Lesson 15

Review

(1) Solve system $Ax=0$ by row operations

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 6 & 7 & 4 & 9 \\ 1 & 2 & 3 & 5 & 5 \end{pmatrix}$$

$Ax = 0$

$$A \to \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & -4 & -8 & -16 & -16 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \qquad x_3, x_5 \text{ free}$$

Basis for solution space (null space)

Take $x_3 = 0 \quad x_5 = 1$

$\Rightarrow \quad x_4 = 0 \qquad x_2 = -4 \qquad x_1 = 8 - 5 = 3$

Take $x_3 = 1 \quad x_5 = 0$

$\Rightarrow \quad x_4 = 0 \qquad x_2 = -2 \qquad x_1 = 4 - 3 = 1$

<u>Basis</u>

$$\begin{pmatrix} 3 \\ -4 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

The rank of A is 3

The nullity of A is 2

$$3 + 2 = \# \text{ col's of } A = 5$$

✓

(2) $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 3 & 6 \\ 0 & 0 & 3 \end{pmatrix}$

Eigenvalues

$\lambda = 1$ (alg. mult. 1) $\qquad \lambda = 3$ (alg. mult 2)

$A - \lambda I = \begin{pmatrix} 0 & 2 & 3 \\ 0 & 2 & 6 \\ 0 & 0 & 2 \end{pmatrix} \qquad \begin{pmatrix} -2 & 2 & 3 \\ 0 & 0 & 6 \\ 0 & 0 & 0 \end{pmatrix}$

Basis for eigenspace $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

$\lambda = 3$ has geometric mult. 1

A is defective.

A is not diagonalizable.

Computing A^{-1},

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 3 & 6 & 0 & 1 & 0 \\ 0 & 0 & 3 & 0 & 0 & 1 \end{array}\right) \rightarrow \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1/3 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1/3 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1/3 & -2/3 \\ 0 & 0 & 1 & 0 & 0 & 1/3 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -2/3 & 1/3 \\ 0 & 1 & 0 & 0 & 1/3 & -2/3 \\ 0 & 0 & 1 & 0 & 0 & 1/3 \end{array}\right)$$

Notice that the eigenvalues of A^{-1} here are reciprocals of those of A. Is that always the case?

Answer – yes. To see this, suppose λ is an eigenvalue of A nonsingular. Then $\lambda \neq 0$,

$$\det(A - \lambda I) = 0$$

$$\Rightarrow \det((A - \lambda I)A^{-1}) = 0$$

$$\Rightarrow \quad \det\left(I - \lambda A^{-1}\right) = 0$$

$$\Rightarrow \quad \det\left(\tfrac{1}{\lambda} I - A^{-1}\right) = 0$$

$$\Rightarrow \quad \det\left(A^{-1} - \tfrac{1}{\lambda} I\right) = 0$$

So $\frac{1}{\lambda}$ is an eigenvalue of A^{-1}.

③ Change matrix slightly to

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

$\lambda = 1$ $\qquad\qquad\qquad\qquad$ $\lambda = 2$

$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ — basis for eigenspace — $\begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$

A is nondefective

Therefore, A is diagonalizable:

If $P = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$,

and $P^{-1}AP = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$.

(4) Consider the nonlinear system

$$\frac{dy_1}{dt} = y_1 - 2y_2 + y_1 y_2 - 2$$

$$\frac{dy_2}{dt} = y_1 - 2y_2$$

Find critical points:

$$y_1 - 2y_2 + y_1 y_2 - 2 = 0$$
$$y_1 - 2y_2 \qquad = 0$$

$$\Rightarrow \left.\begin{array}{l} y_1 - 2y_2 = 0 \\ y_1 y_2 = 2 \end{array}\right\} \Rightarrow \quad y_2 \neq 0 \text{ so we can divide}$$

$$\cancel{\frac{2}{y_2}} - \cancel{2}y_2 = 0 \Rightarrow y_2 = \pm 1 \Rightarrow$$

$(2, 1)$ $(-2, -1)$ critical points

at $(2,1)$: $\tilde{y}_1 = y_1 - 2$ $\tilde{y}_2 = y_2 - 1$

$$\frac{d\tilde{y}_1}{dt} = \hat{y}_1 + 2 - 2(\tilde{y}_2 + 1) + (\hat{y}_1 + 2)(\tilde{y}_2 + 1) - 2$$

$$\frac{d\tilde{y}_2}{dt} = \hat{y}_1 + 2 - 2(\tilde{y}_2 + 1)$$

$$\frac{d\tilde{y}_1}{dt} = 2\tilde{y}_1 + \tilde{y}_1\tilde{y}_2$$

$$\frac{d\tilde{y}_2}{dt} = \tilde{y}_1 - 2\tilde{y}_2$$

The linear approximation has $A = \begin{pmatrix} 2 & 0 \\ 1 & -2 \end{pmatrix}$

$\lambda_1 = 2$ $\qquad\qquad$ $\lambda_2 = -2$

Saddle Point $\qquad$ (unstable)

At $(-2,-1)$,

$$\frac{d\tilde{y}_1}{dt} = \tilde{y}_1 - 2 - 2(\tilde{y}_2 - 1) + (\tilde{y}_1 - 2)(\tilde{y}_2 - 1) - 2$$

$$\frac{d\tilde{y}_2}{dt} = \tilde{y}_1 - 2 - 2(\tilde{y}_2 - 1)$$

so
$$\frac{d\tilde{y}_1}{dt} = -4\tilde{y}_2 + \tilde{y}_1\tilde{y}_2$$

$$\frac{d\tilde{y}_2}{dt} = \tilde{y}_1 - 2\tilde{y}_2$$

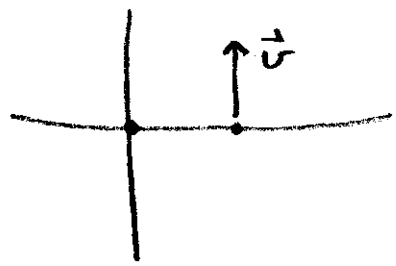
Approximating $A = \begin{pmatrix} 0 & -4 \\ 1 & -2 \end{pmatrix}$ and

$p(\lambda) = \lambda^2 + 2\lambda + 4$

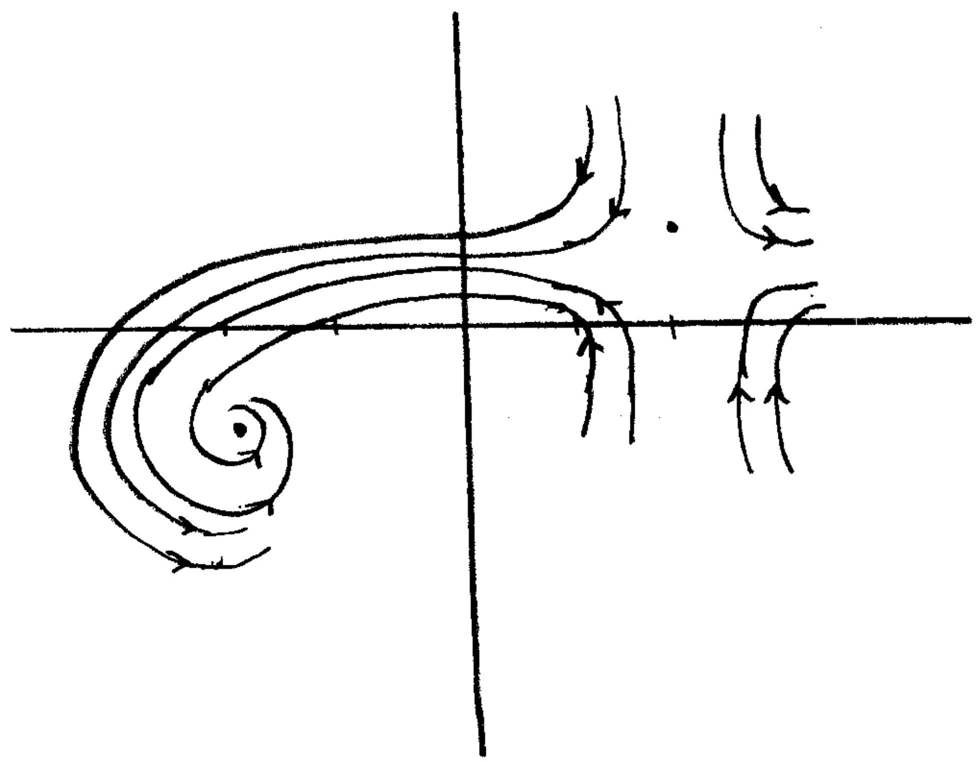
$\lambda = -1 \pm \sqrt{3}\, i$ Therefore, the critical point is a spiral (stable and attractive)

find direction on spiral

$$\frac{d\tilde{y}}{dt} = \begin{pmatrix} 0 & -4 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} \tilde{y}_1 \\ \tilde{y}_2 \end{pmatrix}$$



Consider point $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Velocity vector $\vec{v}$ is $\begin{pmatrix} 0 & -4 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ so spiral is counterclockwise



Let's find the real general solution to the linear approximation at $(-2,-1)$

$$A = \begin{pmatrix} 0 & -4 \\ 1 & -2 \end{pmatrix} \qquad \text{Take } \lambda = -1+\sqrt{3}\,i$$

Then, $A - \lambda I = \begin{pmatrix} 1-\sqrt{3}\,i & -4 \\ 1 & -1-\sqrt{3}\,i \end{pmatrix}$

The echelon form of $A - \lambda I$ is $\begin{pmatrix} 1-\sqrt{3}\,i & -4 \\ 0 & 0 \end{pmatrix}$

so an eigenvector is $\begin{pmatrix} 4 \\ 1-\sqrt{3}\,i \end{pmatrix}$

$$= \begin{pmatrix} 4 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ \sqrt{3} \end{pmatrix} i$$

$$y = e^{-t}\left[c_1\left(\cos\sqrt{3}\,t \begin{pmatrix} 4 \\ 1 \end{pmatrix} + \sin\sqrt{3}\,t \begin{pmatrix} 0 \\ \sqrt{3} \end{pmatrix} \right) + c_2\left(-\cos\sqrt{3}\,t \begin{pmatrix} 0 \\ \sqrt{3} \end{pmatrix} + \sin\sqrt{3}\,t \begin{pmatrix} 4 \\ 1 \end{pmatrix} \right) \right]$$

(6) Convert the equation $y'' - 9y + y^3 = 0$ to a system, find location and types of critical points.

$x_1 = y$

$x_2 = y'$

$$\frac{dx_1}{dt} = x_2$$

$$\frac{dx_2}{dt} = 9x_1 - x_1^3$$

$x_2 = 0$, $x_1(9 - x_1^2) = 0$. Critical points

$(0,0)$ $(3,0)$ $(-3,0)$

At $(0,0)$ approximating A is $\begin{pmatrix} 0 & 1 \\ 9 & 0 \end{pmatrix}$

so $\lambda = \pm 3$ and we have an unstable saddle point.

At $(3,0)$

$$\frac{d\tilde{x}_1}{dt} = \tilde{x}_2$$

$$\frac{d\tilde{x}_2}{dt} = 9(\tilde{x}_1 + 3) - (\tilde{x}_1 + 3)^3$$

$$= -18\tilde{x}_1 + \text{higher order terms}$$

$$A = \begin{pmatrix} 0 & 1 \\ -18 & 0 \end{pmatrix} \qquad \lambda = \pm 3\sqrt{2}\, i$$

<u>center</u> (<u>stable</u>)

at $(-3,0)$

$$\frac{d\tilde{x}_1}{dt} = \tilde{x}_2$$

$$\frac{d\tilde{x}_2}{dt} = 9(\tilde{x}_1 - 3) - (\tilde{x}_1 - 3)^3$$

$$= -18\tilde{x}_1 + \text{higher order terms}$$

so again

$$A = \begin{pmatrix} 0 & 1 \\ -18 & 0 \end{pmatrix}$$

and we have a <u>center</u> (<u>stable</u>).

(7) For the system

$$\frac{dx}{dt} = \cos(x+y) - xy^2$$

$$\frac{dy}{dt} = x^2 - y^2 ,$$

determine the linearized system at $(1,-1)$ and the type of critical point.

$\tilde{x} = x - 1 \quad \tilde{y} = y + 1$

$$\frac{d\tilde{x}}{dt} = \cos(\tilde{x}+\tilde{y}) - (\tilde{x}+1)(\tilde{y}-1)^2$$

$$\frac{d\tilde{y}}{dt} = (\tilde{x}+1)^2 - (\tilde{y}-1)^2$$

linearized system:

$$\left.\frac{\partial f_1}{\partial \tilde{x}}\right|_{(0,0)} = \left. -\sin(\tilde{x}+\tilde{y}) - (\tilde{y}-1)^2 \right|_{(0,0)} = -1$$

$$\left.\frac{\partial f_1}{\partial \tilde{y}}\right|_{(0,0)} = \left. -\sin(\tilde{x}+\tilde{y}) - 2(\tilde{x}+1)(\tilde{y}-1) \right|_{(0,0)} = 2$$

$$\begin{pmatrix} d\tilde{x}/dt \\ d\tilde{y}/dt \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix}$$

$$p(\lambda) = (-1-\lambda)(2-\lambda) - 4 = \lambda^2 - \lambda - 6$$

$$\lambda = \frac{1 \pm \sqrt{1+24}}{2} = 3, -2$$

saddle point, unstable