Linear systems of differential equations

Review of linear differential equations with constant coefficients. An n^{th} order equation is

$$a_n \frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + \cdots + a_1 \frac{dy}{dt} + a_0 y = 0$$

homogeneous

Characteristic polynomial

$$p(r) = a_n r^n + a_{n-1} r^{n-1} + \cdots + a_0$$

Roots $r_1, \ldots, r_k$ (may be repeated - may be complex)

Associated with a root r of order l we get solutions

$$e^{rt}, te^{rt}, \ldots, t^{l-1}e^{rt}$$

If r is complex then the pair $r = a+bi$ and $a-bi$ generate solutions

$$e^{at}\cos bt,\ e^{at}\sin bt,\ te^{at}\cos bt,\ te^{at}\sin bt,$$

$$\cdots\ t^{l-1}e^{at}\cos bt,\ t^{l-1}e^{at}\sin bt.$$

The set of all functions $f_1, f_2, \ldots, f_n$ obtained this way is a basis for the vector space of solutions, ie any solution can be written as a linear combination

$$c_1 f_1(t) + c_2 f_2(t) + \cdots + c_n f_n(t).$$

This expression is called the general solution.

The initial value problem is the differential equation along with initial conditions

$$y(t_0) = y_0$$

$$y'(t_0) = y_1$$

$$\vdots$$

$$y^{(n-1)}(t_0) = y_{n-1}$$

and using these, the $c_1, \dots, c_n$ can be solved for yielding a particular solution, the solution to the initial value problem.

Ex. $\frac{d^4y}{dt^4} - 2\frac{d^3y}{dt^3} + 2\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + y = 0$

$p(r) = r^4 - 2r^3 + 2r^2 - 2r + 1 = (r-1)^2(r^2+1)$

General solution

$$c_1 e^t + c_2 t e^t + c_3 \cos t + c_4 \sin t$$

We now move on to first order linear systems with constant coefficients

$$\frac{dx_1}{dt} = a_{11} x_1(t) + a_{12} x_2(t) \cdots + a_{1n} x_n(t)$$

$$\frac{dx_2}{dt} = a_{21} x_1(t) + \cdots + a_{2n} x_n(t)$$

$$\vdots$$

$$\frac{dx_n}{dt} = a_{n1} x_1(t) + \cdots + a_{nn} x_n(t).$$

This can be written in vector notation. If we write the vector function $x(t) = \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix}$

and matrix $A = (a_{ij})$, then system becomes

$$\frac{dx}{dt} = Ax.$$

Motivated by the case of equations we can try solutions $x = e^{\lambda t} v$ where λ is a constant number and v is a constant vector. Substituting into the equation we get

$$\lambda e^{\lambda t} v = A e^{\lambda t} v.$$

Now, $e^{\lambda t}$ is a (nonzero) number so it can be cancelled.

$$Av = \lambda v.$$

Now, $v = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$ always works. But if we want a nontrivial solution we take λ

as an eigenvalue and v a corresponding eigenvector.

The set of all solutions to the system, that is the vector functions $\begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix}$ satisfying the system is a <u>vector space</u> of dimension n. If $x^{(1)}(t), \ldots, x^{(n)}(t)$ is a basis, then any solution can be written

$$c_1 x^{(1)}(t) + c_2 x^{(2)}(t) + \cdots + c_n x^{(n)}(t).$$

This expression is the <u>general solution</u>.

<u>Ex.</u>

$$\frac{dx_1}{dt} = x_1 + 2x_2 \qquad\qquad \frac{dx}{dt} = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} x$$

$$\frac{dx_2}{dt} = 3x_2$$

$\lambda = 1$ $\qquad$ $\lambda = 3$ $\qquad$ General Soln.

$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\qquad$ $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\qquad$ $c_1 e^{t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$

n^{th} order equations can be converted to first order systems via bookkeeping.

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_0 y = f(t)$$

forcing function

This conversion works whether or not the a_n's are constant.

Bookkeeping

$x_1 = y$

$x_2 = \dfrac{dy}{dt}$

$x_3 = d^2y/dt^2$

$\vdots$

$x_n = d^{n-1}y/dt^{n-1}$

System

$$\frac{dx_1}{dt} = x_2$$

$$\frac{dx_2}{dt} = x_3$$

$$\vdots$$

$$\frac{dx_{n-1}}{dt} = x_n$$

$$\frac{dx_n}{dt} = -\frac{a_{n-1}}{a_n} x_n - \dots - \frac{a_0}{a_n} x_1 + \frac{f(t)}{a_n}$$

The system is

$$\frac{dx}{dt} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & \ddots & \\ 0 & \cdots & & & 1 \\ -\frac{a_0}{a_n} & -\frac{a_1}{a_n} & \cdots & & -\frac{a_{n-1}}{a_n} \end{pmatrix} x + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ \frac{f(t)}{a_n} \end{pmatrix}$$

$$\frac{dx}{dt} = Ax + F(t)$$

nonhomogeneous if $f(t) \neq 0$.

Ex. Find the general solution of

$$y''' + 2y'' - y' - 2y = 0$$

directly and by converting to a system.

Direct: $p(r) = r^3 + 2r^2 - 2r - 2 = (r+2)(r+1)(r-1)$

$$y = c_1 e^{-2t} + c_2 e^{-t} + c_3 e^{t}.$$

System

$$\frac{dx}{dt} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 1 & -2 \end{pmatrix} x$$

$$p(\lambda) = \begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 2 & 1 & -2-\lambda \end{vmatrix} = -\lambda\left(-\lambda(2+\lambda)-1\right) - 1(-2)$$

$$= -\lambda^3 - 2\lambda^2 + \lambda + 2$$

$\lambda_1 = -2$ $\qquad \lambda_2 = -1$ $\qquad \lambda_3 = 1$

$v_1 = \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix}$ $\qquad v_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ $\qquad v_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

General solution

$$X = c_1 e^{-2t}\begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + c_3 e^{t}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Picking off the first component of the general solution to the system (which is $y(t)$ by the bookkeeping)

$$y = c_1 e^{-2t} + c_2 e^{-t} + c_3 e^{t}$$

as before.