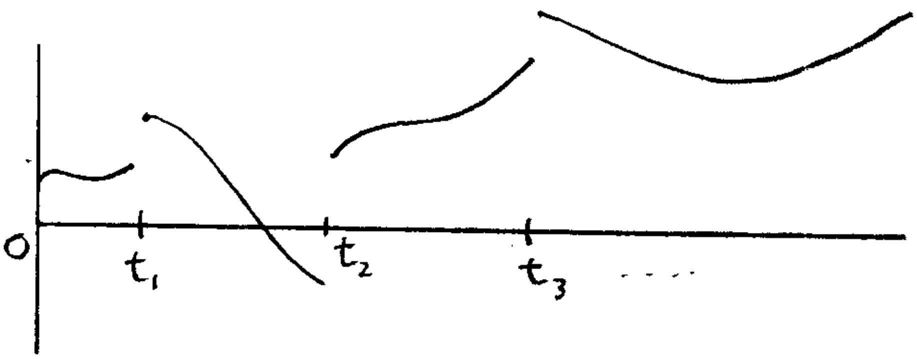
Lesson 17

Laplace Transforms

$f(t)$ piecewise continuous for $t \geq 0$



("one sided limits" at $t_1, t_2, \ldots$)

The Laplace transform $\mathcal{L}(f) = F(s)$ is

$$F(s) = \int_0^\infty e^{-st} f(t)\,dt, \quad (\text{if the integral converges})$$

Ex. $f(t) = e^t$, $F(s) = \int_0^\infty e^{-st} e^t \, dt$

$$= \int_0^\infty e^{t(1-s)}\,dt = \frac{1}{1-s} e^{t(1-s)} \Big|_0^\infty = \frac{1}{s-1}$$

for $s > 1$.

In order to be sure that the improper integral in the definition of $\mathcal{L}\{f\}$ converges we assume

$$|f(t)| < Me^{kt}$$

for some constants M and k. If this condition is not satisfied, we may not be able to use the Laplace transform. For example, e^{e^t} does not have a Laplace transform since $\int_0^\infty e^{-st} e^{e^t} dt$ diverges for every s.

The inverse Laplace transform $\mathcal{L}^{-1}(F(s))$ goes the other way.

Ex. $\mathcal{L}^{-1}\left(\frac{1}{s-1}\right) = e^t$.

One objective we have is to build a modest Library of Laplace transforms.

$f(t) = e^{at}$. As we did already with $a=1$ we get

$$\boxed{\mathcal{L}(e^{at}) = \frac{1}{s-a}} \qquad (s > a)$$

Also,

$$\mathcal{L}(1) = \int_0^\infty e^{-st}\,dt = \frac{1}{s} \qquad s>0$$

(of course this is the special case e^{at} $a=0$)

If we are willing to do calculus with complex numbers, we can simplify many computations such as $\mathcal{L}(\sin at)$ $\mathcal{L}(\cos at)$.

Since $\cos at = \operatorname{Re} e^{iat}$

$\sin at = \operatorname{Im} e^{iat}$

We can compute

$$\mathcal{L}(e^{iat}) = \int_0^{\infty} e^{-st} e^{iat}\, dt$$

$$= \frac{1}{s - ia} = \frac{s+ia}{s^2+a^2}$$

Taking real part, we get

$$\boxed{\mathcal{L}(\cos at) = \frac{s}{s^2+a^2}}.$$

Taking imaginary part

$$\boxed{\mathcal{L}(\sin at) = \frac{a}{s^2+a^2}}.$$

In addition to building a library we also have to develop the calculus of Laplace Transforms.

To begin with, $\mathcal{L}(f)$ is a linear transformation:

$$\mathcal{L}(f+g) = \mathcal{L}(f) + \mathcal{L}(g)$$
$$\mathcal{L}(cf) = c\,\mathcal{L}(f).$$

Next we observe the s-shifting property

Theorem. If $\mathcal{L}(f(t)) = F(s)$, then

$$\boxed{\mathcal{L}(e^{at} f(t)) = F(s-a)}.$$

Proof. $F(s-a) = \int_0^\infty e^{-(s-a)t} f(t)\,dt$

$$= \int_0^\infty e^{-st} e^{at} f(t)\,dt = \mathcal{L}(e^{at} f(t))$$

Ex. Find $\mathcal{L}^{-1}\left(\frac{s}{s^2-4s+13}\right)$

(Completing square for x^2+ax+b : $(x+\frac{a}{2})^2 - \frac{a^2}{4} + b$.)

$$= \mathcal{L}^{-1}\left(\frac{s}{(s-2)^2+9}\right) =$$

$$= \mathcal{L}^{-1}\left(\frac{s-2}{(s-2)^2+9} + \frac{2}{(s-2)^2+9}\right)$$

$$= \mathcal{L}^{-1}\left(\frac{s-2}{(s-2)^2+9}\right) + \frac{2}{3}\mathcal{L}^{-1}\left(\frac{3}{(s-2)^2+9}\right)$$

$$= e^{2t}\cos 3t + \tfrac{2}{3}e^{2t}\sin 3t.$$

We could compute $\mathcal{L}(\cosh(at))$ and $\mathcal{L}(\sinh(at))$ from their definitions

$$\boxed{\mathcal{L}(\cosh(at)) = \frac{s}{s^2-a^2}}$$

$$\boxed{\mathcal{L}(\sinh(at)) = \frac{a}{s^2-a^2}}.$$

The powers t^n $n = 0, 1, 2, \dots$ can be done using integration by parts.

$$\mathcal{L}(1) = 1/s$$

$$\mathcal{L}(t) = \int_0^\infty t e^{-st}\,dt$$

$$u = t \qquad dv = e^{-st}\,dt$$

$$du = dt \qquad v = -\frac{1}{s}e^{-st}$$

$$-\frac{1}{s}e^{-st}t \Big|_{t=0}^{t\to\infty} + \frac{1}{s}\int_0^\infty e^{-st}\,dt$$

$$= \quad 0 + \frac{1}{s}\mathcal{L}(1) = \frac{1}{s^2}$$

$$\mathcal{L}(t^2) = \int_0^\infty t^2 e^{-st}\,dt$$

$$u = t^2 \qquad dv = e^{-st}\,dt$$

$$du = 2t\,dt \qquad v = -\frac{1}{s}e^{-st}$$

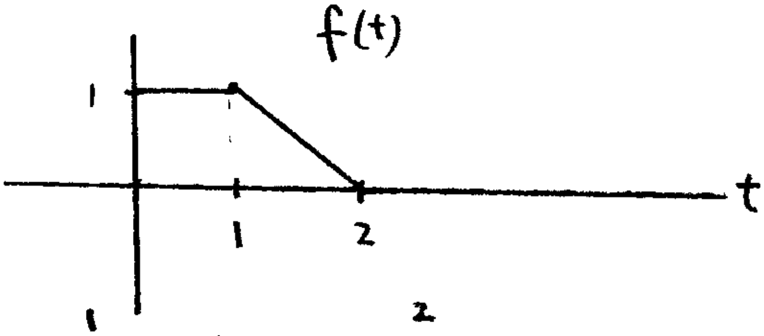
$$-\frac{1}{s}t^2 e^{-st} \Big|_{t=0}^{t\to\infty} + \frac{2}{s}\int t\, e^{-st}\,dt$$

$$= 0 + 2/s\,\mathcal{L}(t) = 2/s^3$$

Continuing this way we get

$$\boxed{\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}}.$$

Ex.



$$\mathcal{L}(f) = \int_0^1 e^{-st}\,dt + \int_1^2 (-t+2)\,e^{-st}\,dt$$

$$= -\frac{1}{s} e^{-st}\Big|_0^1 - \int_1^2 t\,e^{-st}\,dt - \frac{2}{s} e^{-st}\Big|_1^2$$

$$u = t \qquad dv = e^{-st}$$
$$du = dt \qquad v = -\frac{1}{s} e^{-st}$$

$$= \frac{1}{s}\left(1 - e^{-s}\right) + \frac{t}{s} e^{-st}\Big|_1^2 - \frac{1}{s}\int_1^2 e^{-st}\,dt - \frac{2}{s}\left(e^{-2s} - e^{-s}\right)$$

$$= \frac{1}{s} + \frac{e^{-2s}}{s^2} - \frac{e^{-s}}{s^2}$$