LEVEL CURVES OF MINIMAL GRAPHS

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Abstract. We consider minimal graphs $u = u(x, y) > 0$ over domains $D \subset R^2$ bounded by an unbounded Jordan arc γ on which $u = 0$. We prove an inequality on the curvature of the level curves of u, and prove that if D is concave, then the sets $u(x, y) > C$ $(C > 0)$ are all concave. A consequence of this is that solutions, in the case where D is concave, are also superharmonic.

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1. INTRODUCTION

Let D be a plane domain bounded by an unbounded Jordan arc γ. In this paper we consider the boundary value problem for the minimal surface equation

$$\begin{cases} \operatorname{div}\dfrac{\nabla u}{\sqrt{1+|\nabla u|^2}} = 0 \quad \text{and } u > 0 \quad \text{in } D \\ u = 0 \quad \text{on } \gamma \end{cases} \tag{1.1}$$

We shall study the curvature $\kappa = \pm|d\varphi/ds|$ for level curves $u = C$ $(C > 0)$ where φ is the angle of the tangent vector to the curve, and the sign will be taken to be + when the curve bends away from the set where $u > C$.

Theorem 1. *There exists a constant K depending on u such that, if u as in (1.1) and $C > 0$, the curvature $\kappa = \kappa(C)$ of the level curve $u = C$ satisfies the inequality*

$$|\kappa| \le \frac{K}{C}. \tag{1.2}$$

Further comments regarding the constant K are given in §6.

Our next result concerns solutions whose domains are concave. There is a literature (see [3] and references cited there) regarding the propogation of convexity for level curves of solutions to partial differential equations over convex domains.

However, regarding the possible geometry of D in (1.1), it follows from a theorem of Nitsche [6, p.256] that D cannot be convex unless D is a halfplane since (1.1) cannot have nontrivial solutions over domains contained in a sector of opening less than π. On the other hand, amongst the examples given in [5], there is a continuum of graphs which do have concave domains; specifically those given parametrically in the right half plane $\mathbf{H}$ by

$$z(\zeta) = (\zeta+1)^{\gamma} - \frac{1}{\gamma(2-\gamma)}(\bar{\zeta}+1)^{2-\gamma} \quad (\zeta \in \mathbf{H}, \quad 1 < \gamma < 2) \tag{1.3}$$

together with the height function $2\Re e\,\zeta$. A concave domain D is taken to be one whose complement is an unbounded convex domain. The boundary of D is then a curve which bends away from the domain.

In §6 we will verify that the domains for the graphs of (1.3) are concave. In this note we shall prove the following

Theorem 2. *If u is a solution to (1.1) with D concave and bounded by a C^2 curve γ, then the sets where $u > C$ are concave for each $C > 0$.*

This has the curious consequence

Corollary. *If u is as in Theorem 2 above, then u is also superharmonic in D.*

2. PRELIMINARIES

For a solution u to the minimal surface equation over a simply connected domain D we shall slightly abuse notation by using u to also denote the solution to (1.1) when given in parametric form. We shall make use of the parametrization of the surface given by u in isothermal coordinates using Weierstrass functions $(x(\zeta), y(\zeta), u(\zeta))$ with ζ in the right half plane $\mathbf{H}$. Our notation will then be given by

$$f(\zeta) = x(\zeta) + iy(\zeta) \quad \zeta = \sigma + i\tau \in \mathbf{H}. \tag{2.1}$$

Then $f(\zeta)$ is univalent and harmonic, and since D is simply connected it can be written in the form

$$f(\zeta) = h(\zeta) + \overline{g(\zeta)} \quad \zeta = \sigma + i\tau \in \mathbf{H} \tag{2.2}$$

where $h(\zeta)$ and $g(\zeta)$ are analytic in $\mathbf{H}$,

$$|h'(\zeta)| > |g'(\zeta)|, \tag{2.3}$$

and

$$u(\zeta) = 2\Re e\, i \int \sqrt{h'(\zeta)g'(\zeta)}\, d\zeta. \tag{2.4}$$

(cf. [2, §10.2]).

Now, $u(\zeta)$ is harmonic and positive in $\mathbf{H}$ and vanishes on $\partial\mathbf{H}$. Thus, (cf. [7, p. 151]),

$$u(\zeta) = k_0\, \Re e\, \zeta, \tag{2.5}$$

where k_0 is a positive constant. This with (2.4) gives

$$g'(\zeta) = -\frac{k}{h'(\zeta)} \qquad (k = k_0^2/4). \tag{2.6}$$

Then from (2.3) we have, in particular, that

$$|h'(\zeta)| \geq \sqrt{k}. \tag{2.7}$$

It follows from (2.5) that the level curves of u can be parametrized by $f(\sigma_0 + i\tau)$ for $-\infty < \tau < \infty$ and fixed values σ_0. Then the curvature κ corresponding to height σ_0 with the sign convention given at the begining for

$$\varphi = \arctan(y_\tau/x_\tau)$$

is given by

$$\kappa = \kappa(\sigma_0, \tau) = \frac{d\varphi}{ds} = \frac{1}{(x_\tau^2 + y_\tau^2)^{3/2}}(x_\tau y_{\tau\tau} - y_\tau x_{\tau\tau}). \tag{2.8}$$

To compute (2.8) we use (2.1) and (2.6) to write

$$x_\tau = \frac{\partial}{\partial\tau}\Re e(h + \overline{g}) = \Re e\, i(h' - k/h') = -\Im m(h' - k/h') = -(|h'|^2 + k)\Im m\frac{1}{\overline{h}'} \tag{2.9}$$

$$x_{\tau\tau} = -\frac{\partial}{\partial\tau}\Im m(h' - k/h') = -\Re e(h'' + kh''/h'^2) \tag{2.10}$$

$$y_\tau = \frac{\partial}{\partial\tau}\Im m(h + \overline{g}) = \Im m\, i(h' + k/h') = \Re e(h' + k/h') = (|h'|^2 + k)\Re e\frac{1}{\overline{h}'} \tag{2.11}$$

$$y_{\tau\tau} = \frac{\partial}{\partial\tau}\Re e(h' + k/h') = -\Im m(h'' - kh''/h'^2) \tag{2.12}$$

Substituting (2.9)-(2.12) into (2.8) we get

$$\kappa = \frac{|h'|^3}{4(|h'|^2 + k)^2}\left(-(\frac{1}{\overline{h}'} - \frac{1}{h'})(h'' - k\frac{h''}{h'^2} - \overline{h}'' + k\frac{\overline{h}''}{\overline{h}'^2}) + (\frac{1}{\overline{h}'} + \frac{1}{h'})(h'' + k\frac{h''}{h'^2} + \overline{h}'' + k\frac{\overline{h}''}{\overline{h}'^2})\right)$$

which simplifies down to

$$\kappa = \frac{|h'|}{|h'|^2 + k}\,\Re e\,\frac{h''}{h'}. \tag{2.13}$$

Summarizing this, we have

Lemma 1. *With u as in (1.1) and k_0 as in (2.5), then the locus of $u = C$ is the set $\zeta = \sigma_0 + i\tau$, where $\sigma_0 = C/k_0$ and $-\infty < \tau < \infty$. The curvature κ at each point of this level set satisfies (2.13).*

The proof of Theorem 2 uses the comparison of κ in (2.13) with the corresponding curvature κ_1 of the image of the line $\sigma_0 + i\tau \quad (-\infty < \tau < \infty)$ under h. Since $\arg h' = \Im m \log h'$, the formula (2.8) gives

$$\kappa_1 = \frac{1}{|h'|}\Re e \frac{h''}{h'}. \tag{2.14}$$

3. PROOF OF THEOREM 1

Since f in (2.2) is a univalent harmonic mapping, we may convert the estimate from [1, Lemma 1] (cf. also ([2, p. 153])) for a univalent harmonic mapping $F = H + \overline{G}$ in the unit disk $\mathbf{U}$ to a mapping of the half plane $\mathbf{H}$.

Lemma 2. *Let u be as in (1.1) and $f = h + \overline{g}$ as in (2.2). Then*

$$\left|\frac{h''(\zeta)}{h'(\zeta)}\right| \le A/\sigma$$

for some absolute constant A.

Proof of Lemma 2. For the univalent harmonic mapping $F = H + \overline{G}$ of $\mathbf{U}$, the estimate of [1] is

$$\left|\frac{H''(w)}{H'(w))}\right| \le \frac{A_1}{1-|w|}, \qquad w \in \mathbf{U}$$

for some absolute constant A_1. Now, for $f(\zeta) = h(\zeta) + \overline{g(\zeta)}$, let

$$F(w) = f\left(\frac{1+w}{1-w}\right), \quad w \in \mathbf{U}.$$

Then,

$$h(\zeta) = H\left(\frac{\zeta-1}{\zeta+1}\right),$$

$$h'(\zeta) = H'\left(\frac{\zeta-1}{\zeta+1}\right)\frac{2}{(\zeta+1)^2},$$

and

$$h''(\zeta) = H''\left(\frac{\zeta-1}{\zeta+1}\right)\frac{4}{(\zeta+1)^4} - H'\left(\frac{\zeta-1}{\zeta+1}\right)\frac{4}{(\zeta+1)^3}.$$

Thus,

$$\left|\frac{h''(\zeta)}{h'(\zeta)}\right| \le \frac{2}{|\zeta+1|}\left(\frac{1}{|\zeta+1|}\frac{A_1}{1-\left|\frac{\zeta-1}{\zeta+1}\right|}+1\right)$$

$$\le \frac{2}{|\zeta+1|}\left(\frac{A_1}{|\zeta+1|-|\zeta-1|}+1\right) \le \frac{2}{|\zeta+1|}\left(\frac{A_2(|\zeta+1|+|\zeta-1|)}{4\sigma}+1\right)$$

$$\leq A/\sigma$$

for some absolute constant A. □

Proof of Theorem 1. From Lemma 1, Lemma 2, and (2.7) it follows that, on the level set $u = C$,

$$|\kappa| \leq \frac{A}{\sqrt{k}C}. \tag{3.1}$$

□

4. PROOF OF THEOREM 2

For convenience, we dismiss the trivial case where u is planar, and hence we may assume that h' is nonconstant.

From the given hypothesis, it follows that γ must have asymptotic angles in both directions as $z \to \infty$. By a rotation we may assume that the asymptotic tangent vectors have directions $\pm\alpha$ for some $0 \leq \alpha \leq \pi/2$.

From the concavity of D and the assumption that the asymptotic tangents to γ have angles $\pm\alpha$, it follows that $y_\tau \geq 0$ for $\sigma = 0$. Thus, from (2.11) it follows that for $\sigma = 0$, $\Re e\, 1/\overline{h}' \geq 0$, and hence $\Re e\, 1/h' \geq 0$. Since, by (2.7) $1/h'$ is bounded in $\mathbf{H}$, this means that $\Re e\, 1/h' > 0$ thoughout $\mathbf{H}$. This in turn gives

$$\Re e\, h'(\zeta) > 0 \qquad \zeta \in \mathbf{H}. \tag{4.1}$$

Let $\psi(\tau) = \arg h'(i\tau)$. It follows from (2.13) and (2.14) that $0 \leq \kappa_1 \not\equiv 0$ on $\partial\mathbf{H}$ so that

$$\frac{d\psi}{d\tau} = \frac{\partial}{\partial\tau}\Im m(\log h') = \Re e\frac{h''}{h'} \geq 0 \quad \text{when } \tau = 0. \tag{4.2}$$

By (4.1)

$$-\pi/2 \leq \psi(\tau) \leq \pi/2. \tag{4.3}$$

Now, $-\pi/2 < \Im m(\log h') < \pi/2$ in $\mathbf{H}$, and in particular is a bounded harmonic function in $\mathbf{H}$. So for $\zeta = \sigma + i\tau \in \mathbf{H}$,

$$\Im m\, \log h'(\zeta) = \frac{\sigma}{\pi}\int_{-\infty}^{\infty} \frac{\psi(t)dt}{\sigma^2 + (t-\tau)^2}.$$

Then

$$\Re e\frac{h''(\zeta)}{h'(\zeta)} = \frac{\partial}{\partial\tau}\Im m\, \log h'(\zeta) = \frac{\partial}{\partial\tau}\left(\frac{\sigma}{\pi}\int_{-\infty}^{\infty} \frac{\psi(t)dt}{\sigma^2 + (t-\tau)^2}\right) = \frac{2\sigma}{\pi}\int_{-\infty}^{\infty} \frac{(t-\tau)\psi(t)dt}{(\sigma^2 + (t-\tau)^2)^2}.$$

An integration by parts yields

$$\Re e \frac{h''}{h'} = \frac{\sigma}{\pi}\left(\frac{-\psi(t)}{\sigma^2+(t-\tau)^2}\Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \frac{\psi'(t)dt}{\sigma^2+(t-\tau)^2}\right).$$

By (4.3) it follows that the first term on the right vanishes, and by (4.2) the second term is positive. Thus κ_1 in (2.14) and hence κ in (2.13) are positive in $\mathbf{H}$. □

5. PROOF OF THE COROLLARY

We may write the minimal surface equation for u as

$$\frac{\Delta u + F}{|\nabla u|^3} = 0$$

where $F = F(u, x, y) = u_y^2 u_{xx} + u_x^2 u_{yy} - 2u_x u_y u_{xy}$.

Now, for a given function $v(x, y) > 0$ the curvature of the level set $v(x, y) = 0$ is given by $F(v, x, y)/|\nabla v|^3$ [4, p. 72] which is positive when the curve bends away from the interior of the domain. Since Theorem 2 shows that the level sets $u = c$ which bound the sets $u > c$ each have positive curvature, then applying this to $F(u - c, x, y)$ we find that $\Delta u < 0$ and hence u is superharmonic in D □

6. Concluding Remarks.

For the examples (1.3) of §1,

$$\Re e \frac{h''}{h'} = \Re e\, \frac{\gamma - 1}{\zeta + 1} > 0.$$

for $1 < \gamma < 2$ so that by (2.13) these have concave domains.

Furthermore, using (2.13), this shows that Theorem 1 is sharp. Regarding the constant K in Theorem 1, the scaling factor k in (3.1) is consistent with the fact that κ would be rescaled by replacing $u(x, y)$ by $cu(x/c, y/c)$ for $0 < c < \infty$.

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