Lesson 12 

## Stability

We have been studying $\frac{dy}{dt} = Ay$ where $A$ is a $2\times2$ constant matrix.

This is the simplest example of an <u>autonomous system</u>. That is a system

$$\frac{dy_1}{dt} = f_1(y_1, y_2)$$

$$\frac{dy_2}{dt} = f_2(y_1, y_2)$$

where the formulas on the right hand side involve only position but not time. A critical point is a point $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ in the phase plane where $f_1(y_1, y_2) = f_2(y_1, y_2) = 0$. In the $\frac{dy}{dt} = Ay$ case, if $A$ is a nonsingular matrix, the origin is the only critical point.

Ex.
$$\frac{dy_1}{dt} = \cos y_1 \cos y_2 + y_1^2 - \frac{\pi}{2} y_1$$
$$\frac{dy_2}{dt} = y_2 \sin y_1$$

This is a nonlinear autonomous system. The points $\begin{pmatrix} 0 \\ \pi/2 \end{pmatrix}$ and $\begin{pmatrix} \pi/2 \\ 0 \end{pmatrix}$ are critical points but $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is not.

An important concept in the study of dynamics is stability. Roughly speaking, a critical point is <u>stable</u> if whenever a particle begins near enough to the point, it stays near the point. It is <u>stable and attractive</u> if whenever it begins near enough to the point, it tends to the point in the limit.

However, if it is not stable, then it is called <u>unstable</u>.

Consider the cases from our previous lesson for $\frac{dy}{dt} = Ay$. Let $\lambda_1, \lambda_2$ be eigenvalues of $A$, with corresponding eigenvectors $v_1, v_2$.

① $\begin{pmatrix}0\\0\end{pmatrix}$ Proper Node. $\lambda_1 = \lambda_2 = \lambda$ and $A$ nondefective. General solution

$$c_1 e^{\lambda t} v_1 + c_2 e^{\lambda t} v_2$$

where $v_1, v_2$ are basis vectors for the eigenspace.

This can be written $e^{\lambda t}(c_1 v_1 + c_2 v_2)$.

<u>Stable</u> <u>and attractive</u> if $\lambda < 0$

<u>Unstable</u> if $\lambda > 0$

② $\begin{pmatrix}0\\0\end{pmatrix}$ Improper Node. $\lambda_1 > \lambda_2 > 0$ or $\lambda_1 < \lambda_2 < 0$. Looking at

$$c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2$$

we find that the origin is

Stable and attractive if $\lambda_1 < \lambda_2 < 0$.

Unstable if $\lambda_1 > \lambda_2 > 0$.

③ $\begin{pmatrix}0\\0\end{pmatrix}$ Saddle point $\lambda_1 > 0$, $\lambda_2 < 0$.

Then the origin is unstable.

④ $\begin{pmatrix}0\\0\end{pmatrix}$ Center $\lambda = \pm ci$

The trajectories are ellipses around the origin, so the origin is stable.

(5) $\begin{pmatrix}0\\0\end{pmatrix}$ Spiral point. $\lambda = a \pm bi \quad a \neq 0$

$$c_1 e^{(a+bi)t} v_1 + c_2 e^{(a-bi)t} v_2$$

$$= e^{at}\left( c_1 e^{ibt} v_1 + c_2 e^{-ibt} v_2 \right)$$

so origin is

stable and attractive if $\operatorname{Re}\lambda < 0$

unstable if $\operatorname{Re}\lambda > 0$.

(6) $\begin{pmatrix}0\\0\end{pmatrix}$ Degenerate Node. $\lambda_1 = \lambda_2 = \lambda$

and a 1-dimensional eigenspace with basis vector $v$. The general solution is

$$c_1 e^{\lambda t} v + c_2 \left( t e^{\lambda t} v + u e^{\lambda t} \right),$$

where $u$ is called a "generalized eigenvector". We see that the origin is

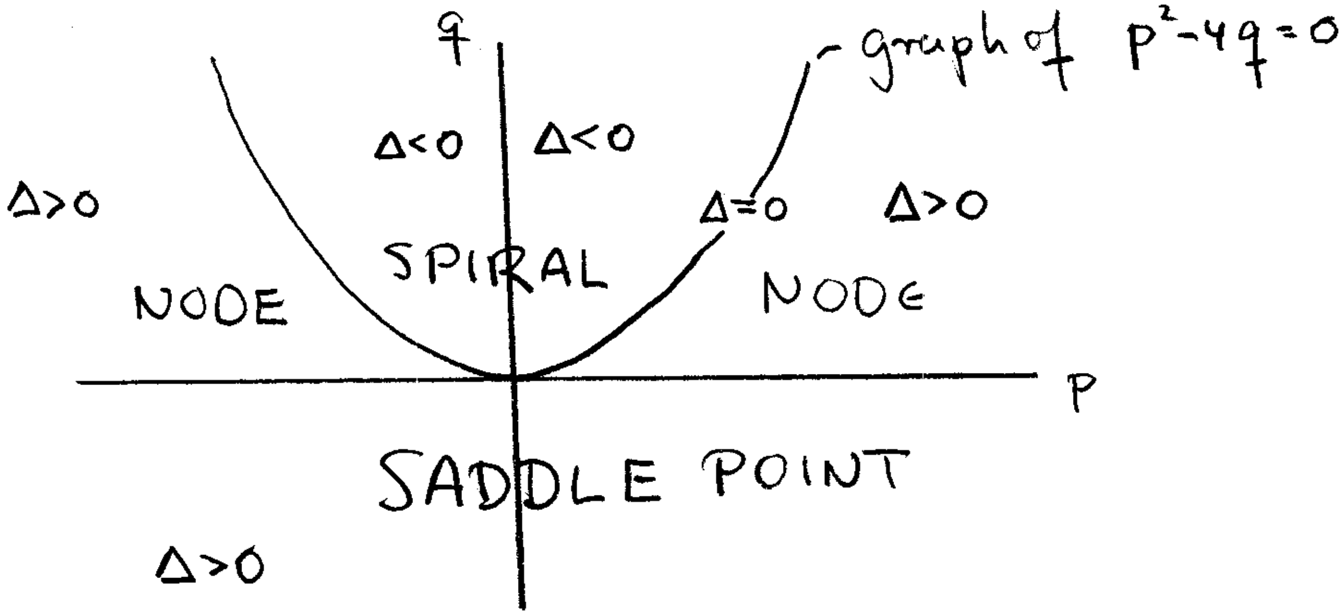
stable and attractive if $\lambda < 0$

unstable if $\lambda > 0$.

There is a schematic which summarizes these ideas.

$$\frac{dy}{dt} = Ay \qquad A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

$$p = \operatorname{tr} A \qquad q = \det A \qquad \Delta = p^2 - 4q$$




Origin

a) Stable and Attractive if $p<0$ and $q>0$

b) Stable if $p \leq 0$ and $q>0$

c) Unstable $p>0$ or $q<0$.

Ex. $\frac{dy}{dt} = \begin{pmatrix} 0 & 1 \\ -9 & 6 \end{pmatrix} y$

$\lambda = 3$ mult. 2

$A - \lambda I = \begin{pmatrix} -3 & 1 \\ -9 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \qquad x = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

defective.

One basis solution is

$$e^{3t} \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

To find another, substitute $te^{3t}\begin{pmatrix} 1 \\ 3 \end{pmatrix} + e^{3t} u$

$$\frac{dy}{dt} = \cancel{e^{3t}}\begin{pmatrix} 1 \\ 3 \end{pmatrix} + 3t\cancel{e^{3t}}\begin{pmatrix} 1 \\ 3 \end{pmatrix} + 3\cancel{e^{3t}} u$$

$$= A\left(t\cancel{e^{3t}}\begin{pmatrix} 1 \\ 3 \end{pmatrix} + \cancel{e^{3t}} u\right) = 3t\cancel{e^{3t}}\begin{pmatrix} 1 \\ 3 \end{pmatrix} + \cancel{e^{3t}} Au$$

$$\begin{pmatrix} 1 \\ 3 \end{pmatrix} + 3u = Au$$

$$(A - \lambda I) u = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \qquad \lambda = 3$$

$$\begin{pmatrix} -3 & 1 \\ -9 & 3 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \qquad u = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$y = c_1 e^{3t}\begin{pmatrix} 1 \\ 3 \end{pmatrix} + c_2\left(te^{3t}\begin{pmatrix} 1 \\ 3 \end{pmatrix} + e^{3t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) \quad \underline{\text{unstable}}$$