on 11

Phase plane analysis.

We consider $\frac{dy}{dt} = Ay$ where A is a 2×2 matrix. Now $y(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}$ may be thought of as the position of a particle at time t in the y_1, y_2 plane called the <u>phase plane</u>. The graph of $y(t)$ is the <u>trajectory</u> or <u>orbit</u>. Then, the equation gives the <u>velocity</u> vector on the left in terms of the <u>position</u> vector on the right.

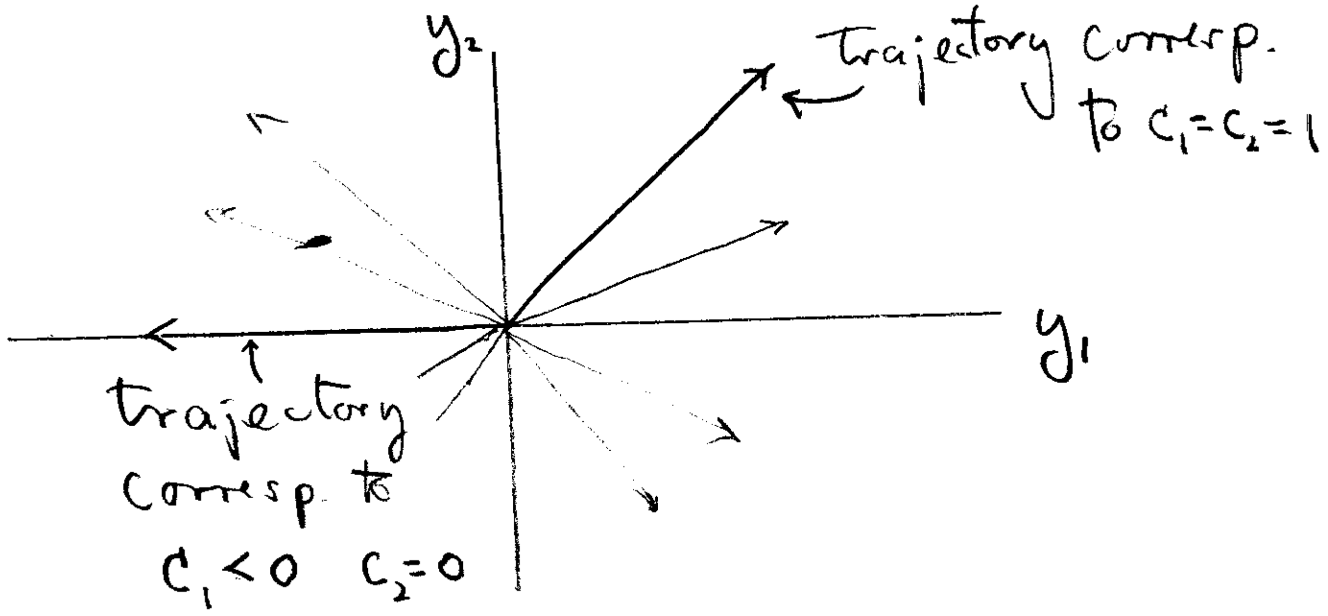
<u>Ex</u>. $\frac{dy}{dt} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} y$

$$\frac{dy_1}{dt} = y_1$$

$$\frac{dy_2}{dt} = y_2$$

Now, $\frac{dy_1}{dt} = y_1$ is one of the first differential equations in elementary courses (radioactive decay, population growth, etc) and has solution $y_1 = ce^t$. So the solutions to the system are

$$y = c_1 \begin{pmatrix} e^t \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ e^t \end{pmatrix}.$$



When all trajectories go away (such as here) or toward (if eg. $A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$) with limiting directio[n] the origin is a <u>proper</u> <u>node</u>.

The analysis of the "phase portrait" for a system centers on the critical points, ie those points $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ such that $Ay = 0$.

These are points in the "flow" where the particle stops (has 0 velocity). If A is nonsingular, this will only be the origin.

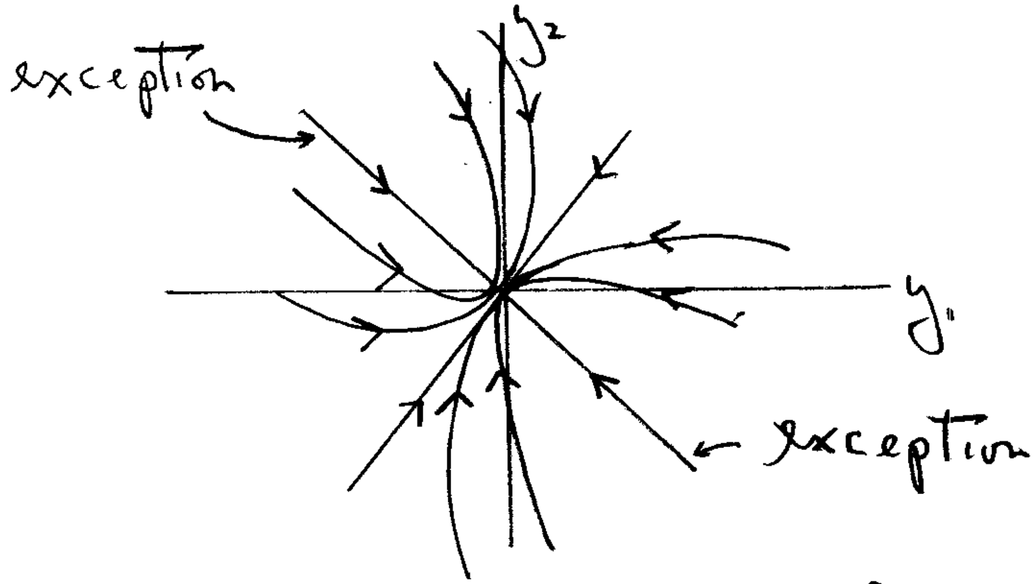
Ex. $\frac{dy}{dt} = Ay \qquad A = \begin{pmatrix} -3 & 1 \\ 1 & -3 \end{pmatrix}$

$$\lambda_1 = -2 \qquad\qquad \lambda_2 = -4$$

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad\qquad \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$y = c_1 e^{-2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-4t} \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Then the origin is called an im~~proper~~ node since all trajectories except 2 have the same limiting direction (in this example inward).

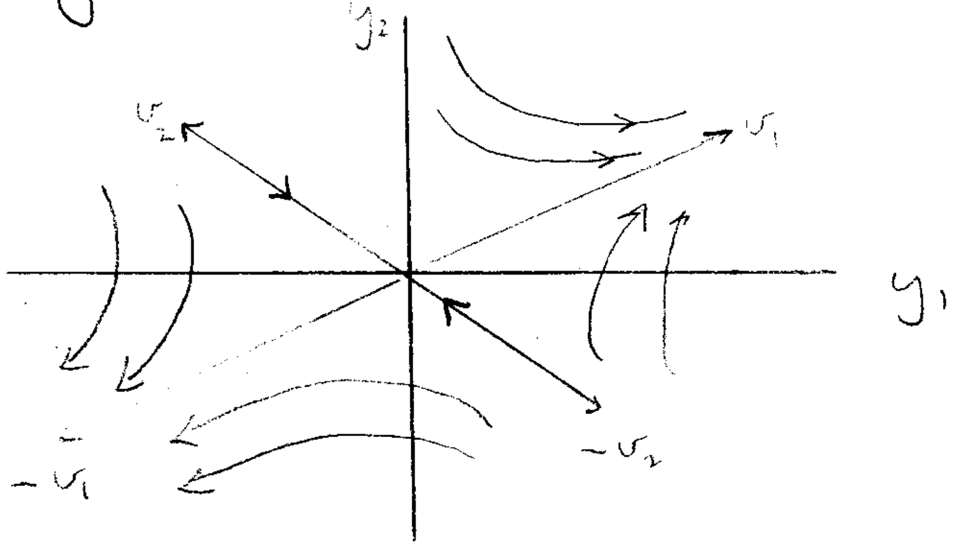


(all except 2 are tangent to line $y_1 = y_2$).

Ex. If the eigenvalues have different sign, the origin is a saddle point.

$$\frac{dy}{dt} = Ay \qquad \text{say } \lambda_1 > 0 \text{ eigenvector } v_1$$
$$\lambda_2 < 0 \quad \text{"} \quad \text{"} \quad v_2$$

Solution $y = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2$



Ex. If in $\frac{dy}{dt} = Ay$ A has complex roots $\lambda = a \pm bi$ with corresponding respective eigenvectors $x = u \pm iv$ where u is the real part of the vector x and v the imaginary part, then the system has complex basis vectors $e^{(a+bi)t}(u+iv)$ and $e^{(a-bi)t}(u-iv)$.

Now, Taking real and imaginary parts in the equation $dy/dt = Ay$ we see that the real part and imaginary parts are solutions. To find the real and imaginary parts of $e^{(a+bi)t}(u+iv)$ we write

$$e^{at}(\cos bt + i \sin bt)(u+iv).$$

Taking real part we get

$$e^{at}\left((\cos bt)u - (\sin bt)v\right)$$

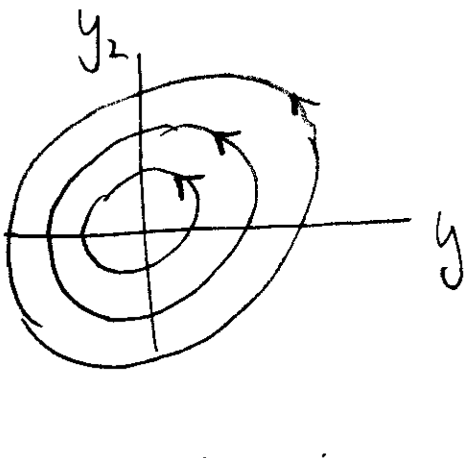
Taking imaginary parts

$$e^{at}\left((\cos bt)v + (\sin bt)u\right).$$

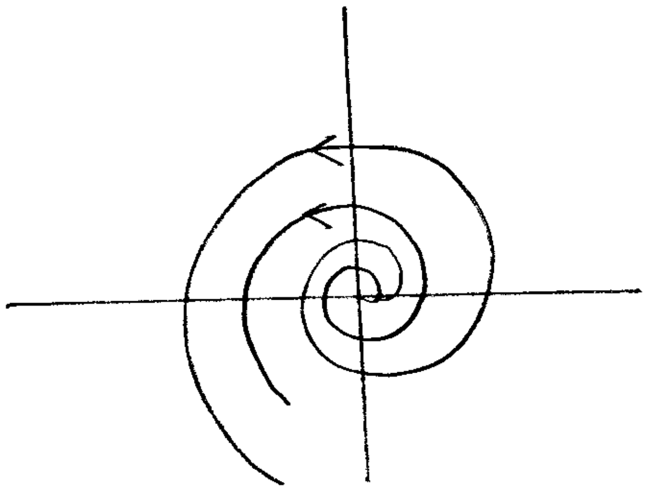
Thus the real general solution is

$$e^{at}\Big(C_1\big((\cos bt)u - (\sin bt)v\big) + C_2\big((\cos bt)v + (\sin bt)u\big)\Big)$$

If $a = 0$ (ie. λ pure imaginary) the origin is called a center and the trajectories are ellipses.



If $a \neq 0$ the origin is a spiral point



Ex. (Text p. 144) $\frac{dy}{dt} = \begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix} y$

$\lambda_1 = -1 + i \qquad \lambda_2 = -1 - i$

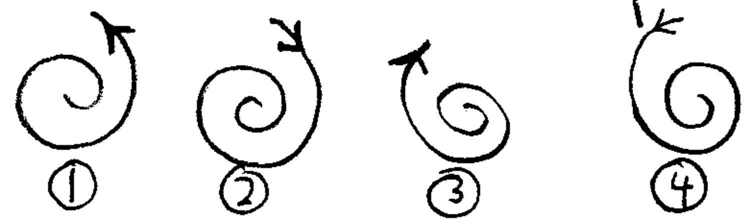
$\begin{pmatrix} 1 \\ i \end{pmatrix} \qquad \begin{pmatrix} 1 \\ -i \end{pmatrix}$

Using $a = -1$ $b = 1$ $u = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $v = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

we find the real general solution

$$e^{-t}\left(c_1\left((\cos t)\begin{pmatrix} 1 \\ 0 \end{pmatrix} - (\sin t)\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) + c_2\left((\cos t)\begin{pmatrix} 0 \\ 1 \end{pmatrix} + (\sin t)\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)\right)$$

How do we know which way the spiral goes??

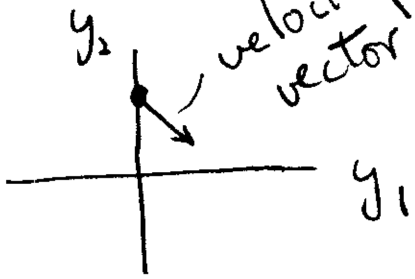


It's easy!

First, since $e^{-t} \to 0$ as $t \to \infty$, we can eliminate ① and ③.

Next, we take a convenient point, say $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ in the phase plane and check the velocity at that point by $dy/dt = Ay$

$$\begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$



so the answer is ②.

Ex. If in $dy/dt = Ay$ both eigenvalues are the same $\lambda_1 = \lambda_2 = \lambda$ then the origin is a degenerate node.

In this case, if v is a corresponding eigenvector, we get one basis solution

$$e^{\lambda t} v.$$

To get a second, we substitute

$$tve^{\lambda t} + ue^{\lambda t}$$

where u is unknown.

$$\lambda tve^{\lambda t} + ve^{\lambda t} + \lambda ue^{\lambda t} = A(tve^{\lambda t} + ue^{\lambda t}).$$

Since λ is an eigenvalue with eigenvector v,
$\lambda tve^{\lambda t} = Atve^{\lambda t}$ in the above eqn so

$$v\cancel{e^{\lambda t}} + \lambda u\cancel{e^{\lambda t}} = Au\cancel{e^{\lambda t}}$$

$$\Rightarrow \quad v = Au - \lambda u$$

$$\Rightarrow \quad (A - \lambda I)u = v$$

Recall that λ and v were given eigenvalue and eigenvector and u is unknown. We solve this system for u.