. The function $V_n(z) = \tau^n V_0(\tau^{-n} z)$ satisfies the minimal surface equation in the scaled domains $\Delta_n = \tau^n \Delta$. Then V_n also has the corresponding boundary values, and

$$V_n(z) \leq C\tau^n \quad |z| \leq \tau^{n-1}, \tag{3.2}$$

where C is a constant independent of n.

From the hypotheses, the variation of $\arg z$ on the chosen arc satisfies $\arg z \backsimeq 4\pi n\sigma(\beta)$ where the arc intersects $|z| = \tau^n$, and $\arg z \backsimeq 4\pi(n+1)\sigma(\beta)$ where $|z| = \tau^{n+1}$. Thus, by the maximum principle (cf. [JS; p. 325]), for n large, $u(z) < V_n(z)$ in the portion of D inside $|z| \leq \tau^n$, and by (3.2) we then have

$$u(z) < C\tau^n \quad z \in D,\ |z| = \tau^{n-1}, \quad n \geq N. \tag{3.3}$$

This proves (3.1). $\square$

Returning to the proof of Theorem 1, we may assume again that $\sigma(\beta) \geq 1$. Fix a and b as in §2 and for sufficiently large $t > 0$ let $S(t)$ be the component of $D \cap \{z : |z| = t\}$ separating a from ∞ in D. Choosing r large enough so $S(r)$ separates b from ∞ in D, and $R > r$, let T be the subdomain of D between $S(r)$ and $S(R)$. Let $\Gamma = \Gamma(r, R)$ be the family of curves in T that join $S(r)$ and $S(R)$.

Following [M2], we define a density function

$$\rho(z) = (|z|^2 + u^2(z))^{-1/2},$$

for $z \in B = \overline{T} \cap \{r \leq |z| \leq R)\}$ and $\rho(z) = 0$ for all the remaining values $z \in D$. Hence

$$\mathrm{mod}_F\Gamma \leq \frac{\displaystyle\iint_B (|z|^2 + u^2(z))^{-1} dS_F}{\left(\inf\limits_{\gamma\in\Gamma} \int_\gamma (|z|^2 + u^2(z))^{-1/2} ds_F\right)^2}. \tag{3.4}$$

A general bound for the numerator in (3.4) has been given in [M2] (see also [W; pp. 622-623])

$$\iint_B \frac{dS_F}{|z|^2 + u^2(z)} \leq \frac{\pi}{2} \iint_B (1 + o(1)) \frac{dx_1 dx_2}{|z|^2} + O(1) \quad (R \to \infty),$$

from which we obtain

$$\iint_B \frac{dS_F}{|z|^2+u^2(z)} \le \pi^2(1+o(1))\log R \quad (R\to\infty). \tag{3.5}$$

Let $\tau = e^{4\pi}$ as in the proof of Lemma 1. For the denominator in (3.4), let γ be a curve in Γ and γ_n be the portion of γ whose initial point is the last point at which $|z| = \tau^n$ and terminal point is the first point at which $|z| = \tau^{n+1}$. Now, as in Lemma 1, the variation of $\arg z$ on γ_n is (asymptotically) at least $4\pi\sigma(\beta)$.

Thus, for sufficiently large n, γ_n encircles the origin at least $[\sigma]$ times, where $[\sigma]\ge 1$ is the greatest integer in $\sigma(\beta)$. Therefore, if l_n is the (Euclidean) length of γ_n, we have

$$l_n \ge 2\pi[\sigma]\tau^n \quad (n\ge N). \tag{3.6}$$

Let γ_n project up to $\tilde{\gamma}_n$ in F. Then, it follows from Lemma 1 and (3.6) that

$$\int_{\gamma_n} \rho\, ds_F = \int_{\tilde{\gamma}_n} \frac{|dx|}{\sqrt{x_1^2+x_2^2+x_3^2}} \ge \int_{\tilde{\gamma}_n} \frac{|dx|}{\sqrt{\tau^{2n+2}+A^2\tau^{2n+2}}}$$

$$\ge \int_{\gamma_n} \frac{|dz|}{\sqrt{1+A^2}\tau^{n+1}} \ge \frac{2\pi[\sigma]\tau^n}{\sqrt{1+A^2}\tau^{n+1}} = K[\sigma] \quad (n\ge N)$$

where K is an absolute constant.

Thus, with $R=\tau^{n+1}$, we obtain

$$\int_\gamma \rho\, ds_F \ge nK[\sigma](1+o(1)) = (1+o(1))\frac{K[\sigma]}{4\pi}\log R \quad (R\to\infty).$$

Using this with (3.5) in (3.4) we obtain

$$\operatorname{mod}_F\Gamma \le (1+o(1))\frac{16\pi^4}{K^2[\sigma]^2\log R}. \tag{3.7}$$

We now use the conformal invariance of the mapping $\mathcal{H}\to F$ as described in §2 together with (3.7). With $a,\ b,\ f(\zeta)$ as in §2, continuing with [M2; p.67] we take $r>0$ so that $S(r)$ separates b and ∞ in D. For $t\ge r$, let $S^*(t) = f^{-1}(S(t))$ so that $\overline{S^*(t)}$ has endpoints on $\partial\mathcal{H}$ in the ζ plane. Let $l(t)$ denote the Jordan curve formed by $\overline{S^*(t)}$ along with its reflection across $\partial\mathcal{H}$ and G the annular domain between $l(r)$ and $l(R)$. Let $\tilde{\Gamma}(r,R)$ be the family of curves separating $l(r)$ and $l(R)$ in G. Then since $l(r)$ and $l(R)$ separate 0 $(=f^{-1}(a))$ and 1 $(=f^{-1}(b))$ from ∞, the modulus (in the Euclidean metric) satisfies [LV; pp.32, 56 and 61 (2.10)]

$$\operatorname{mod}\tilde{\Gamma}(r,R) \le \frac{1}{2\pi}\log(16(P+1)), \tag{3.8}$$

where

$$P = \min_{\zeta \in l(R)} |\zeta|.$$

Now let $\Gamma^*(r,R)$ be the curves joining $l(r)$ and $l(R)$ in G. Then $\operatorname{mod}\Gamma^*(r,R) = 1/\operatorname{mod}\tilde{\Gamma}(r,R)$. This follows from conformal invariance and the fact that this is the case for a true annulus [A; pp. 12,13]. Therefore,

$$\operatorname{mod}\Gamma^*(r,R) \geq \frac{2\pi}{\log(16(m(R)+1))}. \tag{3.9}$$

Let

$$m(t) = \min_{\substack{|z|=t \\ z\in D}} |\zeta(z)|.$$

Then by (3.9), the symmetry principle [A; p.16], and conformal invariance,

$$\operatorname{mod}_F \Gamma(r,R) \geq \frac{\pi}{\log(16(m(R)+1))}. \tag{3.10}$$

Thus, (3.10) taken together with (3.7) yields

$$\log m(R) \geq (1+o(1))C[\sigma]^2 \log R \quad (R\to\infty), \tag{(3.11)}$$

where C is an absolute constant.

Let $s = m(R)$. Then (3.11) gives

$$R \leq s^{(1+o(1))/C[\sigma]^2} \quad (R\to\infty). \tag{3.12}$$

By the maximum principle and Lemma 1 we have

$$\max_{|\zeta|=s} u(f(\zeta)) \leq \max_{|z|=R \ z\in D} u(z) \leq A\,R,$$

which when combined with (3.12) gives

$$\max_{|\zeta|=s} u(f(\zeta)) \leq As^{(1+o(1))/C[\sigma]^2} \quad (s\to\infty).$$

Since $u(f(\zeta)) = c\Im m\zeta$ it must be that $C[\sigma]^2 \leq 1$ which completes the proof. $\square$

4. Example of spiraling minimal graph. The goal of this section is to prove the following

Theorem 2. *There exist spiraling domains D with corresponding solutions satisfying (1.1) over D.*

In order to construct the spiraling minimal graph, we shall construct a univalent harmonic function $F(\zeta)$ defined in the upper half plane $\mathcal{H}$, and mapping onto a spiraling domain. The function F can be written in the form $H(\zeta) + \overline{G(\zeta)}$ where H and G are analytic in $\mathcal{H}$, and then a minimal graph can be represented in parametric form $(\Re e F(\zeta), \Im m F(\zeta), 2\Im m \int \sqrt{H'(\zeta)G'(\zeta)} d\zeta)$, (see [Du; pp. 177-178]) as long as the $\sqrt{}$ is well defined. In the construction which follows, that will be the case.

We first construct a 1-1 conformal mapping $h(z)$ of the upper half plane $\mathcal{H}$ using an approximating quasiconformal mapping. To achieve this we use the following (cf. [D; Lemma 5.8])

Lemma A. *Let φ be quasiconformal in the plane such that $\varphi(0) = 0$, $\varphi(1) = 1$, $\varphi(\infty) = \infty$, and the dilatation*

$$\mu(z) = \varphi_{\overline{z}}(z)/\varphi_z(z)$$

satisfies

$$\int_0^{2\pi} |\mu(re^{i\theta})| \, d\theta \to 0 \qquad r \to \infty.$$

Then, in any fixed annulus $A_R = \{R^{-1} \le |z| \le R\}$ $(R > 1)$,

$$\frac{\varphi(tz)}{\varphi(t)} \to z$$

uniformly in $A(R)$ as $0 < t \to \infty$.

Using Lemma A and Lemma 1 we shall prove

Lemma 1. *There exist one to one conformal mappings $h(\zeta)$ mapping the upper half plane $\mathcal{H}$ onto spiraling domains D such that for $0 \le \theta \le \pi$,*

$$h(te^{i\theta}) = t^{3/2+o(1)} e^{i((3/2)\theta + \tau(t))}(1 + o(1)) \quad (\tau(t) \to \infty \text{ as } t \to \infty) \tag{4.1}$$

and

$$h'(te^{i\theta}) = (3/2)t^{1/2+o(1)} e^{i((1/2)\theta + \tau(t))}(1 + o(1)) \quad (\tau(t) \to \infty \text{ as } t \to \infty). \tag{4.2}$$

Proof. Let

$$\alpha(z) = \int_0^{|z|} \delta(s)/(s + c) ds \tag{4.3}$$

where $\delta(s) > 0$ is a continuous function which tends to 0 subject only to the condition that $\alpha(z) \to \infty$ as $z \to \infty$, and c is a positive constant to be determined later. Then for fixed $k > 1$,

$$\alpha(kt) - \alpha(t) = \delta(\tilde{t}) \int_t^{kt} 1/(s+c)\, ds \quad (t \le \tilde{t} \le kt),$$

so that

$$(\alpha(kt) - \alpha(t))/\alpha(t) = \frac{\delta(\tilde{t})}{\alpha(t)} \log \frac{kt+c}{t+c},$$

In particular

$$\alpha(kt)/\alpha(t) \to 1 \quad \text{as } t \to \infty. \tag{4.4}$$

Let $\varphi(z)$ be defined in $\mathcal{H}$ by

$$\varphi(z) = z^{3/2} e^{i\alpha(z)} \qquad z \in \mathcal{H},\ \ 0 < \arg z^{3/2} < 3\pi/2, \tag{4.5}$$

and $D_o = \varphi(\mathcal{H})$. Then D_o is a spiraling domain. Now let $\psi(w)$ be the 1-1 conformal mapping of $\mathcal{H}$ onto D_o so that $\psi(0) = 0$, $\psi(1) = \varphi(1)$, and $\psi(\infty) = \infty$.

Define $Q : \mathcal{H} \to \mathcal{H}$ by

$$Q(z) = \psi^{-1} \circ \varphi(z). \tag{4.6}$$

Then $Q(z)$ is quasiconformal in $\mathcal{H}$. In fact,

$$Q_{\overline{z}}(z) = \psi_w^{-1}(\varphi(z)) e^{i\alpha(z)} \cdot z^{3/2} \alpha_{\overline{z}}(z) i,$$

(4.7)

$$Q_z(z) = \psi_w^{-1}(\varphi(z)) e^{i\alpha(z)} \cdot z^{1/2}(z\alpha_z(z)i + 3/2) \quad 0 < \arg z^{1/2} < \pi/2.$$

Now, from (4.3),

$$\alpha_{\overline{z}}(z) = kz, \ \ \alpha_z(z) = k\overline{z}, \ \ k = \frac{\delta(|z|)}{2|z|(|z|+c)}. \tag{4.8}$$

By (4.7) and (4.8) it follows that the dilatation $\mu(z)$ of Q satisfies

$$\mu(z) = \frac{Q_{\overline{z}}(z)}{Q_z(z} = \frac{z\alpha_{\overline{z}} i}{z\alpha_z i + 3/2} = \frac{z^2 k i}{|z|^2 k i + 3/2}.$$

Thus, from (4.8) it follows that $|\mu(z)| < 1$ for sufficiently large c which we so fix,, and thus Q is quasiconformal in $\mathcal{H}$

Now, by reflection (cf. [LV;p.47]) Q can be extended as a quasiconformal mapping $\mathbb{C} \to \mathbb{C}$ which we continue to denote by Q. Furthermore, $\mu(z) \to 0$ as $z \to \infty$.

From Lemma A, we see that

$$|z|^{1-o(1)} < |Q(z)| < |z|^{1+o(1)} \quad (|z| \to \infty), \tag{4.9}$$

and Lemma A applies to the inverse Q^{-1} as well. Applying Lemma A to Q^{-1} we get for any fixed $R > 1$ and $0 < t < \infty$,

$$\varphi^{-1} \circ \psi(tz)) = c_t(z + \epsilon_t(z)) \qquad (c_t = \varphi^{-1}(\psi(t)) > 0),$$

or

$$\psi(tz) = \varphi(c_t(z + \epsilon_t(z))) = (c_t(z + \epsilon_t(z)))^{3/2} e^{i\alpha(c_t(z+\epsilon_t(z)))}$$

for z in $\mathcal{H} \cap \{\, 0 \le \arg z + \epsilon_t(z) \le \pi\} \cap A(R)$. Here and in the continuation, $\epsilon_t(z)$ will denote quantities having the property that $\epsilon_t(z) \to 0$ uniformly for $1/R \le |z| \le R$ as $t \to \infty$.

As in (4.9), it follows from Lemma A that

$$c_t = t^{1+o(1)} \quad (t \to \infty). \tag{4.10}$$

With the previous choice of $\sqrt{\,}$, we define $h(z)$ by

$$h(z) = e^{-3i\pi/4} \psi(e^{i\pi/4}\sqrt{z})^2$$

so that for $z \in \mathcal{H} \cap A(R)$ $(t > t_o)$,

$$h(tz) = (c_{\sqrt{t}}(\sqrt{z} + \epsilon_{\sqrt{t}}(\sqrt{z}))^3 e^{2i\alpha(c_{\sqrt{t}}(e^{i\pi/4}\sqrt{z} + \epsilon_{\sqrt{t}}(e^{i\pi/4}\sqrt{z})))},$$

which with (4.4) simplifies to

$$h(tz) = (c_{\sqrt{t}}\sqrt{z})^3 e^{2i\alpha(c_{\sqrt{t}}))}(1 + o(1)) \quad (t \to \infty,\ z \in \mathcal{H} \cap A(R)). \tag{4.11}$$

Notice that (4.11) can be extended to a larger region $\{-\pi/2 + \epsilon < \arg z < 3\pi/2 - \epsilon\} \cap A(2R)$ for small $\epsilon > 0$ and $t > t_o$ since h is defined in this extended region. Thus, for $t > t_o$, we may compute $dh(tz)/dz$ for points $|z| \ge 1$ in $\overline{\mathcal{H}} \cap A(R)$ by Cauchy's formula

$$\frac{d}{dz} h(tz) = \frac{1}{2\pi i} \int_C \frac{h(tw)}{(z-w)^2}\, dw, \tag{4.12}$$

where C is the circle $|w - z| = 1/2$.

Inserting (4.11) into (4.12) we obtain

$$\frac{d}{dz} h(tz) = (3/2) c_{\sqrt{t}}^3 z^{1/2} e^{2i\alpha(c_{\sqrt{t}})}(1 + o(1)) \quad (t \to \infty,\ z \in \mathcal{H} \cap A(R)). \tag{4.13}$$

Let $\zeta = te^{i\theta}$ with $0 \le \theta \le \pi$. The function $h(\zeta)$ has most of the properties of the desired conformal mapping. Specifically, since $dh(\zeta)/d\zeta = (1/t)dh(tz)/dz$, it follows from (4.10), (4.11) and (4.13) that for $\zeta \in \mathcal{H}$,

$$h(\zeta) = \frac{c^3_{\sqrt{t}}}{t^{3/2}}\, \zeta^{3/2} e^{2i\alpha(c_{\sqrt{t}})}(1 + o(1)), \quad (t \to \infty), \tag{4.14}$$

$$h'(\zeta) = (3/2)\frac{c^3_{\sqrt{t}}}{t^{3/2}}\, \zeta^{1/2} e^{2i\alpha(c_{\sqrt{t}})}(1 + o(1)), \quad (t \to \infty), \tag{4.15}$$

and

$$c_{\sqrt{t}} = t^{1/2+o(1)} \quad (t \to \infty). \tag{4.16}$$

Then (4.1) and (4.2) follow from (4.14), (4.15), and (4.16). □

Proof of Theorem 2. If we now Let h be as in Lemma 1 and define

$$f(\zeta) = h(\zeta) + \overline{g(\zeta)} \qquad \text{where } \ g(\zeta) = \int_1^\zeta 1/h'(w)\, dw, \tag{4.17}$$

then, from (4.2) it follows that

$$\left| \int_1^\zeta 1/h'(w)\, dw \right| = o(|\zeta|^{1/2+\epsilon}), \quad (\epsilon > 0,\ |\zeta| \to \infty). \tag{4.18}$$

so that from (4.1) and (4.18) we have

$$f(te^{i\theta}) = t^{3/2+o(1)} e^{i((3/2)\theta + \tau(t))}(1 + o(1)), \quad (t \to \infty). \tag{4.19}$$

We now show that f is univalent for $|\zeta|$ sufficiently large in $\mathcal{H}$. To see this, consider first points $x > R$ on the positive real axis for R sufficiently large. Then, from (4.2), (4.17), and (4.19) we have

$$\Re e f'(x)/f(x) = \frac{3}{2x^{(1+o(1))}} > 0 \qquad (x \to \infty)$$

so that for large x, $d/dx \log |f(x)| > 0$. A similar result holds for $x < 0$. Thus f is 1-1 on $\partial H \cap \{|x| > R\}$ for sufficiently large R. Also, it is clear from (4.19) that the two boundary spirals Γ_1 and Γ_2 in $f(\partial H \cap \{|x| > R\})$ are disjoint for R sufficiently large.

Since $|f(x)|$ is increasing on $\{|x| > R\}$ for large R, we may take a circular arc Γ_ρ in $f(\mathcal{H})$ connecting a point $z_1 = \rho e^{i\theta_1}$ on Γ_1 and $z_2 = \rho^{i\theta_2}$ on Γ_2. Without loss of generality, we assume that $\theta_1 < \theta_2 \le \theta_1 + 2\pi$ and may take a (unique) continuation of f^{-1} from z_1 to z_2 on Γ_ρ for sufficiently large ρ. Let γ_ρ denote the resulting curve.

Now from (4.17)-(4.19) we have that on γ_ρ

$$\frac{\partial}{\partial\theta}\arg f(te^{i\theta}) = \frac{\partial}{\partial\theta}\Im m\log f(te^{i\theta}) = \Im m\frac{\partial f/\partial\theta}{f} = \Re e\frac{(te^{i\theta}h'(te^{i\theta}) - te^{-i\theta}/\overline{h'(te^{i\theta})})}{f}$$

$$= \Re e\frac{(3/2)t^{3/2+o(1)}e^{i(3\theta/2+\tau(t))}(1+o(1)) - (2/3)t^{1/2+o(1)}e^{i(-\theta/2+\tau(t))}(1+o(1))}{t^{3/2+o(1)}e^{i(3\theta/2+\tau(t))}(1+o(1))}$$

$$= (3/2)t^{o(1)}(1+o(1)) \qquad (t\to\infty).$$

Thus, $\arg f$ is increasing on γ_ρ for large ρ. Since, by (4.19), the variation of the argument of f on γ_ρ will be at most $(3\pi/2)(1+o(1))$ it follows that f is 1-1 on γ_ρ.

Let $\mathcal{H}_R = \mathcal{H}\cap\{|\zeta| > R\}$. Then for large R, f is 1-1 on $\partial\mathcal{H}_R$, $f(\infty)=\infty$, and the Jacobian of f does not vanish in $\mathcal{H}_R$. Thus, f is 1-1 on H_R. If we replace $f(\zeta)$ by $F(\zeta) = H(\zeta) + \overline{G(\zeta)}$, where $H(\zeta) = h(\zeta + iR)$ and $G(\zeta) = g(\zeta + iR)$, then F maps $\mathcal{H}$ univalently onto a spiraling domain, and by means of F we can give a parametrization $X : \mathcal{H} \to S$ in conformal coordinates by $(\Re eF(\zeta), \Im mF(\zeta), 2\Im m\int\sqrt{H'(\zeta)G'(\zeta)}d\zeta)$, which from (4.16) we may write

$$X(\zeta) = (F(\zeta), \Im m2\zeta)$$

as desired. □

5. Concluding Remarks. In addition to finding the best constant σ_0 as mentioned in §1, it would be interesting to see how to replace the lim in (1.3) by lim sup. Also, it can be seen from the construction that the tangent plane to the surface tends to the horizontal at infinity. It would be interesting to know if this is dictated by the spiraling.

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