Lesson 14

Nonhomogeneous linear systems —
Variation of Parameters

We consider the system

$$\frac{dy}{dt} = Ay + g(t)$$

"forcing function"

$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$. For the theory, a_{ij}'s may also be functions of t, but for solving we would have no methods.

We first consider the homogeneous equation

$$\frac{dy}{dt} = Ay$$

and let Y be a 2×2 matrix whose columns are a basis for the solution space of the homogeneous equation.

<u>Ex.</u> $\frac{dy}{dt} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} y \qquad Y = \begin{pmatrix} e^t & 0 \\ 0 & e^{2t} \end{pmatrix}$

$\lambda = 1 \qquad\qquad \lambda = 2$

$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad\qquad \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

The general solution to the homogeneous solution will be denoted by $y^{(h)}$. Here

$$y^{(h)}(t) = c_1 e^t \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The matrix Y above is called a <u>fundamental</u> <u>matrix</u>.

Now suppose somehow we can find one solution to the nonhomogeneous eqn.

<u>Ex.</u> $\frac{dy}{dt} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} y + \begin{pmatrix} e^t \\ e^t \end{pmatrix}$

Then one particular solution which we denote by $y^{(p)}$ might be

$$y^{(p)} = \begin{pmatrix} te^t \\ -e^t \end{pmatrix}$$

$$\left(\text{Check: } \frac{dy^{(p)}}{dt} = \begin{pmatrix} te^t + e^t \\ -e^t \end{pmatrix} \stackrel{?}{=} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} te^t \\ -e^t \end{pmatrix} + \begin{pmatrix} e^t \\ e^t \end{pmatrix} \right)$$

The general solution to the nonhomogeneous equation is then $y = y^{(h)} + y^{(p)}$.

Since we have studied how to find $y^{(h)}$ (at least in the constant coefficient case), it remains to find a method for $y^{(p)}$.

There are 3 popular methods

i) undetermined coefficients

ii) variation of parameters

iii) diagonalization

In special cases undetermined coefficients is simpler, but in general the method of variation of parameters is more powerful.

We shall only study the method of variation of parameters. Our objective is to find a $y^{(p)}$.

The first step is to solve for $y^{(h)}$ and construct a fundamental matrix Y.

After we do this, our solution $y^{(p)}$ can then be computed by the variation of parameters formula:

$$\boxed{y^{(p)} = Y \int Y^{-1}(t)\, g(t)\, dt}$$

Check: Our equation is $\frac{dy}{dt} = Ay + g$ and Y satisfies

$$\frac{dY}{dt} = AY$$

$$\frac{dy^{(p)}}{dt} = \frac{dY}{dt}\int Y^{-1}(t)g(t)\,dt + Y(t)Y^{-1}(t)g(t)$$

$$= A\underbrace{Y\int Y^{-1}(t)g(t)\,dt}_{y^{(p)}} + I\,g(t)$$

$$= A\,y^{(p)}(t) + g(t) \quad \checkmark$$

Consider the previous example.

$$\frac{dy}{dt} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} y + \begin{pmatrix} e^t \\ e^t \end{pmatrix}.$$

$$Y = \begin{pmatrix} e^t & 0 \\ 0 & e^{2t} \end{pmatrix} \qquad Y^{-1} = \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{pmatrix}$$

$$\int Y^{-1}(t)g(t)\,dt = \int \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{pmatrix}\begin{pmatrix} e^t \\ e^t \end{pmatrix} dt$$

$$= \int \begin{pmatrix} 1 \\ e^{-t} \end{pmatrix} dt = \begin{pmatrix} t \\ -e^{-t} \end{pmatrix} \qquad (+\,C)$$

So

$$y^{(p)} = \begin{pmatrix} e^t & 0 \\ 0 & e^{2t} \end{pmatrix} \int Y^{-1}(t)\, g(t)\, dt$$

$$= \begin{pmatrix} e^t & 0 \\ 0 & e^{2t} \end{pmatrix} \begin{pmatrix} t \\ -e^{-t} \end{pmatrix} = \begin{pmatrix} te^t \\ -e^t \end{pmatrix}.$$

The general solution is then

$$y^{(h)} + y^{(p)}$$

$$= c_1 e^t \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} te^t \\ -e^t \end{pmatrix}$$