ON THE GROWTH OF SOLUTIONS TO THE MINIMAL SURFACE EQUATION OVER DOMAINS CONTAINING A HALFPLANE

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ABSTRACT. We consider minimal graphs $u = u(x, y) > 0$ over unbounded domains D with $u = 0$ on ∂D. Assuming D contains a sector properly containing a halfplane, we obtain estimates on growth and provide examples illustrating a range of growth.
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1. Introduction

Let D be an unbounded plane domain. In this paper we consider the boundary value problem for the minimal surface equation

$$\begin{cases} \operatorname{div}\dfrac{\nabla u}{\sqrt{1+|\nabla u|^2}} = 0 \quad \text{and } u > 0 \quad \text{in } D \\ u = 0 \quad \text{on } \partial D \end{cases} \tag{1.1}$$

We shall study the constraints on growth of nontrivial solutions to (1.1) as determined by the maximum

$$M(r) = \max u(x, y),$$

where the max is taken over the values $r = \sqrt{x^2 + y^2}$ and $(x, y) \in D$.

Perhaps the first relevant theorem in this direction was proved by Nitsche [8, p. 256] who observed that if D is contained in a sector of opening strictly less than π, then $u \equiv 0$. For domains contained in a half plane, but not contained in any such sector, there are a host of solutions to (1.1) which will be discussed later. However, in this case, it has been shown [14] that if D is bounded by a Jordan arc,

$$Cr \le M(r) \le e^{Cr} \quad (r > r_0)$$

for some positive constants C and r_0.

If, on the other hand, the domain D contains a sector of opening α bigger than π, we shall show that the growth of $M(r)$ is at most linear (see Theorem 2.1 in Section 2). Regarding the bound from below, with the order ρ of u defined by

$$\rho = \lim_{r\to\infty} \sup \frac{\log M(r)}{\log r},$$

it follows by using the module estimates of Miklyukov [6] (see also chapter 9 in [7]) as in [13] that if D omits a sector of opening $2\pi - \alpha$, ($\pi \leq \alpha \leq 2\pi$, the omitted set in the case $\alpha = 2\pi$ being a line), then the order of any nontrivial solution to (1.1) is at least π/α,

The paper concludes with a list of problems and conjectures.

2. Estimates on Growth

For later convenience we shall use complex notation $z = x + iy$ for points (x, y) when describing solutions to the minimal surface equation. As such, we are given a minimal graph with positive height function $u(z)$ over a domain D as in (1.1).

Theorem 2.1. *Let D be a simply connected domain whose boundary is a Jordan arc, and D contains a sector $S_\lambda := \{z : |\arg z| \leq \lambda\}$, with $\lambda > \pi/2$. With $M(r)$ defined as above, if u satisfies (1.1) in D, then there exist positive constants K and R such that*

$$M(r) \leq Kr, \quad |z| > R. \tag{2.1}$$

Throughout, we will make use of the parametrization in isothermal coordinates by the Weierstrass functions $(x(\zeta), y(\zeta), U(\zeta))$ with ζ in the right half plane H, $U(\zeta) = u(x(\zeta), y(\zeta))$ and (up to additive constants)

$$\begin{cases} x(\zeta) = \Re e \, \dfrac{1}{2} \displaystyle\int_{\zeta_0}^{\zeta} \omega(\bar{\zeta})(1 - G^2(\bar{\zeta}))d\bar{\zeta} \\ y(\zeta) = \Re e \, \dfrac{i}{2} \displaystyle\int_{\zeta_0}^{\zeta} \omega(\bar{\zeta})(1 + G^2(\bar{\zeta}))d\bar{\zeta} \\ U(\zeta) = \Re e \displaystyle\int_{\zeta_0}^{\zeta} \omega(\bar{\zeta})G(\bar{\zeta})d\bar{\zeta}. \end{cases} \tag{2.2}$$

Here ω is analytic and G meromorphic in H, with ω nonvanishing except for a zero of order $2n$ when G has a pole of order n.

With this parameterization, the height function $U(\zeta)$ pulled back to the halfplane H becomes a positive harmonic function in H which is 0 on the imaginary axis, and thus is simply $U(\zeta) = C\Re e\{\zeta\}$ for a real positive constant C. We may assume without loss of generality that $C = 2$.

Since $f(\zeta) := x(\zeta) + iy(\zeta)$ is harmonic in H, there exist analytic functions $h(\zeta)$ and $g(\zeta)$ in H such that (see [1, p. 176])

$$f(\zeta) = h(\zeta) + \overline{g(\zeta)}.$$

With this formulation, the height function then satisfies

$$U(\zeta) = 2\Re e\, i \int \sqrt{h'(\zeta)g'(\zeta)}\, d\zeta,$$

and since $U(\zeta) = 2\Re e\{\zeta\}$ in (2.2), it follows that

$$g'(\zeta) = -\frac{1}{h'(\zeta)}. \tag{2.3}$$

2.1. **Proof of Theorem 2.1.** First we establish the bound (2.1) inside a sector.

Lemma 2.2. *Let $S_\alpha := \{z : |\arg z| \le \alpha < \pi/2\}$ be a sector contained in $H \subset D$. Then for some $K > 0$ the upper bound (2.1) holds in S_α for all r sufficiently large:*

$$\max_{|z|=r, z\in S_\alpha} u(z) \le Kr.$$

Proof of Lemma. Let $f(\zeta)$, $U(\zeta)$ be as above. So, $u(f(\zeta)) = U(\zeta) = 2\Re e\,\zeta$.
Let $P := \{\zeta : \Re e\, f(\zeta) > 0\}$ be the preimage of the right halfplane, and introduce a new variable $\tilde\zeta$ and let $\psi(\tilde\zeta)$ be a conformal map from the right half $\tilde\zeta$-plane $H := \{\tilde\zeta : \Re e\,(\tilde\zeta) > 0\}$ onto P.
Define

$$\begin{cases} \tilde f(\tilde\zeta) := f(\psi(\tilde\zeta)) \\ \tilde g(\tilde\zeta) := g(\psi(\tilde\zeta)) \\ \tilde h(\tilde\zeta) := h(\psi(\tilde\zeta)) \end{cases}$$

Then $\tilde f$ is a harmonic map, and

$$\tilde f(\tilde\zeta) = \tilde h(\tilde\zeta) + \overline{\tilde g(\tilde\zeta)}.$$

We wish to show that for all $|z| > R$ in S_α,

$$\frac{u(z)}{|z|} = \frac{U(\zeta)}{|f(\zeta)|} = \frac{2\Re e\,\zeta}{|f(\zeta)|} = \frac{2\Re e\,\psi(\tilde\zeta)}{|\tilde f(\tilde\zeta)|} < K.$$

Let $\tilde F(\tilde\zeta) = \tilde h(\tilde\zeta) + \tilde g(\tilde\zeta)$ be the analytic function with the same real part as $\tilde f$. Then $\Re e\,\tilde F$ is positive in H and vanishes on ∂H, and therefore, without loss of generality we may write (see [12, p. 151])

$$\tilde F(\tilde\zeta) = \tilde\zeta \implies \tilde F'(\tilde\zeta) = 1. \tag{2.4}$$

The proof hinges on (2.4) along with the chain rule combined with (2.3). Now,

$$\tilde h'(\tilde\zeta) = h'(\psi(\tilde\zeta)) \cdot \psi'(\tilde\zeta),$$

and

$$\tilde g'(\tilde\zeta) = -\frac{\psi'(\tilde\zeta)}{h'(\psi(\tilde\zeta))} = -\frac{\psi'(\tilde\zeta)^2}{\tilde h'(\tilde\zeta)}. \tag{2.5}$$

Combining this with (2.4) we have

$$1 = \tilde{F}'(\tilde{\zeta}) = \tilde{h}'(\tilde{\zeta}) - \frac{\psi'(\tilde{\zeta})^2}{\tilde{h}'(\tilde{\zeta})}$$

which implies

$$\tilde{h}'(\tilde{\zeta})^2 - \tilde{h}'(\tilde{\zeta}) - \psi'(\tilde{\zeta})^2 = 0.$$

Thus,

$$\tilde{h}'(\tilde{\zeta}) = \frac{1+\sqrt{1+4\psi'(\tilde{\zeta})^2}}{2}. \tag{2.6}$$

Since $\psi(\tilde{\zeta})$ is a conformal map with $\Re e\, \psi(\tilde{\zeta}) > 0$ in H, we can apply the following result restated from [12, Thm. IV.19]:

Theorem A. *Let $f(z) = u + iv$ be analytic in $x > 0$ where $u > 0$. Then*

$$\lim_{z\to\infty} \frac{f(z)}{z} = c, \quad \lim_{z\to\infty} f'(z) = c, \quad 0 \le c < \infty,$$

uniformly as $z \to \infty$ from the inside of any fixed angular domain $|\arg(z)| \le \phi_0 < \pi/2$.

Applying Theorem A directly to ψ, we conclude that there exists a real constant $0 \le c < \infty$ such that in any sector $S_\beta := \{\tilde{\zeta} : |\arg\tilde{\zeta}| \le \beta < \pi/2\}$ the limit $\psi'(\tilde{\zeta}) \to c$ exists as $\tilde{\zeta} \to \infty$ in S_β.

Case 1: $\psi'(\tilde{\zeta}) \to c = 0$ as $\tilde{\zeta} \to \infty$ (with $\tilde{\zeta}$ in S_β).

From (2.6) we have $\tilde{h}'(\tilde{\zeta}) \to 1$ as $\tilde{\zeta} \to \infty$, and using (2.5) we have $\tilde{g}'(\tilde{\zeta}) \to 0$. Thus, $\tilde{h}(\tilde{\zeta}) \approx \tilde{\zeta}$ and $\tilde{g}(\tilde{\zeta}) = o(1)$, which implies that $\tilde{f}(\tilde{\zeta}) = \tilde{h}(\tilde{\zeta}) + \overline{\tilde{g}(\tilde{\zeta})} \approx \tilde{\zeta}$.

Since $\tilde{f} : H \to H$ is asymptotic to the identity map, given α, we may choose $\beta < \pi/2$ so that $S_\alpha \cap \{|z| > R\}$ is contained in the image of the sector S_β for R large enough. Thus, the estimate $\psi'(\tilde{\zeta}) \to 0$ applies in the region S_α, and we have

$$\frac{u(z)}{|z|} = \frac{\Re e\, \psi(\tilde{\zeta})}{|\tilde{f}(\tilde{\zeta})|} \le \frac{|\psi(\tilde{\zeta})|}{|\tilde{f}(\tilde{\zeta})|} = o(1), \quad \text{for } z \in S_\alpha \cap \{|z| > R\},$$

since $\tilde{f}(\tilde{\zeta}) \approx \tilde{\zeta}$, and $\psi'(\tilde{\zeta}) = o(1)$.

Case 2: $\psi'(\tilde{\zeta}) \to c > 0$ as $\tilde{\zeta} \to \infty$.

From (2.4) we have $\Re e\, \{\tilde{h}(\tilde{\zeta}) + \overline{\tilde{g}(\tilde{\zeta})}\} = \Re e\, \tilde{\zeta}$. Let us also estimate $\Im m\, \tilde{f}(\tilde{\zeta}) = \Im m\, \tilde{h}(\tilde{\zeta}) - \Im m\, \tilde{g}(\tilde{\zeta})$. We use (2.6) and (2.5):

$$\tilde{h}'(\tilde{\zeta}) \to \frac{1+\sqrt{1+4c^2}}{2},$$

$$\tilde{g}'(\tilde{\zeta}) \to \frac{-2c^2}{1+\sqrt{1+4c^2}},$$

which imply

$$\tilde{h}'(\tilde{\zeta}) - \tilde{g}'(\tilde{\zeta}) \to \frac{(1+\sqrt{1+4c^2})^2 + 4c^2}{2(1+\sqrt{1+4c^2})} = 1 + \frac{4c^2}{1+\sqrt{1+4c^2}}.$$

Putting this together, we have

$$\tilde{h}(\tilde{\zeta}) + \overline{\tilde{g}(\tilde{\zeta})} = \Re e\, \tilde{\zeta} + i\left(1 + \frac{4c^2}{1+\sqrt{1+4c^2}} + o(1)\right)\Im m\, \tilde{\zeta}.$$

As in the first case, given α, we may thus choose $\beta < \pi/2$ and $R > 0$ so that $S_\alpha \cap \{|z| > R\}$ is contained in the image $\tilde{f}(S_\beta)$ of the sector S_β. Then we have

$$\frac{u(z)}{|z|} = \frac{2\Re e\, \psi(\tilde{\zeta})}{|\tilde{f}(\tilde{\zeta})|} \le \frac{|2\psi(\tilde{\zeta})|}{|\tilde{f}(\tilde{\zeta})|} = O(1), \quad \text{for } z \in S_\alpha \cap \{|z| > R\}.$$

Indeed, $|\tilde{f}(\tilde{\zeta})| = \left|\Re e\, \tilde{\zeta} + i\left(1 + \frac{4c^2}{1+\sqrt{1+4c^2}} + o(1)\right)\Im m\, \tilde{\zeta}\right|$, and $\psi'(\tilde{\zeta}) = O(1) \implies \psi(\tilde{\zeta}) = O(|\tilde{\zeta}|)$.

□

Applying Lemma 2.2 to two sectors, one rotated clockwise and the other counterclockwise, in order that their union covers S_λ, the upper bound (2.1) is established in S_λ. It remains to prove the estimate in the rest of D.

Let $\pi/2 < \alpha < \lambda$. We will show that the upper bound (2.1) holds in $D \setminus \overline{S_\alpha}$.

In order to prove this, we will apply the following result from [2, Main Theorem]:

Theorem B. *Let $\Omega \subset \Omega_1 = \{(x,y) : x > 0, -f(x) < y < f(x)\}$, where $f, g \in C[0,\infty)$, $f, g \ge 0$, $g(0) = 0$, $f(t), g(t)/t$ increase as t increases, and let $u \in C(\overline{\Omega}) \cap C^2(\Omega)$. Suppose that*

i) $\mathrm{div}\dfrac{\nabla u}{\sqrt{1+|\nabla u|^2}} \ge 0$ in Ω,

ii) $u|_{\partial\Omega \cap (\{x\} \times [-f(x), f(x)])} \le g(x)$ for $x \in [0,\infty)$,

iii) $0 < \kappa(x) := f(x)/(2g(x)) < 1$ for all x larger than some $x_1 > 0$,

iv) $\kappa(x)$ is decreasing on $[x_1, \infty)$.

Then $u(x,y) \le g(x/(1-\kappa(x)))$ for every $(x,y) \in \Omega$ with $x > x_1$.

We apply this to $\Omega = D \setminus \overline{S_\alpha}$, while taking $\Omega_1 = \mathbb{C} \setminus \overline{S_\alpha}$. In order to relate to the setup in the theorem, reflect these domains about the y-axis, so that Ω and Ω_1 are in the right halfplane. Then $\Omega_1 = \{(x,y) : x > 0, -f(x) < y < f(x)\}$, where $f(x) = \tan(\pi - \alpha)x$. If $C > 0$ is sufficiently large, then $g(x) = Cx(1 - \exp(-x)/2)$ satisfies both (iii) and (iv). We check that for C large enough, (ii) is also satisfied. Note that $\partial\Omega$ contains points on ∂D and points on ∂S_α. For points on ∂D, $u = 0$, and for points on ∂S_α, u has at most linear growth by Lemma 2.2. Thus, in both cases (ii) is satisfied, and Theorem B may be applied. The result is that $u(x,y) \leq g(x/(1-\kappa(x)))$ for all large enough $x \in \Omega$. Since

$$\frac{x}{1-\kappa(x)} = \frac{x}{1-\tan(\pi-\alpha)/C}(1+o(1)),$$

and $\tan(\pi-\alpha)/C$ is a small constant provided C is large, we have

$$u(x,y) < Cx,$$

for all large enough $x \in \Omega$. This completes the proof of (2.1).

2.2. **A lower bound.**

Proposition 2.3. *Suppose D is a domain with $\partial D \neq \emptyset$, and $u(z) > 0$ satisfies (1.1) with $u(z) = 0$ on ∂D. Then $u(z)$ has at least logarithmic growth.*

Proof. Without loss of generality assume that $0 \in \partial D$, and consider the top half of the vertical catenoid centered at $z = 0$ as a "barrier" (cf. [9, p. 92]). Explicitly, let $\cosh^{-1}$ denote the positive branch of the inverse of $\cosh : \mathbb{R} \to \mathbb{R}$, and define

$$G(z; r_1) := r_1 \cosh^{-1}\left(\frac{|z|}{r_1}\right), \quad |z| \geq r_1.$$

For each r_1, $G(z; r_1)$ satisfies (1.1).

Let $\varepsilon > 0$ and choose a δ-neighborhood $B(\delta, 0)$ of $z = 0$ small enough that $u(z) < \varepsilon$ throughout $B(\delta, 0) \cap D$.

Define $u_\varepsilon(z) = u(z) - \varepsilon$. For $r_1 > 0$ small enough, $G(|z|; r_1) > u_\varepsilon(z)$ on $\partial B(\delta, 0) \cap D$. For $R > 0$, let

$$K_R := D \cap B(R, 0) \setminus B(\delta, 0).$$

Fix $R = R_0$. Suppose $\max_{|z|=R} |u(z)|$ grows slower than logarithmically, so it grows slower than $G(|z|, r_1)$. Then for $R > R_0$ sufficiently large, $G(|z|; r_1) > u_\varepsilon(z)$ on ∂K_R. This implies the same inequality throughout $K_{R_0} \subset K_R$. In particular, $u_\varepsilon(z) < r_1 \cosh^{-1}\left(\frac{R_0}{r_1}\right)$ in K_{R_0}. But $r_1 > 0$ is arbitrary, and $r_1 \cosh^{-1}\left(\frac{R_0}{r_1}\right) \to 0$ as $r_1 \to 0$. Thus, $u_\varepsilon(z) \leq 0$ in K_{R_0} which implies that $u(z) \leq 0$ since ε was arbitrary. This contradicts that $u(z) > 0$ in D. $\square$

3. Examples

In this Section, we provide examples that together with the above (and previously known) results give a broad picture of the possible growth rates of minimal graphs. One notices three "regimes" illustrated in Fig. 1. When D contains a halfplane we find nontrivial examples, but their growth rates appear to be determined by the asymptotic angle $\pi < \beta < 2\pi$. This is reminiscent of the behavior of positive harmonic functions, hence we deem this the "Phragmén-Lindelöf regime". However, the geometry of D plays a subtle role, since if D is a true sector of opening β, even in the range $\pi < \beta < 2\pi$, then (1.1) has only the trivial solution $u \equiv 0$ [5, p.993].

When D is contained in a sector $\beta < \pi$, we have a "completely rigid regime", due to Nitsche's theorem. At the critical angle $\beta = \pi$, an interesting phase transition occurs; there are examples with D contained in a halfplane with $\beta = \pi$ exhibiting a full spectrum of possible growth rates anywhere from linear to exponential thus interpolating the known upper and lower bounds. It is of interest to note further that the domains D for these examples are contained within powerlaw growth. That is, $D \subset \{(x, y) : |y| \leq |x|^\rho\}$. This answers in the affirmative a question attributed to Michael Beeson in [10, Example 1.8] that asked about the existence of such examples.

We note also that it follows from Theorem B that these examples exhibit the maximum possible growth rate for their respective boundary curves (see also Problem 7 of Section 4).

3.1. **Examples in the "Phragmén-Lindelöf" regime $\pi < \beta < 2\pi$:** In [14], there appears an example of a minimal graph with height function (pulled back to ζ-plane) $U(\zeta) = 2\Re e\,\zeta$, and harmonic map from the half plane $H := \{z = x + iy : x > 0\}$

$$z(\zeta) = \frac{(\zeta + 1)^2}{2} - \log(\bar{\zeta} + 1).$$

This example has asymptotic angle 2π and growth of order $1/2$. (See §4 for the definition of asymptotic angle.)

Let us demonstrate a whole one-parameter family of examples with asymptotic angles $\pi < \beta < 2\pi$ having growth of orders π/β. Let $\gamma = \beta/\pi$ (so $1 < \gamma < 2$). Then such a minimal surface is given by the harmonic map from the half plane H to a region D

$$z(\zeta) = (\zeta + 1)^\gamma - \frac{1}{\gamma(2 - \gamma)}(\bar{\zeta} + 1)^{2-\gamma}$$

together with the height function $U(\zeta) = 2\Re e\,\zeta$.

Assuming $z(\zeta)$ is univalent, then we have growth of order $1/\gamma = \pi/\beta$ as desired, since

$$\frac{u(z)}{|z|^{1/\gamma}} = \frac{U(\zeta)}{|z(\zeta)|^{1/\gamma}} = \frac{2\Re e\,\zeta}{|(\zeta + 1)^\gamma - \frac{1}{\gamma(2-\gamma)}(\bar{\zeta} + 1)^{2-\gamma}|^{1/\gamma}}.$$

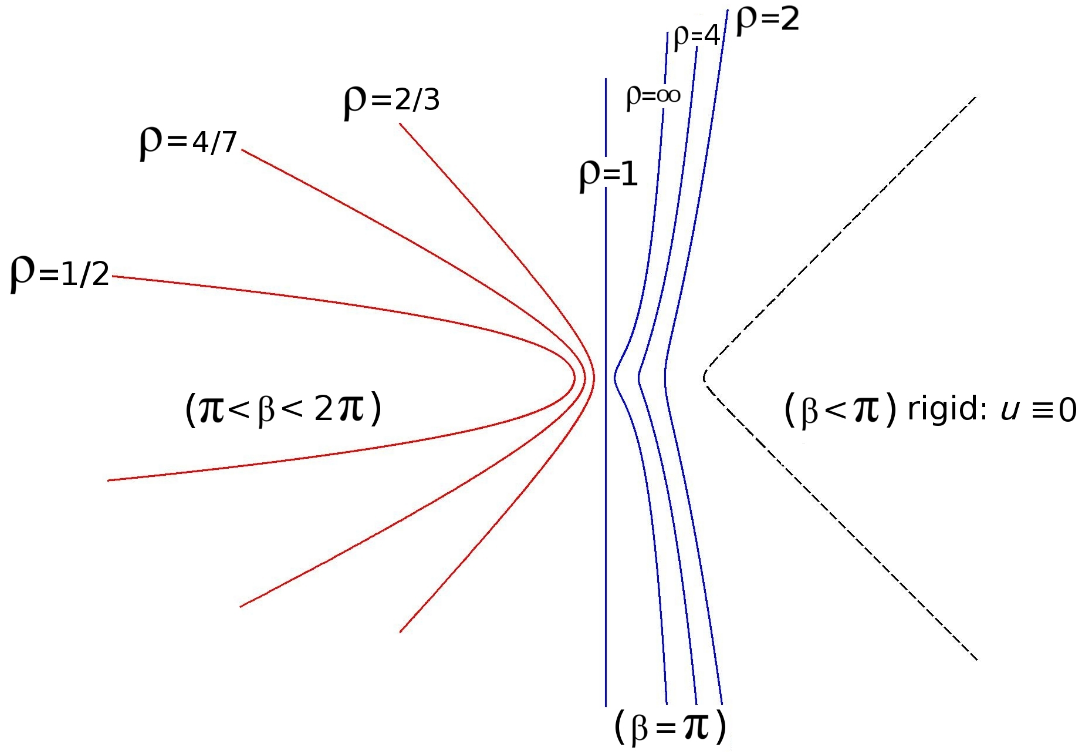

Figure 1. A plot of the boundary of D labeled with order ρ. Phragmén-Lindelöf regime: $\pi < \beta < 2\pi$, Critical regime: $\beta = \pi$, and Rigid regime: $\beta < \pi$. For the curves, from left to right the angles are $\beta = 2\pi$, $7\pi/4$, and $3\pi/2$.

Thus, the only thing to check is that $z(\zeta)$ is univalent in H. Its Jacobian is

$$\gamma^2|\zeta+1|^{2(\gamma-1)} - \frac{1}{\gamma^2|\zeta+1|^{2(\gamma-1)}} > 0$$

since

$$\gamma^2|\zeta+1|^{2(\gamma-1)} > 1.$$

Thus, global univalence can be ensured by checking the boundary behavior. We will show that the imaginary part of $z(\zeta)$ is increasing on the boundary $\zeta = it$, $-\infty < t < \infty$. The imaginary part of $z(it)$ is

$$\Im m\,\{z(it)\} = (1+t^2)^{\gamma/2}\sin(\gamma\tan^{-1} t) + \frac{1}{\gamma(2-\gamma)}(1+t^2)^{(2-\gamma)/2}\sin((2-\gamma)\tan^{-1} t).$$

This is an odd function, so we just consider the interval $0 < t < \infty$. The second term is increasing, since it is a product of increasing functions. Indeed, $0 < 2-\gamma < 1$ so that $0 < (2-\gamma)\tan^{-1} t < \pi/2$ for $0 < t < \infty$ so that $\sin((2-\gamma)\tan^{-1} t)$ is increasing. In order to show that $(1+t^2)^{\gamma/2}\sin(\gamma\tan^{-1}t)$ is increasing, we check that the derivative

$$\gamma(1+t^2)^{\gamma/2-1}t\sin(\gamma\tan^{-1}t)+\gamma(1+t^2)^{\gamma/2-1}\cos(\gamma\tan^{-1}t)$$

is positive, or equivalently that

$$t\sin(\gamma\tan^{-1}t)+\cos(\gamma\tan^{-1}t)>0.$$

For this let $0<\theta<\pi/2$ and take $t=\tan\theta$. Then we see that

$$\tan\theta\sin(\gamma\theta)+\cos(\gamma\theta)=\frac{\cos(\gamma-1)\theta}{\cos\theta},$$

which is positive since $0<\theta<\pi/2$ and $1<\gamma<2$.

3.2. **The critical angle $\beta=\pi$: Examples from linear growth to exponential.** A plane and a horizontal catenoid sliced by a plane parallel to its axis provide two examples of minimal graphs over a domain contained in a half plane. These examples have linear and exponential growth respectively.

For each given $\rho>1$, we provide an example contained in a halfplane (each having asymptotic angle $\beta=\pi$) with order of growth ρ. Let $b=1/\rho$. Then, once again, $z(\zeta)$ has the form

$$z(\zeta)=h(\zeta)-\overline{\int\frac{1}{h'(\zeta)}d\zeta},$$

so that $U(\zeta)=2\Re e\,\zeta$.

Taking $h(\zeta)=\zeta+\frac{1}{b}\zeta^b$,

$$z(\zeta)=\zeta+\frac{1}{b}\zeta^b-\bar{\zeta}+\overline{\int\frac{1}{1+\zeta^{1-b}}d\zeta},$$

Assuming $z(\zeta)$ is univalent, $u(z)$ has order ρ, since

$$\frac{u(z)}{|z|^\rho}=\frac{U(\zeta)}{|z(\zeta)|^\rho}=\frac{2\Re e\,\zeta}{|z(\zeta)|^\rho},$$

which tends to a constant on the real axis.

It remains to check that $z(\zeta)$ is univalent in H. Its Jacobian is

$$|1+\zeta^{b-1}|^2-\frac{1}{|1+\zeta^{b-1}|^2}>0$$

since

$$|1+\zeta^{b-1}|^2>1,\quad \text{for } \zeta\in H.$$

Thus, global univalence can be ensured by checking the boundary behavior. As in the previous examples we show that $\Im m\,\{z(\zeta)\}$ is increasing on the boundary $\zeta=it$,

$-\infty < t < \infty$. This is an odd function, so we just consider the interval $0 < t < \infty$. It suffices to show that the derivative

$$\frac{d}{dt}\Im m\left\{z(it)\right\}$$

is positive. We use the identity

$$\frac{d}{dt}\Im m\left\{z(it)\right\} = \frac{d}{dt}\Im m\left\{h(it)\right\} - \frac{d}{dt}\Im m\left\{g(it)\right\} = \Re e\left\{h'(it)\right\} - \Re e\left\{g'(it)\right\},$$

to compute

$$\begin{aligned}\frac{d}{dt}\Im m\left\{z(it)\right\} &= 1 + \Re e\,\frac{1}{(it)^{1-b}} + 1 - \Re e\,\frac{1}{1+(it)^{1-b}}\\ &> 2 - \frac{1}{1+\Re e\left\{(it)^{1-b}\right\}}\\ &> 1.\end{aligned}$$

We note that the domain D for this example has a corner at the point $z(0)$. This can be removed by shifting the minimal graph $(x, y, u(x, y))$ in the negative u-direction.

4. Problems and conjectures

I. When dealing with a nonlinear equation, issues of existence and uniqueness are often complex. A survey of uniqueness results can be found in [4]. A natural question to ask here is

Problem 1. Is it possible for (1.1) to have more than one nontrivial (nonplanar) solution?

II. As discussed in the introduction, for domains D contained in the half plane, at least when bounded by a Jordan arc, the growth of solutions to (1.1) is at most exponential. However, it seems likely that this is true in general.

Problem 2. If u is a solution to (1.1), then does its maximum $M(r)$ satisfy

$$M(r) \le e^{Cr} \quad (r > r_0),$$

for some positive constants C and r_0

III. As noted above, the maximum growth rate for solutions to (1.1) in a halfplane is exponential, and this is achieved by horizontal catenoids over domains contained in the set $\{(x, y) : -Ce^{Cx} < y < Ce^{Cx}\}$.

Problem 3. If D contains a set $\{(x, y) : x > 0, -f(x) < y < f(x)\}$ where $f(x) \to \infty$ faster than any exponential function e^{Cx}, can (1.1) have a nontrivial solution, and if so, must any such solution have linear growth?

IV. Theorem 2.1 requires that D contain a sector of opening bigger than π.

Problem 4. Does the conclusion of Theorem 2.1 still hold under the assumption that D contains a halfplane?

V. In this paper we have shown that if D contains a sector of opening $\alpha > \pi$, then any nontrivial solution has order at most 1. However, it seems likely that this might be be improved.

Problem 5. If D contains a sector of opening $\alpha > \pi$, then is it true that the order of any nontrivial solution to (1.1) is bounded above by π/α? The interpretation as with the minimum bound discussed in §1 has the case $\alpha = 2\pi$ taken to mean that the omitted set is a line.

VI. The results in [14] are phrased in terms of the *asymptotic angle* β defined as follows. Let $\Theta(r)$ be the angular measure of the set $D \cap \{|z| = r\}$, and $\Theta^*(r) = \Theta(r)$ if D does not contain the circle $|z| = r$, and $+\infty$ otherwise. Then

$$\beta = \lim_{r\to\infty} \sup \Theta^*(r).$$

Consideration of the case $\beta = 2\pi$ raises the following question

Problem 6. If D is an unbounded simply connected region bounded by a Jordan arc (taken to mean a proper curve which does not self intersect or close), then is it true that the maximum of a nontrivial solution satisfies

$$M(r) \geq C\sqrt{r} \quad (r > r_0)$$

for some positive constants C and r_0?

VII. Returning to Nitsche's theorem as mentioned in §1, in terms of the asymptotic angle β it seems likely that a corresponding result should hold.

Problem 7. If D has asymptotic angle $\beta < \pi$, and u is a solution to (1.1), then must it be that $u \equiv 0$?

VIII. As noted, the examples of Section 3 in the critical regime have maximal growth for their respective boundaries.

Problem 8. Suppose that D is contained in $\{(x, y) : x > 0,\ -Ce^{Cx} < y < Ce^{Cx}\}$ and also that D contains a set $\{(x, y) : x > x_0 > 0, -x^n < y < x^n\}$. Is it possible for a solution u to (1.1) over D to have growth smaller than x^k with $k < n$? In particular, can u have linear growth?

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