Lesson 1

Grading	2 midterms	200
	Homework	100
	Final	150

Matrices $m \times n$ matrix A

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & & \cdots & a_{2n} \\ & \vdots & & \\ a_{m1} & & \cdots & a_{mn} \end{pmatrix} \leftarrow m \text{ rows}$$

$\uparrow$ n column

<u>Shorthand</u> <u>notation</u>

$$A = (a_{ij})$$

a_{ij} = element in i^{th} row and j^{th} column

A is a <u>square matrix</u> if $m = n$.

<u>Transpose</u> A^T switches rows and columns.

<u>Ex</u> $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$ $\quad A^T = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$

If $A^T = A$, A is called symmetric.

Thus, A is symmetric if and only if $a_{ij} = a_{ji}$ (and in particular this implies that A is a square matrix)

If $A^T = -A$, A is skew-symmetric

Addition of matrices

Ex. $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} + \begin{pmatrix} -1 & 2 & -3 \\ -4 & 5 & -6 \end{pmatrix} = \begin{pmatrix} 0 & 4 & 0 \\ 0 & 10 & 0 \end{pmatrix}$

Multiplication by scalar

Ex. $2\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} = \begin{pmatrix} 2 & 4 & 6 \\ 8 & 10 & 12 \end{pmatrix}$.

It is easy to see that

$$(A+B)^T = A^T + B^T$$

$$(cA)^T = cA^T$$

$$(A^T)^T = A$$

Matrix multiplication

$A = (a_{ij})$ $m \times n$ matrix

$B = (b_{ij})$ $n \times p$ matrix

Then $AB = C$ where $C = (c_{ij})$ and

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} .$$

Thus, C is an $m \times p$ matrix. In order for AB to be defined, # columns of A = # rows of B.

The formula for c_{ij} is useful for theory, but in practice we view the rows of A as row vectors and columns of B as column vectors and use dot products:

eg. $a = (a_1, \ldots, a_n)$ $b = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$

$$a \cdot b = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n .$$

Thus, the formula for C can be visualized as follows.

$$A = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{pmatrix} \qquad \left(\begin{array}{l} a_1 = \text{first row of } A \\ a_2 = \text{second} \quad " \quad " \\ \quad \text{etc} \end{array} \right)$$

$$B = (b_1 \cdots b_p) \qquad \left(\begin{array}{l} b_1 = \text{first column of } B \\ b_2 = \text{second} \quad " \quad " \\ \quad \text{etc} \end{array} \right)$$

$$AB = \begin{pmatrix} a_1 \cdot b_1 & a_1 \cdot b_2 & \cdots & a_1 \cdot b_p \\ a_2 \cdot b_1 & \cdots & \cdots & a_2 \cdot b_p \\ & \vdots & & \\ a_m \cdot b_1 & \cdots & \cdots & a_m \cdot b_p \end{pmatrix}$$

Ex.

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 4 & 2 \\ 10 & 5 \\ 16 & 8 \end{pmatrix}$$

Ex.

$$(1 \ \ 2 \ \ 3) \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} = 32$$

$$\begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} (1 \ \ 2 \ \ 3) = \begin{pmatrix} 4 & 8 & 12 \\ 5 & 10 & 15 \\ 6 & 12 & 18 \end{pmatrix}$$

If the dimensions are appropriate so the operations are defined, matrix multiplication is <u>associative</u>:

$$(AB)C = A(BC).$$

However it is <u>not</u> <u>commutative</u>:

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 3 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} -2 & -2 \\ 3 & 4 \end{pmatrix}$$

The rule for transposes is $(AB)^T = B^T A^T$.

The $n \times n$ <u>identity</u> <u>matrix</u> I is

$$I = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ & & \ddots & & \\ & 0 & & \ddots & 1 \end{pmatrix}.$$

Then if A is $n \times n$,

$$AI = IA = A.$$

An $n \times n$ matrix $A = (a_{ij})$ is <u>upper triangular</u> if

$a_{ij} = 0$ when $j < i$

<u>lower triangular</u> if $a_{ij} = 0$ when $j > i$.

Ex. Suppose A, B are both symmetric. Question: when is AB symmetric?

① If AB is symmetric, then $(AB)^T = AB$. But $(AB)^T = B^T A^T = BA$ also. Thus A and B commute.

② Conversely, suppose A and B commute. Then

$$(AB)^T = (BA)^T = A^T B^T = AB$$

Putting ① and ② together we find that for symmetric A and B,

AB symmetric if and only if A and B commute.

Note that A is diagonal if and only if it is upper triangular and lower triangular.