 (Nice software at www.math.uprm.edu/~portna/difeq/phase.html)

Nonlinear autonomous systems

We now explore general autonomous systems using the linear theory to approximate the dynamics in neighborhoods of critical points. This enables us to qualitatively do a phase plane analysis.

We have

$$\frac{dy_1}{dt} = f_1(y_1, y_2)$$

$$\frac{dy_2}{dt} = f_2(y_1, y_2).$$

Suppose that $\begin{pmatrix} a \\ b \end{pmatrix}$ is a critical point. For simplicity it is convenient to make

Change of variables $\tilde{y}_1 = y_1 - a \quad \tilde{y}_2 = y_2 - b$

so the system can be written

$$\frac{d\tilde{y}_1}{dt} = f_1(\tilde{y}_1 + a, \tilde{y}_2 + b) = \tilde{f}_1(\tilde{y}_1, \tilde{y}_2)$$

$$\frac{d\tilde{y}_2}{dt} = f_2(\tilde{y}_1 + a, \tilde{y}_2 + b) = \tilde{f}_2(\tilde{y}_1, \tilde{y}_2).$$

From multivariable calculus we know that if $f(x,y)$ has continuous partials and $f(0,0) = 0$ (like $\tilde{f}_1, \tilde{f}_2$ above) then

$$f(x,y) = \underbrace{a_1 x + a_2 y}_{\text{first order approximation}} + \text{higher order terms}$$

Since the higher order terms involve higher powers, they are insignificant as $(x,y) \to (0,0)$.

Also, $\quad a_1 = \frac{\partial f}{\partial x}(0,0) \quad a_2 = \frac{\partial f}{\partial y}(0,0).$

We now use this first order approximation in conjunction with the linear theory we have developed.

Ex. A very famous elementary problem is the predator - prey model.

y_1 = rabbit population y_2 = fox population

$$\frac{dy_1}{dt} = ay_1 - by_1y_2$$

$$\frac{dy_2}{dt} = ky_1y_2 - ly_2$$

a, b, k, l positive constants

First we must find the critical points:

$$ay_1 - by_1y_2 = 0$$
$$ky_1y_2 - ly_2 = 0$$

$\Longrightarrow$ $y_1(a - by_2) = 0$ and $y_2(ky_1 - l) = 0$

so we have $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} l/k \\ a/b \end{pmatrix}$.

The point $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is easiest since we do not need to translate to $\tilde{y}_1$ $\tilde{y}_2$ coordinates. The linear approximation is

$$\frac{dy_1}{dt} = a y_1$$

$$\frac{dy_2}{dt} = -l y_2$$

so

$$A = \begin{pmatrix} a & 0 \\ 0 & -l \end{pmatrix} \qquad \lambda_1 = a \quad \lambda_2 = -l$$

and we have a saddle point.

for $\begin{pmatrix} l/k \\ a/b \end{pmatrix}$, $\quad \tilde{y}_1 = y_1 - l/k \quad \tilde{y}_2 = y_2 - \frac{a}{b}$,

and

$$\frac{d\tilde{y}_1}{dt} = a(\tilde{y}_1 + l/k) - b(\tilde{y}_1 + l/k)(\tilde{y}_2 + a/b)$$

$$= (\tilde{y}_1 + l/k)(-b\tilde{y}_2)$$

$$\frac{d\tilde{y}_2}{dt} = k(\tilde{y}_1 + l/k)(\tilde{y}_2 + a/b) - l(\tilde{y}_2 + a/b)$$

$$= (\tilde{y}_2 + a/b)\, k\tilde{y}_1$$

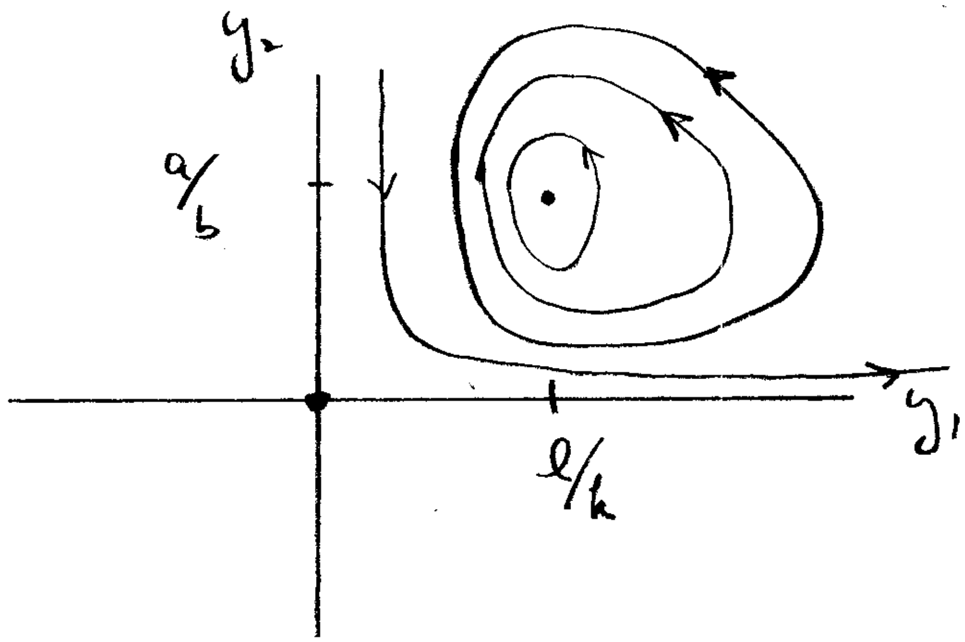
The linearized system is then

$$\frac{d\tilde{y}_1}{dt} = -\frac{lb}{k}\tilde{y}_2$$

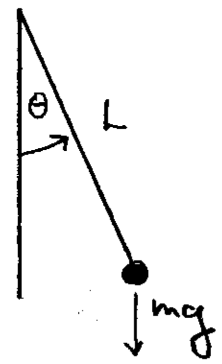
$$\frac{d\tilde{y}_2}{dt} = \frac{ak}{b}\tilde{y}_1$$

so $\quad \lambda = \pm i\sqrt{al}$

This gives us an idea of how the phase plane might look. In reality it looks like



Ex. Another important example is the undamped pendulum.



$$\frac{d^2\theta}{dt^2} + k\sin\theta = 0 \qquad k = g/L.$$

$y_1 = \theta$

$y_2 = \frac{d\theta}{dt}$

$$\frac{dy_1}{dt} = y_2$$

$$\frac{dy_2}{dt} = -k\sin y_1$$

Critical points $\begin{pmatrix} n\pi \\ 0 \end{pmatrix}$ $\quad n = 0, \pm 1, \pm 2, \ldots$

Consider $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$. The linear approximation

is
$$\frac{dy_1}{dt} = y_2$$
$$\frac{dy_2}{dt} = -ky_1 \qquad \left(\frac{\partial f_2}{\partial y_1}(0,0) = -k,\ \frac{\partial f_2}{\partial y_2}(0,0) = 0\right)$$

Thus, the linear approximation has $A = \begin{pmatrix} 0 & 1 \\ -k & 0 \end{pmatrix}$ and the origin is a center.

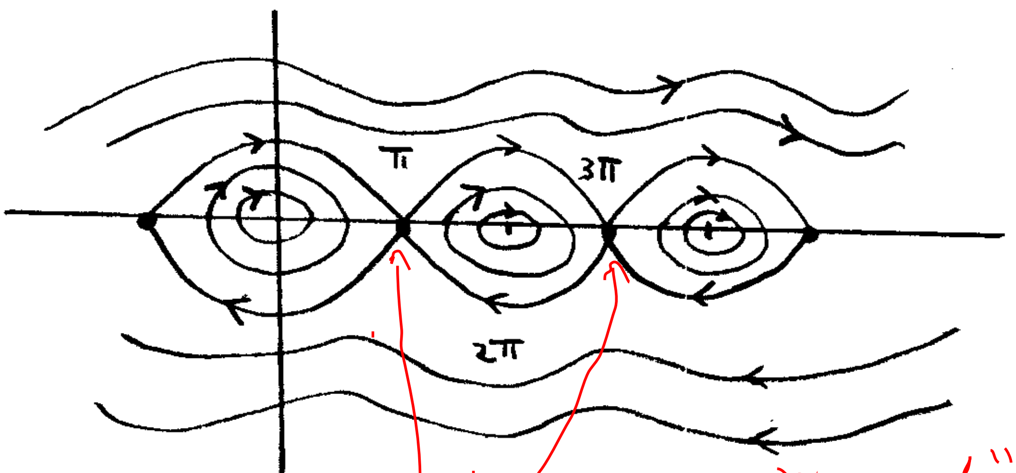
For $\begin{pmatrix} \pi \\ 0 \end{pmatrix}$

$$\frac{d\tilde{y}_1}{dt} = \tilde{y}_2$$

$$\frac{d\tilde{y}_2}{dt} = -k \sin(\tilde{y}_1 + \pi)$$

Now the approximating A is $\begin{pmatrix} 0 & 1 \\ k & 0 \end{pmatrix}$ which gives a saddle point.

If we continue this analysis we find that the centers and saddle points alternate, and we get a phase portrait



No Trajectories "pass through" these points