Carleman's method and solutions to the minimal surface equation

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Abstract

A general form of Carleman's method is given for harmonic functions on surfaces. This is applied to the surfaces given by minimal graphs and growth estimates for solutions to the minimal surface equation with vanishing boundary values are obtained.

1 Introduction

Let D be an unbounded simply connected domain in $\mathbf{R}^2$ and f a solution of the minimal surface equation

$$\sum_{i=1}^{2} \frac{\partial}{\partial x_i} \left(\frac{\nabla f}{\sqrt{1 + |\nabla f|^2}} \right) = 0 \tag{1.1}$$

in D with

$$f = 0 \quad \text{on } \partial D. \tag{1.2}$$

In this paper we shall prove some estimates (Theorems 1.1 and 1.3) for solutions to (1.1) and (1.2) and give some consequences. Estimates of this type were derived earlier in [6] and [7] for regions bounded by a single Jordan arc using the method of extremal length. In the present work, Sections 2 to 4 are devoted to developing Carleman's method [2] in a general form, which can then be applied to the surface of the graph given by $f(x)$. (See also Section 8.) Because of its generality, this portion of the paper may be of independent interest. Our application of Carleman's method enables us to treat minimal graphs over general simply connected domains. Corresponding estimates for general domains remain open.

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Theorem 1.1 *Let D be an unbounded simply connected domain in $\mathbf{R}^2$ and f a solution of the minimal surface equation* (1.1) *in D with* (1.2)*. If ∂D is C^2 and*

$$\liminf_{R\to\infty} R^2 \exp\left\{-4(\log R)^2 \Big/ \int_{D\cap\{1<|x|<R\}} \frac{dx_1dx_2}{|x|^2}\right\} = 0\,, \tag{1.3}$$

then $f \equiv 0$ in D.

If also, $f(x)/|x| \to 0$ as $|x| \to \infty$, and for some $\epsilon > 0$

$$\liminf_{R\to\infty} R^{2+\epsilon} \exp\left\{-2\pi(\log R)^2 \Big/ \int_{D\cap\{1<|x|<R\}} \frac{dx_1dx_2}{|x|^2}\right\} = 0\,, \tag{1.4}$$

then $f \equiv 0$ in D.

Regarding applications of Theorem 1.1, we consider first a conjecture of Meeks. In his talk at the Clay Mathematics Institute's Summer School on "The Global Theory of Minimal Surfaces" at MSRI in the summer of 2001, Meeks conjectured that there can be at most two such solutions over disjoint domains. In Section 6, we prove the following theorem.

Theorem 1.2 *There can be at most three disjoint simply connected domains D in $\mathbf{R}^2$ with f satisfying* (1.1) *and* (1.2) *in each domain, unless $f \equiv 0$ in at least one domain.*

In [9], Spruck proved the Meeks conjecture, but under stringent side conditions on the behavior of f. Earlier, Li and Wang [5] proved, without restrictions, that there could be at most 12 disjoint domains.

Note that Theorem 1.1 is purely geometrical, that is, it specifies conditions on domains under which there can be nontrivial solutions to (1.1) with (1.2). However, the same proof (see Section 7) gives information in terms of the growth as measured by

$$M(r, f) = \max_{|x|=r,\ x\in D} |f(x)|.$$

Theorem 1.3 *Let D be an unbounded simply connected domain in $\mathbf{R}^2$ and f a solution of the minimal surface equation* (1.1) *in D with* (1.2)*. If ∂D is C^2 and f satisfies*

$$\liminf_{R\to\infty} M(2R, f)^2 \exp\left\{-4(\log R)^2 \Big/ \int_{D\cap\{1<|x|<R\}} \frac{dx_1dx_2}{|x|^2}\right\} = 0\,, \tag{1.5}$$

then $f \equiv 0$ in D.

If also, $f(x)/|x| \to 0$ *as* $|x| \to \infty$*, and for some* $\epsilon > 0$

$$\liminf_{R\to\infty} M(2R,f)^{2+\epsilon} \exp\left\{ -2\pi(\log R)^2 \Bigg/ \int_{D\cap\{1<|x|<R\}} \frac{dx_1dx_2}{|x|^2} \right\} = 0\,, \tag{1.6}$$

then $f \equiv 0$ *in* D*.*

A consequence of Theorem 1.3 pertains to the *order* of f given by

$$\limsup_{|x|=r\to\infty,\ x\in D} \frac{\log M(r,f)}{\log r}.$$

and the *asymptotic angle* α of D defined by

$$\alpha = \limsup_{r\to\infty} \text{meas}_\theta(D\cap\{|x|=r\})$$

where $0 < \text{meas}_\theta \le 2\pi$ is the angular measure of the arc.

Theorem 1.4 *If* $f \not\equiv 0$ *satisfies* (1.1) *and* (1.2) *in a simply connected domain* D *which has asymptotic angle* $\alpha \ge \pi$*, then the order of* f *in* D *must be at least* π/α*.*

In particular, any nontrivial solution of (1.1) *with* (1.2) *in a simply connected domain must have order at least* $1/2$*.*

Theorem 1.4 was proved earlier [9], [10] under additional side conditions.

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2 Laplace-Beltrami equation

Let $D \subset \mathbf{R}^2$ be a domain. Let g_{ij} $(i,j=1,2)$ be measurable functions such that for any relatively compact set $Q \subset\subset D$ there are some constants $0 < \nu_1(Q) \le \nu_2(Q) < \infty$ such that a.e. on Q, for every $\xi \in \mathbf{R}^2$ we have

$$\nu_1(Q)\,|\xi|^2 \le \sum_{i,j=1}^{2} g_{ij}(x)\xi_i\xi_j \le \nu_2(Q)\,|\xi|^2\,. \tag{2.1}$$

Consider an abstract surface $F = (D, ds_F^2)$ with line element

$$ds_F^2 = \sum_{i,j=1}^{2} g_{ij}(x)\,dx_i dx_j \qquad g_{ij} = g_{ji}. \tag{2.2}$$

We put

$$g = \det(g_{ij}) = g_{11}g_{22} - g_{12}^2\,.$$

Then, from (2.1), it follows that $g > 0$ a.e. in D. The area element of F has the form

$$d\sigma_F = \sqrt{g}\,dx_1dx_2\,.$$

Next we set

$$(g^{ij}) = (g_{ij})^{-1}.$$

Then, from (2.1), we have

$$(1/\nu_2(Q))\,|\xi|^2 \le \sum_{i,j=1}^{2} g^{ij}(x)\xi_i\xi_j \le (1/\nu_1(Q))\,|\xi|^2\,. \tag{2.3}$$

The equation

$$L[\phi] = \frac{1}{\sqrt{g}}\sum_{i=1}^{2}\frac{\partial}{\partial x_i}\left(\sqrt{g}\sum_{j=1}^{2} g^{ij}\frac{\partial\phi}{\partial x_j}\right) = 0 \tag{2.4}$$

describes the *harmonic functions* ϕ on F and is called the *Laplace-Beltrami* equation [8, Chapter I, Section 1].

In addition to the usual scalar product $\langle\xi,\eta\rangle = \xi_1\eta_1 + \xi_2\eta_2$ and norm $|\xi| = \sqrt{\langle\xi,\xi\rangle}$, we use the notation

$$\langle\xi,\eta\rangle_F = \sum_{i,j=1}^{2} g^{ij}\xi_i\,\eta_j \tag{2.5}$$

and

$$|\xi|_F^2 = \sum_{i,j=1}^{2} g^{ij}\xi_i\,\xi_j\,.$$

We now define the generalized solutions of (2.4). Let B_F be the set of all $x \in D$ in which the matrix (g_{ij}) is not defined or does not satisfy (2.1). By $\mathcal{D} = \{D'\}$ we denote a set of the subdomains $D' \subset\subset D$ with rectifiable boundaries $\partial D'$ such that the linear measure $\mathrm{mes}_1(\partial D' \cap B_F) = 0$.

Let $\psi : \overline{D'} \to \mathbf{R}$ be a Lipschitz function for $D \in \mathcal{D}$ Then, it follows that $\nabla\psi = (\psi_{x_1}, \psi_{x_2})$ exists a.e. in D' [3, Theorem 3.1.6].

Let $x_1 = x_1(t),\ x_2 = x_2(t)\ \ (a \le t \le b)$ be a positively oriented parametrization of $\partial D'$. Let $\overline{n}$ be the outer normal vector, that is,

$$\langle\overline{n},\overline{v}\rangle = 0 \text{ for } \overline{v} = (dx_1/dt, dx_2/dt), \tag{2.6}$$

normalized in the metric of F, that is, $|\overline{n}|_F = 1$. Since $\partial D'$ is rectifiable, $\overline{n}$ exists a.e. on $\partial D'$. A locally Lipschitz function $\phi : D \to \mathbf{R}$ is called a *generalized solution of* (2.4) if for every subdomain D' of the class $\mathcal{D}$ and for every Lipschitz function $\psi : \overline{D'} \to \mathbf{R}$,

$$\int_{D'} \langle \nabla\phi, \nabla\psi \rangle_F \, d\sigma_F = \int_{\partial D'} \psi \langle \nabla\phi, \overline{n} \rangle_F ds_F \, . \tag{2.7}$$

3 Carleman type estimates

Let ϕ be a generalized solution of (2.4) in a simply connected domain D, with $\phi = 0$ on ∂D. For our purposes, we may assume that the components of ∂D are C^2 Jordan arcs and that ϕ and the g_{ij} are all continuous in $\overline{D}$. By standard elliptic theory [4, p. 179], then ϕ satisfies a maximum principle so that D must be unbounded unless $\phi \equiv 0$.

Also, in our applications we will have $\nu_1(Q) \geq 1$ for all Q so that half of (2.1) and (2.3) can be written

$$\sum_{i,j=1}^{2} g_{ij}(x)\xi_i\xi_j \geq |\xi|^2, \qquad \sum_{i,j=1}^{2} g^{ij}(x)\xi_i\xi_j \leq |\xi|^2 \, . \tag{3.1}$$

Let B be a bounded domain in D and A an open subset of B such that the following conditions are satisfied:

a) Every connected component of A is simply connected which is bounded by arcs of ∂D and C^2 crosscut arcs in D,

b) B is bounded by arcs of ∂D, C^2 closed curves disjoint from $\overline{A}$, and C^2 crosscut arcs in D.

By crosscut arcs in D, we mean arcs α_j for which $\overline{\alpha}_j \backslash \alpha_j \neq \emptyset$ with

$$x \in \overline{\alpha}_j \backslash \alpha_j \Rightarrow \phi(x) = 0. \tag{3.2}$$

Suppose that along with A and B as above, there is a continuous function $h : D \to \mathbf{R}$ such that:

(i) the function h satisfies

$$h\,|_A = 0, \quad h\,\big|_{D\backslash B} = 1, \quad 0 < h(x) < 1 \quad \text{for all} \quad x \in B \setminus \overline{A}\,, \tag{3.3}$$

(ii) h is C^2 in $\overline{B} \backslash A$,

(iii) h satisfies a maximum principle in $B \backslash A$. That is, if Q is a subdomain of $B \backslash A$, then $\max_{x\in Q} h(x) \leq \max_{x\in\partial Q} h(x)$.

We shall call a function satisfying (i) through (iii) an *exhaustion function* corresponding to A and B, and our computations will be made with reference to such an h.

We put $D_0 = A,\ D_1 = B$,

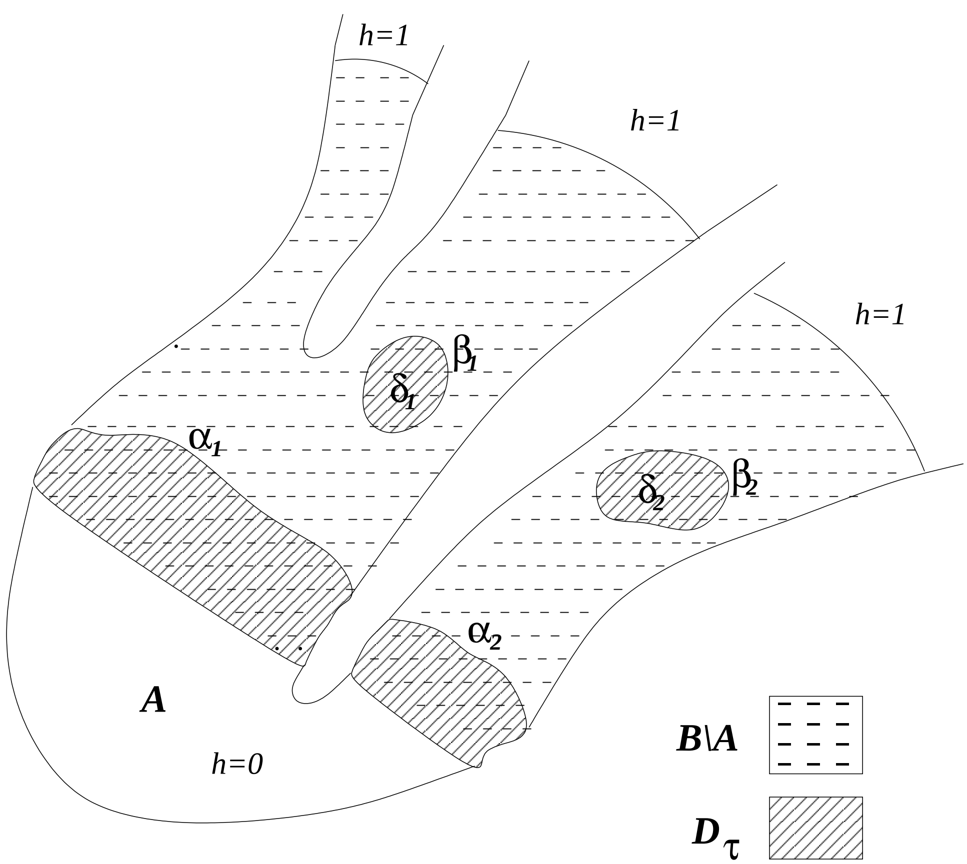
$$D_\tau = \{x \in D; h(x) < \tau\}, \quad \tau \in (0,1)\,,$$

and define $\partial D_\tau \cap D = E_\tau$. We assume throughout that $E_\tau \neq \emptyset$. Then, there is a set $T \subset [0,1]$ such that meas $T = 1$ and $\nabla h \neq 0$ on $\overline{E}_\tau$ for $\tau \in T$, so that for each fixed $\tau \in T$,

$$\inf_{x \in E_\tau} |\nabla h(x)| > 0. \tag{3.4}$$

In fact, since $h : D \to [0,1]$ is C^2, by Sard's theorem, the set of levels $t \in [0,1]$ for which there is a point in $x \in D$ such that $\nabla h(x) = 0$ and $h(x) = t$, has measure 0. Statement (3.4) then follows by continuity.

Since h has a maximum principle, for a given $\tau \in T$, the set E_τ consists of crosscuts $\alpha_j = \alpha_j(\tau)$ as in (3.2) and C^2 Jordan curves $\beta_k = \beta_k(\tau)$ which enclose simply connected subdomains $\delta_k = \delta_k(\tau)$.



In the classical case where F is flat and (2.4) is the Laplacian, the standard exhaustion function is

$$h(x) = \min\left\{\left(\log\frac{R_2}{R_1}\right)^{-1}\log^+\frac{|x|}{R_1},\ 1\right\} \tag{3.5}$$

with $A = D\cap\{x; |x| < R_1\}$ and $B = D\cap\{x; |x| < R_2\}$ where $0 < R_1 < R_2$; here $\log^+|a|$ means $\max\{\log|a|, 0\}$. In this case, the arcs α_j are simply arcs of circles and, since h in (3.5) is harmonic, it has a minimum principle which precludes sets of the form δ_k.

We write

$$I(D_\tau,\,\phi) = \int_{D_\tau} |\nabla\phi|_F^2\, d\sigma_F\,. \tag{3.6}$$

Theorem 3.1 *Let ϕ be a generalized solution of* (2.4) *in a simply connected domain D with $\phi = 0$ on ∂D. Assume that the components of ∂D are C^2 and that ϕ and the g_{ij} are continuous in $\overline{D}$. Then with* (3.1) *through* (3.6) *we have*

$$I(D_0,\phi) \le I(D_1,\phi)\,\exp\left\{-2\pi\ \Big/ \int_{B\setminus A} |\nabla h|_F^2\, d\sigma_F\right\}. \tag{3.7}$$

The proof of Theorem 3.1 will be carried out through the rest of this section along with Section 4.

Let D'_τ denote the union of those subdomains δ_k enclosed by β_k and, on each β_k, let x_k be a point where ϕ takes its minimum $c_k = \phi(x_k)$. Let

$$X_\tau = \bigcup_k \{x_k\}, \tag{3.8}$$

and

$$\psi(x) = \begin{cases} \phi(x) & x\in D_\tau\setminus D'_\tau \\ \phi(x) - c_k & x\in\delta_k\subset D'_\tau, \end{cases} \tag{3.9}$$

so that

(3.10) $\qquad\psi(x) = 0$ at a point on each component of $\partial D'_\tau$.

Using this choice of ψ in (2.7), we have for $\tau\in T$,

$$I(D_\tau,\,\phi) = \int_{E_\tau} \psi\,\langle\nabla\phi, \overline{n}\rangle_F\, ds_F\,,$$

and by the Cauchy inequality,

$$I(D_\tau,\,\phi) \le \left(\int_{E_\tau} \psi^2\,|\nabla h|_F\,ds_F\right)^{1/2} \left(\int_{E_\tau} \langle\nabla\phi,\overline{n}\rangle_F^2\,\frac{ds_F}{|\nabla h|_F}\right)^{1/2}. \tag{3.11}$$

Let $\overline{m}$ be a vector field on E_τ such that a.e. $\overline{m}$ is continuous and satisfies

$$\langle\overline{n},\overline{m}\rangle_F = 0, \tag{3.12}$$

and

$$|\overline{m}|_F = 1. \tag{3.13}$$

We will use the *characteristic* defined by

$$\lambda(E_\tau) = \inf_\eta \frac{\left(\displaystyle\int_{E_\tau} \langle\nabla\eta,\overline{m}\rangle_F^2\,\frac{ds_F}{|\nabla h|_F}\right)^{1/2}}{\left(\displaystyle\int_{E_\tau} \eta^2\,|\nabla h|_F\,ds_F\right)^{1/2}}, \tag{3.14}$$

where the infimum in (3.14) is taken over all nontrivial Lipschitz functions $\eta : E_\tau \to \mathbf{R}$ such that

$$\eta = 0 \;\text{ on } X_\tau \cup (\overline{E_\tau} \setminus E_\tau). \tag{3.15}$$

Then, from (3.2), (3.9), (3.10), (3.11), (3.14) and (3.15), we find that, for $\tau \in T$,

$$\lambda(E_\tau) I(D_\tau,\,\phi) \le \left(\int_{E_\tau} \langle\nabla\phi,\overline{m}\rangle_F^2\,\frac{ds_F}{|\nabla h|_F}\right)^{1/2} \left(\int_{E_\tau} \langle\nabla\phi,\overline{n}\rangle_F^2\,\frac{ds_F}{|\nabla h|_F}\right)^{1/2}. \tag{3.16}$$

However,

$$\langle\nabla\phi,\overline{n}\rangle_F^2 + \langle\nabla\phi,\overline{m}\rangle_F^2 = |\nabla\phi|_F^2,$$

so we obtain

$$\left(\int_{E_\tau} \langle\nabla\phi,\overline{m}\rangle_F^2\,\frac{ds_F}{|\nabla h|_F}\right)^{1/2} \left(\int_{E_\tau} \langle\nabla\phi,\overline{n}\rangle_F^2\,\frac{ds_F}{|\nabla h|_F}\right)^{1/2}$$

$$\le \frac{1}{2}\int_{E_\tau} \langle\nabla\phi,\overline{m}\rangle_F^2\frac{ds_F}{|\nabla h|_F} \;+\; \frac{1}{2}\int_{E_\tau} \langle\nabla\phi,\overline{n}\rangle_F^2\,\frac{ds_F}{|\nabla h|_F} \quad = \frac{1}{2}\int_{E_\tau} |\nabla\phi|_F^2\,\frac{ds_F}{|\nabla h|_F}.$$

Thus from (3.16) it follows that, for $\tau \in T$,

$$2\lambda(E_\tau) I(D_\tau,\,\phi) \le \int_{E_\tau} |\nabla\phi|_F^2\,\frac{ds_F}{|\nabla h|_F}. \tag{3.17}$$

Observe now that the coarea formula implies

$$I(D_\tau,\,\phi)-I(D_0,\,\phi)=\int_{D_\tau\setminus A}|\nabla\phi|_F^2\,d\sigma_F=\int_0^\tau dt\int_{E_t}|\nabla\phi|_F^2\,\frac{ds_F}{|\nabla h|_F}\,. \tag{3.18}$$

To see this, let ψ_ϵ be defined to be 0 when $|\nabla h|<\epsilon$ and 1 when $|\nabla h|\geq\epsilon$. Then by the usual coarea formula, we have

$$\begin{aligned}\int_{D_\tau}\psi_\epsilon|\nabla\phi|_F^2\,d\sigma_F&=\int_{D_\tau}\psi_\epsilon|\nabla\phi|_F^2\,\sqrt{g}\,dx_1dx_2\\&=\int_0^\tau dt\int_{E_t}\psi_\epsilon|\nabla\phi|_F^2\,\sqrt{g}\,\frac{|dx|}{|\nabla h|},\end{aligned} \tag{3.19}$$

where $|dx|$ denotes the usual Euclidean length element (see [3, Theorem 3.2.12]). We let $\epsilon\to 0$ in (3.19). By property (ii) of h, the left hand side tends to

$$\int_{D_\tau\setminus\{\nabla h=0\}}|\nabla\phi|_F^2\,d\sigma_F=\int_{D_\tau\setminus A}|\nabla\phi|_F^2\,d\sigma_F.$$

Now,

$$g^{11}=\frac{g_{22}}{g},\quad g^{12}=-\frac{g_{12}}{g},\quad g^{22}=\frac{g_{11}}{g}.$$

At each point $x\in E_t$ with $t\in T$, we have

$$\begin{aligned}|\nabla h|_F^2\;&=\textstyle\sum_{i,j=1}^2 g^{ij}h_{x_i}h_{x_j}=|\nabla h|^2\sum_{i,j=1}^2 g^{ij}\dfrac{h_{x_i}}{|\nabla h|}\dfrac{h_{x_j}}{|\nabla h|}\\&=|\nabla h|^2\left(g^{11}\left(\frac{dx_2}{|dx|}\right)^2-2g^{12}\frac{dx_1}{|dx|}\frac{dx_2}{|dx|}+g^{22}\left(\frac{dx_1}{|dx|}\right)^2\right)\\&=\frac{|\nabla h|^2}{g}\left(g_{22}\left(\frac{dx_2}{|dx|}\right)^2+2g_{12}\frac{dx_1}{|dx|}\frac{dx_2}{|dx|}+g_{11}\left(\frac{dx_1}{|dx|}\right)^2\right)\\&=\frac{|\nabla h|^2}{g\,|dx|^2}\sum_{i,j=1}^2 g_{ij}\,dx_idx_j=\frac{|\nabla h|^2}{g\,|dx|^2}\,ds_F^2\,.\end{aligned}$$

Thus,

$$\frac{ds_F}{|\nabla h|_F}=\sqrt{g}\,\frac{|dx|}{|\nabla h|} \tag{3.20}$$

for $t\in T$. Since meas $T=1$, we obtain (3.18).

It follows from (3.18) that

$$\frac{d}{d\tau} I(D_\tau, \phi) = \int_{E_\tau} |\nabla \phi|_F^2 \, \frac{ds_F}{|\nabla h|_F} \quad \text{for a.e.} \quad \tau \in [0,1]\,,$$

and (3.17) implies

$$2\lambda(E_\tau) I(D_\tau, \phi) \le \frac{d}{d\tau} I(D_\tau, \phi) \quad \text{a.e. on} \quad [0,1]\,. \tag{3.21}$$

By solving (3.21), we obtain an inequality of Carleman type [2]

$$I(D_0, \phi) \le I(D_1, \phi) \, \exp\left\{ -2 \int_0^1 \lambda(E_t)\, dt \right\}\,. \tag{3.22}$$

4 Characteristic estimate

In the classical case mentioned in (3.5), the characteristic $\lambda(E_\tau)$ (with the condition $E_\tau \neq \emptyset$), as it appears in [2, p. 966], is π/ℓ where ℓ is the angular measure of the longest arc α_j.

We now simplify (3.14). Suppose that $\tau \in T$ and $E_\tau = \bigcup_i l_i$, where $l_i = l_i(\tau)$ are connected components of E_τ $(i = 1, 2, \ldots)$. We have

$$\lambda(E_\tau) = \inf_{1 \le i < \infty} \lambda(l_i)\,. \tag{4.1}$$

For the proof, we observe that $l_i \subset E_\tau$ implies

$$\lambda(E_\tau) \le \lambda(l_i) \quad \text{for every} \quad i = 1, 2, \ldots. \tag{4.2}$$

Further from (4.2),

$$\lambda(E_\tau) \le \lambda_0 = \inf_i \lambda(l_i)\,.$$

On the other hand, let η be an arbitrary Lipschitz function on E_τ satisfying (3.15). Consider its nontrivial restrictions $\eta_i = \eta\,|_{l_i}$ $(i = 1, 2, \ldots)$. Each function η_i is admissible in the variational problem (3.14) for the arc l_i. Therefore,

$$\lambda^2(l_i) \int_{l_i} \eta_i^2 |\nabla h|_F \, ds_F \le \int_{l_i} |\nabla \eta_i|_F^2 \frac{ds_F}{|\nabla h|_F}$$

and

$$\lambda_0^2 \sum_i \int_{l_i} \eta_i^2 |\nabla h|_F \, ds_F \le \sum_i \int_{l_i} |\nabla \eta_i|_F^2 \frac{ds_F}{|\nabla h|_F}\,.$$

Thus,

$$\lambda_0^2 \sum_i \int_{E_\tau} \eta_i^2 |\nabla h|_F \, ds_F \le \int_{E_\tau} |\nabla \eta_i|_F^2 \frac{ds_F}{|\nabla h|_F} \,,$$

that is,

$$\lambda_0 \le \lambda(E_\tau)$$

and (4.1) is proved.

For $i = 1, 2, \ldots$, suppose that l_i is a rectifiable arc along which $|\nabla h|$ satisfies (3.4). Let a, b be endpoints of l_i and the arc l_i be given by

$$x = x(t) : [0,1] \to \mathbf{R}^2, \quad x(0) = a, \;\; x(1) = b \,.$$

When l_i is a crosscut of D, then its endpoints are in $\overline{E}_\tau \backslash E_\tau$ as in (3.2), and when l_i is closed, its common endpoint is in X_τ as in (3.8).

Denote by $l_i(t)$ the subarc of l_i lying between points $x(0)$ and $x(t)$. On l_i we introduce a new parameter σ by setting

$$\sigma = \sigma(t) = \frac{1}{\sigma_0} \int_{l_i(t)} |\nabla h|_F \, ds_F \,, \tag{4.3}$$

where $0 \le t \le 1$ and

$$\sigma_0 = \int_{l_i} |\nabla h|_F \, ds_F \,.$$

We have $0 \le \sigma \le 1$ and

$$d\sigma = \frac{1}{\sigma_0} |\nabla h|_F \, ds_F \,.$$

Then, with η as in (3.15),

$$\int_{l_i} \eta^2 \, |\nabla h|_F \, ds_F = \sigma_0 \int_0^1 (\eta^*)^2 \, d\sigma \,, \tag{4.4}$$

where $\eta^*(\sigma) = \eta[x(t(\sigma))]$.

Let $x_1 = x_1(\sigma),\; x_2 = x_2(\sigma)$ be the parametrization of l_i with respect to the parameter σ of (4.3). Then, from (2.6) and (3.12),

$$g^{11} m_1 \frac{dx_2}{d\sigma} - g^{12} m_1 \frac{dx_1}{d\sigma} + g^{21} m_2 \frac{dx_2}{d\sigma} - g^{22} m_2 \frac{dx_1}{d\sigma} = 0,$$

or

$$\frac{dx_1}{d\sigma} (g^{22} m_2 + g^{12} m_1) = \frac{dx_2}{d\sigma} (g^{11} m_1 + g^{21} m_2). \tag{4.5}$$

Furthermore, from (3.13) we have

$$g^{11}m_1^2 + 2g^{12}m_1m_2 + g^{22}m_2^2 = 1. \tag{4.6}$$

Using (4.5) with $\nabla\eta = (\eta_1, \eta_2)$ we obtain

$$\begin{aligned}
\langle \nabla\eta, \overline{m}\rangle_F &= g^{11}m_1\eta_1 + g^{12}m_1\eta_2 + g^{21}m_2\eta_1 + g^{22}m_2\eta_2 \\
&= \eta_1(g^{11}m_1 + g^{21}m_2) + \eta_2(g^{22}m_2 + g^{21}m_1) \\
&= \frac{\eta_1(g^{22}m_2 + g^{12}m_1)\, dx_1/d\sigma}{dx_2/d\sigma} + \eta_2(g^{22}m_2 + g^{12}m_1) \\
&= \left(\eta_1 \frac{dx_1}{d\sigma} + \eta_2 \frac{dx_2}{d\sigma}\right)\left(\frac{g^{22}m_2 + g^{12}m_1}{dx_2/d\sigma}\right).
\end{aligned} \tag{4.7}$$

Similarly,

$$\langle \nabla\eta, \overline{m}\rangle_F = \left(\eta_1 \frac{dx_1}{d\sigma} + \eta_2 \frac{dx_2}{d\sigma}\right)\left(\frac{g^{11}m_1 + g^{21}m_2}{dx_1/d\sigma}\right). \tag{4.8}$$

Multiplying (4.7) by $m_2\, dx_2/d\sigma$ and (4.8) by $m_1\, dx_1/d\sigma$, and using (4.6) we have

$$\langle \nabla\eta, \overline{m}\rangle_F(m_1dx_1/d\sigma + m_2dx_2/d\sigma) = \eta_1dx_1/d\sigma + \eta_2dx_2/d\sigma.$$

Thus,

$$\langle \nabla\eta, \overline{m}\rangle_F = \frac{d\eta^*/d\sigma}{\langle \overline{m}, \overline{v}\rangle},$$

where $\overline{v} = (dx_1/d\sigma, dx_2/d\sigma)$. Using this and (4.3) we obtain

$$\begin{aligned}
\int_{l_i} \langle \nabla\eta, \overline{m}\rangle_F^2 \frac{ds_F}{|\nabla h|_F} &= \int_{l_i} \frac{(d\eta^*/d\sigma)^2}{\langle \overline{m}, \overline{v}\rangle^2} \frac{ds_F}{|\nabla h|_F} \\
= \int_{l_i} \frac{(d\eta^*/d\sigma)^2}{\langle \overline{m}, \overline{v}\rangle^2} \frac{(ds_F/d\sigma)\, d\sigma}{|\nabla h|_F} &= \int_{l_i} \frac{(d\eta^*/d\sigma)^2}{\langle \overline{m}, \overline{v}\rangle^2 d\sigma/ds_F} \frac{d\sigma}{|\nabla h|_F} \\
= \sigma_0 \int_{l_i} \frac{(d\eta^*/d\sigma)^2}{\langle \overline{m}, \overline{v}\rangle^2} \frac{d\sigma}{|\nabla h|_F^2} &= \frac{1}{\sigma_0} \int_{l_i} \frac{(d\eta^*/d\sigma)^2}{\langle \overline{m}, \overline{v}\rangle^2 (d\sigma/ds_F)^2} d\sigma \\
&= \frac{1}{\sigma_0} \int_{l_i} \frac{(d\eta^*/d\sigma)^2\, d\sigma}{\langle \overline{m}, \overline{T}\rangle^2},
\end{aligned} \tag{4.9}$$

where $\overline{T} = (dx_1/ds_F, dx_2/ds_F)$.

Now (3.1) implies that $|\overline{T}| \le 1$ and $|\overline{m}| \le 1$. Thus, $\langle \overline{m}, \overline{T}\rangle^2 \le 1$ which in (4.9) yields

$$\int_{l_i} \langle \nabla\eta, \overline{m}\rangle_F^2 \frac{ds_F}{|\nabla h|_F} \ge \frac{1}{\sigma_0} \int_0^1 \left(\frac{d\eta^*}{d\sigma}\right)^2 d\sigma\,. \tag{4.10}$$

By (3.15) we may use Wirtinger's inequality (see, for example, [1, Chapter V, Theorem 7])

$$\int_0^1 \left(\frac{d\eta^*}{d\sigma}\right)^2 d\sigma \geq \pi^2 \int_0^1 (\eta^*)^2 \, d\sigma \, .$$

With (3.14), (4.4) and (4.10), we then obtain

$$\lambda(l_i) \geq \frac{\pi}{\displaystyle\int_{l_i} |\nabla h|_F \, ds_F}$$

and

$$\lambda(E_\tau) \geq \pi \left[\sup_i \int_{l_i} |\nabla h|_F \, ds_F\right]^{-1} . \tag{4.11}$$

It follows from (3.22) and (4.11) that

$$I(D_0, \phi) \leq I(D_1, \phi) \exp\left\{-2\pi \int_0^1 \frac{dt}{\displaystyle\int_{E_t} |\nabla h|_F \, ds_F}\right\} . \tag{4.12}$$

Now we observe that

$$1 = \left(\int_0^1 dt\right)^2 \leq \int_0^1 \int_{E_t} |\nabla h|_F \, ds_F dt \int_0^1 \frac{dt}{\displaystyle\int_{E_t} |\nabla h|_F \, ds_F}$$

and we can rewrite (4.12) in the form

$$I(D_0, \phi) \leq I(D_1, \phi) \exp\left\{-2\pi \Big/ \int_0^1 \int_{E_t} |\nabla h|_F \, dt \, ds_F\right\} . \tag{4.13}$$

As in the derivation of (3.18), we may use the usual coarea formula to write

$$\int_B \psi_\epsilon |\nabla h|_F^2 d\sigma_F = \int_B \psi_\epsilon |\nabla h|_F^2 \sqrt{g} \, dx_1 dx_2 = \int_0^1 dt \int_{E_t} \psi_\epsilon |\nabla h|^2 \sqrt{g} \frac{|dx|}{|\nabla h|}. \tag{4.14}$$

where ψ_ϵ is as in (3.19). The left hand side tends to $\int_{B\setminus A} |\nabla h|_F^2 \, d\sigma_F$ as $\epsilon \to 0$. Using (3.20) in (4.14), then from (4.13) we obtain

$$I(D_0, \phi) \leq I(D_1, \phi) \exp\left\{-2\pi \Big/ \int_{B\setminus A} |\nabla h|_F^2 \, d\sigma_F\right\} . \tag{4.15}$$

■

5 Proof of Theorem 1.1

Let $F \subset \mathbf{R}^3$ be the graph of a nontrivial function f over an unbounded simply connected domain $D \subset \mathbf{R}^2$ which satisfies (1.1) and (1.2). Here we have

$$(5.1) \qquad ds_F^2 = \sum_{i,j=1}^{2} (\delta_{ij} + f_{x_i} f_{x_j})\, dx_i dx_j\,,$$

so

$$g_{ij} = \delta_{ij} + f_{x_i} f_{x_j}, \quad g^{ij} = \delta_{ij} - \frac{f_{x_i} f_{x_j}}{1 + |\nabla f|^2}, \quad g = 1 + |\nabla f|^2\,.$$

It is easy to see that f satisfies the Laplace-Beltrami equation (2.4) in the metric ds_F. By considering the sets where $f > 0$ and $f < 0$ separately, we may assume that

$$f > 0.$$

Since f is real analytic, it follows from the implicit function theorem that by choosing c to avoid the isolated set of values for which $\nabla f = 0$, and replacing f by $f - c$ in the set $\{x; f(x) > c\}$, we have that the boundary components of the set will be analytic arcs, and the smoothness assumptions in Section 3 are satisfied. Furthermore, by the maximum principle (minimum principle) the new set is still simply connected. We may also assume that the point $x = 0$ belongs to D.

Let $\rho(x) = (|x|^2 + f(x)^2)^{1/2}$. Again, ρ is real analytic and we may choose r and R avoiding the set of critical points of ρ where $\nabla \rho = 0$ so that $0 < r < R$ and

$$(5.2) \qquad f(x) < R \quad \text{if} \quad |x| < r.$$

Let B be the connected component of $\{x \in D : \rho(x) < R\}$ containing the origin. For $0 < t < R$, U_t will denote the subset $\{x \in D : \rho(x) < t\} \cap B$. We take $A = U_r$, which for r chosen sufficiently large, will be nonempty.

Then, for those $r < t < R$ such that the level set $\rho(x) = t$ avoids the critical points of ρ, ∂U_t consists of components of ∂D, C^2 crosscuts α_j, and C^2 Jordan curves β_k as described in §3.

The function

$$h(x) = \begin{cases} 0 & \text{for} \quad x \in A\,, \\ \dfrac{\log(\rho(x)/r)}{\log(R/r)} & \text{for} \quad x \in B \backslash A\,, \\ 1 & \text{for} \quad x \notin B. \end{cases}$$

is then a legitimate exhaustion function corresponding to A and B.

In fact, (i) follows from the definitions of A, B, and h, and (ii) follows from the fact that h is real analytic in $B\backslash A$. Finally, (iii) follows from the fact that h is subharmonic on F. To see this, recall that if the surface were parametrized by isothermal coordinates $(x_1(\zeta), x_2(\zeta), x_3(\zeta))$ for ζ in some parameter set in the complex plane, then these coordinate functions are harmonic in ζ. With these coordinates, ρ is $(x_1(\zeta)^2+x_2(\zeta)^2+x_3(\zeta)^2)^{1/2}$, which is subharmonic in ζ.

Finally, note that (5.2) and the maximum principle imply that connected components of A are simply connected.

Observing that

$$|\nabla \rho|_F \leq 1$$

we find

$$\int_{B\backslash A} |\nabla h|_F^2 \sqrt{g}\, dx_1 dx_2 \leq \frac{1}{(\log R/r)^2} \int_{B\backslash A} \sqrt{1+|\nabla f|^2} \frac{dx_1 dx_2}{|x|^2+f^2(x)} .$$

Denoting as above for an arbitrary $Q \subset D$

$$I(Q,f) = \int_Q |\nabla f|_F^2 d\sigma_F ,$$

from (3.7) we have

$$I(U_r,f) \leq I(U_R,f) \exp\left\{ -\frac{2\pi\left(\log\dfrac{R}{r}\right)^2}{\displaystyle\int\limits_{U_R\backslash U_r} \sqrt{1+|\nabla f|^2}\frac{dx_1dx_2}{|x|^2+f^2(x)}} \right\} . \tag{5.3}$$

We next estimate the right integral. Let

$$r^* = \inf_{x\in D\backslash A} |x| \tag{5.4}$$

and

$$D_{r^*,R} = \{x \in D; r^* < |x| < R\}.$$

Then $r^* \leq \rho(x)$ for $x \in D \setminus A$ and $x \in U_R$ implies that $|x| < R$, so $(U_R \setminus U_r) \subset D_{r^*,R}$, and hence

$$\int_{U_R\backslash U_r} \sqrt{1+|\nabla f|^2} \frac{dx_1 dx_2}{|x|^2+f^2(x)} \leq \int_{D_{r^*,R}} \sqrt{1+|\nabla f|^2} \frac{dx_1 dx_2}{|x|^2+f^2(x)}$$

$$= \int_{D_{r^*,R}} \left(1+|\nabla f|^2\right)^{-1/2} \frac{dx_1 dx_2}{|x|^2+f^2(x)} + \int_{D_{r^*,R}} \frac{|\nabla f|^2}{\sqrt{1+|\nabla f|^2}} \frac{dx_1 dx_2}{|x|^2+f^2(x)} . \tag{5.5}$$

Since f satisfies (1.1) in D with $f = 0$ on ∂D,

$$\int_{\partial D_{r^*,R}} \frac{1}{|x|}\operatorname{arctg}\frac{f}{|x|}\,\frac{\langle \nabla f, \overline{n}\rangle}{\sqrt{1+|\nabla f|^2}}\,|dx|$$

$$= \int_{D_{r^*,R}} \frac{1}{|x|}\operatorname{arctg}\frac{f}{|x|}\sum_{i=1}^{2}\frac{\partial}{\partial x_i}\left(\frac{f_{x_i}}{\sqrt{1+|\nabla f|^2}}\right)dx_1dx_2 - \int_{D_{r^*,R}} \frac{1}{|x|^2}\operatorname{arctg}\frac{f}{|x|}\,\frac{\langle \nabla |x|, \nabla f\rangle}{\sqrt{1+|\nabla f|^2}}\,dx_1dx_2$$

$$- \int_{D_{r^*,R}} \frac{f}{|x|(|x|^2+f^2)}\,\frac{\langle \nabla |x|, \nabla f\rangle}{\sqrt{1+|\nabla f|^2}}\,dx_1dx_2 + \int_{D_{r^*,R}} \frac{1}{|x|^2+f^2}\,\frac{|\nabla f|^2}{\sqrt{1+|\nabla f|^2}}\,dx_1dx_2$$

$$= -\int_{D_{r^*,R}} \operatorname{arctg}\mu(x)\,\frac{\langle \nabla |x|, \nabla f\rangle}{\sqrt{1+|\nabla f|^2}}\,\frac{dx_1dx_2}{|x|^2} - \int_{D_{r^*,R}} \frac{\mu(x)}{1+\mu^2(x)}\,\frac{\langle \nabla |x|, \nabla f\rangle}{\sqrt{1+|\nabla f|^2}}\,\frac{dx_1dx_2}{|x|^2}$$

$$+ \int_{D_{r^*,R}} \frac{1}{|x|^2+f^2(x)}\,\frac{|\nabla f|^2}{\sqrt{1+|\nabla f|^2}}\,dx_1dx_2\,,$$

where

$$\mu(x) = \frac{f(x)}{|x|}\,.$$

Thus,

$$\begin{aligned}\int_{D_{r^*,R}} \frac{1}{|x|^2+f^2(x)}\,\frac{|\nabla f|^2}{\sqrt{1+|\nabla f|^2}}\,dx_1dx_2 \quad &\le \int_{\partial D_{r^*,R}} \frac{1}{|x|}\operatorname{arctg}\frac{f}{|x|}\,|dx| \\ &+ \int_{D_{r^*,R}} \operatorname{arctg}\mu(x)\,\frac{\langle \nabla |x|, \nabla f\rangle}{\sqrt{1+|\nabla f|^2}}\,\frac{dx_1dx_2}{|x|^2} \\ &+ \int_{D_{r^*,R}} \frac{\mu(x)}{1+\mu^2(x)}\,\frac{\langle \nabla |x|, \nabla f\rangle}{\sqrt{1+|\nabla f|^2}}\,\frac{dx_1dx_2}{|x|^2}\,.\end{aligned}$$

Observing that

$$\int\limits_{\partial D_{r^*,R}} \frac{1}{|x|}\operatorname{arctg}\frac{f}{|x|}\,|dx| \le 2\pi^2\,,$$

from (5.5), we obtain

$$\int_{U_R\setminus U_r} \frac{\sqrt{1+|\nabla f|^2}dx_1dx_2}{|x|^2+f^2(x)} \le 2\pi^2 + \int_{D_{r^*,R}} \frac{(1+|\nabla f|^2)^{-1/2}}{1+\mu^2(x)} \frac{dx_1dx_2}{|x|^2}$$

$$+\int_{D_{r^*,R}} \left(\operatorname{arctg}\mu(x) + \frac{\mu(x)}{1+\mu^2(x)}\right) \frac{|\nabla f|}{\sqrt{1+|\nabla f|^2}} \frac{dx_1dx_2}{|x|^2}.$$

However,

$$\frac{1}{\sqrt{1+|\nabla f|^2}}\frac{1}{1+\mu^2(x)} + \left(\operatorname{arctg}\mu(x) + \frac{\mu(x)}{1+\mu^2(x)}\right) \frac{|\nabla f|}{\sqrt{1+|\nabla f|^2}}$$

$$\le \left[\left(\frac{1}{1+\mu^2(x)}\right)^2 + \left(\operatorname{arctg}\mu(x) + \frac{\mu(x)}{1+\mu^2(x)}\right)^2\right]^{1/2} \equiv \Lambda(\mu(x)), \tag{5.6}$$

and we arrive at the inequality

$$\int_{U_R\setminus U_r} \sqrt{1+|\nabla f|^2}\frac{dx_1dx_2}{|x|^2+f^2(x)} \le 2\pi^2 + \int_{D_{r^*,R}} \Lambda(\mu(x))\frac{dx_1dx_2}{|x|^2}. \tag{5.7}$$

We note that the function $\Lambda(\mu)$ is increasing and hence

$$1 = \Lambda(0) \le \Lambda(\mu) < \Lambda(\infty) = \frac{\pi}{2}, \tag{5.8}$$

and that μ tends to 0 if f has order less than 1.

The estimates (5.3) and (5.7) imply

$$I(U_r, f) \le I(U_R, f)\,\exp\left\{-2\pi\left(\log\frac{R}{r}\right)^2 \Big/ \left(2\pi^2 + \int_{D_{r^*,R}} \Lambda(\mu(x))\frac{dx_1dx_2}{|x|^2}\right)\right\}. \tag{5.9}$$

Now we need to estimate $I(U_R, f)$. In fact (cf. [5, Lemma 1]) we have

$$I(U_R, f) = \int_{U_R} \frac{|\nabla f|^2}{\sqrt{1+|\nabla f|^2}}dx_1dx_2 = \int_{U_R} \frac{|\nabla f|^2+1-1}{\sqrt{1+|\nabla f|^2}}dx_1dx_2$$

$$= \int_{U_R} \sqrt{1+|\nabla f|^2}dx_1dx_2 - \int_{U_R} \frac{dx_1dx_2}{\sqrt{1+|\nabla f|^2}} \tag{5.10}$$

$$\le 3\pi R^2.$$

Using (5.10) in (5.9) we have

$$I(U_r, f) \le 3\pi R^2 \exp\left\{-2\pi\left(\log\frac{R}{r}\right)^2 \Bigg/ \left(2\pi^2 + \int_{D_{r^*,R}} \Lambda(\mu(x))\frac{dx_1dx_2}{|x|^2}\right)\right\}. \tag{5.11}$$

From (5.8) we have $\Lambda(\mu(x)) < \pi/2$. Thus, (5.11) becomes

$$I(U_r, f) \le 3\pi R^2 \exp\left\{-2\pi\left(\log\frac{R}{r}\right)^2 \Bigg/ \left(2\pi^2 + \int_{D_{r^*,R}} \frac{\pi}{2}\frac{dx_1dx_2}{|x|^2}\right)\right\}, \tag{5.12}$$

from which (1.3) follows.

If $f(x)/|x| \to 0$, then $\Lambda(\mu(x)) \to 1$ in (5.11) by (5.8). Thus, for any $\epsilon' > 0$, there exists $R_0 > r^*$ such that, for some $C = C(\epsilon', R_0)$,

$$I(U_r, f) \le 3\pi R^2 \exp\left\{-2\pi\left(\log\frac{R}{r}\right)^2 \Bigg/ \left(2\pi^2 + \int_{D_{r^*,R_0}} \frac{\pi}{2}\frac{dx_1dx_2}{|x|^2} + \int_{D_{R_0,R}} (1+\epsilon')\frac{dx_1dx_2}{|x|^2}\right)\right\}$$

$$< CR^2 \exp\left\{-2\pi\left(\log\frac{R}{r}\right)^2 \Bigg/ \int_{D_{R_0,R}} (1+\epsilon')\frac{dx_1dx_2}{|x|^2}\right\} \tag{5.13}$$

$$= C\left(R^{2(1+\epsilon')} \exp\left\{-2\pi\left(\log\frac{R}{r}\right)^2 \Bigg/ \int_{D_{R_0,R}} \frac{dx_1dx_2}{|x|^2}\right\}\right)^{1/(1+\epsilon')}.$$

With $\epsilon = \epsilon'/2$, then (1.4) follows. ■

6 Proof of Theorem 1.2

Suppose there were 4 domains $D_1,\ D_2,\ D_3,\ D_4$. Then for at least one j, we have

$$\int_{D_j \cap \{1<|x|<R\}} \frac{dx_1dx_2}{|x|^2} \le (\pi/2)\log R$$

Then the quantity (1.3) corresponding to this D_j, satisfies

$$R^2 \exp\left(\frac{-8}{\pi}\log R\right) \to 0,$$

which implies that the solution above this D_j vanishes.

■

7 Proofs of Theorems 1.3 and 1.4

We begin by proving Theorem 1.3. To this end, we need only replace the estimate of $I(U_R, f)$ in (5.9) by $M(2R, f)$. Choose a Lipschitz function $\psi : D \to \mathbf{R}$ with properties

$$0 \le \psi(x) \le 1 \text{ for all } x \in D, \quad \psi(x) = 1 \text{ for } |x| \le R, \quad \psi(x) = 0 \text{ for } |x| \ge 2R.$$

Again, by considering the sets where $f > 0$ and $f < 0$ separately, we may assume that $f > 0$ in D. The function $f\psi^2$ has a compact support in $\overline{D}$. By Green's formula,

$$0 = \int_{\partial D} f\psi^2 \frac{\langle \nabla f, n \rangle}{\sqrt{1+|\nabla f|^2}} |dx| = \int_D \psi^2 \frac{|\nabla f|^2}{\sqrt{1+|\nabla f|^2}} dx_1 dx_2 + 2\int_D f\psi \frac{\langle \nabla f, \nabla\psi \rangle}{\sqrt{1+|\nabla f|^2}} dx_1 dx_2.$$

Thus,

$$\begin{aligned} \int_D \psi^2 \frac{|\nabla f|^2}{\sqrt{1+|\nabla f|^2}} dx_1 dx_2 &\le 2\int_D f\psi \frac{|\nabla f|\,|\nabla\psi|}{\sqrt{1+|\nabla f|^2}} dx_1 dx_2 \\ &\le 2M(2R, f) \left(\int_D |\nabla\psi|^2 dx_1 dx_2 \right)^{1/2} \left(\int_D \psi^2 \frac{|\nabla f|^2}{1+|\nabla f|^2} dx_1 dx_2 \right)^{1/2} \\ &\le 2M(2R, f) \left(\int_D |\nabla\psi|^2 dx_1 dx_2 \right)^{1/2} \left(\int_D \psi^2 \frac{|\nabla f|^2}{\sqrt{1+|\nabla f|^2}} dx_1 dx_2 \right)^{1/2}, \end{aligned}$$

so

$$\int_D \psi^2 \frac{|\nabla f|^2}{\sqrt{1+|\nabla f|^2}} dx_1 dx_2 \le 4M(2R, f)^2 \int_D |\nabla\psi|^2 dx_1 dx_2.$$

Remembering that $\psi \equiv 1$ for $|x| \le R$ and $\psi \equiv 0$ for $|x| \ge 2R$, we obtain

$$I(U_R, f) \le \int\limits_{D\cap\{|x|\le R\}} \frac{|\nabla f|^2}{\sqrt{1+|\nabla f|^2}} dx_1 dx_2 \le 4M(2R, f)^2 \int\limits_{D\cap\{r<|x|<R\}} |\nabla\psi|^2 dx_1 dx_2.$$

By taking $\psi(x) = (\log 2)^{-1} \log(|x|/R)$ for $R \le r \le 2R$, we then get

$$I(U_R, f) \le \frac{8\pi M(2R, f)^2}{\log 2}. \tag{7.1}$$

Using (7.1) in place of (5.10) in (5.11), and repeating (5.12) and (5.13) with this estimate, we obtain the proof of Theorem 1.3. ■

Turning now to the proof of Theorem 1.4, we assume that, corresponding to an $\alpha \ge \pi$, the order of f is greater than π/α. Then for some $\epsilon > 0$, there exist R_0 and $C = C(R_0, \epsilon)$ such that

$$M(2R) < CR^{\frac{2\pi}{\alpha(1+\epsilon)}} \qquad (R > R_0), \tag{7.2}$$

and

$$\text{(7.3)} \qquad \int\limits_{D\cap\{R_0<|x|<R\}} \frac{dx_1dx_2}{|x|^2} < \alpha(1+\epsilon/2)\log\frac{R}{R_0} \qquad (R>R_0).$$

Using (7.2) and (7.3) in (1.6) we get $f \equiv 0$, and therefore it must be that the order of f is at least π/α. ∎

8 Concluding Remark.

In his paper [2], Carleman introduced a function $m(r) = \int_0^{2\pi}\phi^2(re^{i\theta})d\theta$ which was then differentiated twice to give a differential inequality involving two derivatives. The first differentiation (with respect to $\log r$) gives the Dirichlet integral.

In the general setting, it is difficult to find a counterpart for the Carleman function $m(r)$. For this reason, it becomes more appropriate to begin with the general Dirichlet integral (3.6) and use just one differentiation as was done in the current work.

References

[1] E. Beckenbach and R.O. Bellman, Inequalities, Ergebnisse der Mathematik und ihrer Grenzgebiete, N .F., Bd. 30 Springer-Verlag, Berlin-Göttingen-Heidelberg, 1961.

[2] T. Carleman, Sur une inégalité differentielle dans la théorie des functions analytiques, C. R. Acad. Sci. Paris **196** (1933), 995–997.

[3] H. Federer, Geometric measure theory, Die Grundlehren der mathematischen Wissenschaften, Band 153, Springer-Verlag, Berlin, 1969.

[4] D. Gilbarg and N. Trudinger, Elliptic partial differential equations of second order, 2nd Ed., Springer-Verlag, Berlin, 1983.

[5] P. Li and J. Wang, Finiteness of disjoint minimal graphs, Math. Res. Lett. **8** (2001), 771–777.

[6] V. M. Miklyukov, On a new approach to Bernstein's theorem and related questions for equations of minimal surface type, Math. Sb. (N.S.) **108** (1979), 268–289; Engl. transl. in Math. USSR Sbornik **36**, (1980), 251–271.

[7] V. M. Miklyukov, Some singularities in the behavior of equations of minimal-surface type in unbounded domains, Mat. Sb. (N.S.) **116** (1981), 72–86; Engl. transl. in Math. USSR Sb. **44** (1983), 61–73.

[8] M. Schiffer and D. Spencer, Functionals of finite Riemann surfaces, Princeton University Press, Princeton, N. J., 1954.

[9] J. Spruck, Two-dimensional minimal graphs over unbounded domains, J. Inst. Math. Jussieu **1** (2002), 631–640.

[10] A. Weitsman, On the growth of minimal graphs, Indiana Univ. Math. J. **54** (2005), 617–625.

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