ssou 18

Laplace transforms and derivatives

If $f(t)$ is continuous for $t \geq 0$, $|f(t)| \leq Me^{kt}$ and f' is continuous,

$$\mathcal{L}(f') = s\mathcal{L}(f) - f(0).$$

<u>Proof</u> $\mathcal{L}(f') = \int_0^\infty e^{-st} f'(t)\,dt$

$$\left(\begin{array}{ll} u = e^{-st} & dv = f'(t)\,dt \\ du = -se^{-st} & v = f(t) \end{array} \right)$$

$$= e^{-st} f(t) \Big|_0^\infty + s\int e^{-st} f(t)\,dt$$

$$= -f(0) + s\mathcal{L}(f).$$

<u>Note</u>: This formula also applies if f' is piecewise continuous.

This formula can be iterated:

$$\mathcal{L}(f^{(n)}) = s^n \mathcal{L}(f) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - f^{(n-1)}(0).$$

This enables us to convert differential equations to algebraic equations:

Ex. $y'' + 4y = 0 \quad y(0) = 1 \quad y'(0) = 1$

$$s^2 Y(s) - s - 1 + 4Y(s) = 0$$

$$(s^2+4)Y = s + 1$$

$$Y = \frac{s}{s^2+4} + \frac{1}{s^2+4}$$

$$y = \cos 2t + \frac{1}{2}\sin 2t.$$

Associated with the differential equation

$$y'' + ay' + by = r(t)$$

we have

a transfer function $Q(s)$:

$$s^2Y - s\,y(0) - y'(0) + a(sY - y(0)) + bY = R(s)$$

$$Y = \frac{(s+a)\,y(0) + y'(0) + R(s)}{s^2 + as + b}$$

Define $Q(s) = \dfrac{1}{s^2 + as + b}$.

Then, if $y(0) = y'(0) = 0$,

$$Q = \frac{Y}{R} = \frac{\mathcal{L}(\text{output})}{\mathcal{L}(\text{input})}$$

Often to invert Laplace transforms we need partial fraction decomposition.

$\dfrac{P(x)}{Q(x)}$ $\deg P < \deg Q$, otherwise do long division.

In the partial fraction decomposition

linear terms $\frac{1}{(x-a)^k}$

generate terms of the form $\frac{A_1}{(x-a)} + \frac{A_2}{(x-a)^2} + \cdots + \frac{A_k}{(x-a)^k}$

and irreducible quadratic terms $\frac{1}{(x^2+ax+b)^k}$ generate

$$\frac{A_1 + B_1 x}{x^2+ax+b} + \frac{A_2 + B_2 x}{(x^2+ax+b)^2} + \cdots + \frac{A_k + B_k x}{(x^2+ax+b)^k}$$

Ex. $\mathcal{L}^{-1}\left(\frac{4s^3+7s+5}{s^2(s^2+2s+5)} \right)$

$$\frac{A_1}{s} + \frac{A_2}{s^2} + \frac{A_3 + A_4 s}{s^2+2s+5} = \frac{4s^3+7s+5}{s^2(s^2+2s+5)}$$

$$A_1 s(s^2+2s+5) + A_2(s^2+2s+5) + (A_3+A_4 s)s^2 = 4s^3+7s+5$$

$$5A_2 = 5 \qquad A_2 = 1$$

$$5A_1 + 2A_2 = 7 \qquad \Rightarrow \qquad A_1 = 1$$

$$2A_1 + A_2 + A_3 = 0 \qquad A_3 = -3$$

$$A_1 + A_4 = 4 \qquad A_4 = 3$$

So we have

$$\mathcal{L}^{-1}\left(1/s\right) + \mathcal{L}^{-1}\left(1/s^2\right) + \mathcal{L}^{-1}\left(\frac{-3+3s}{s^2+2s+5}\right)$$

$$\frac{-3+3s}{s^2+2s+5} = \frac{-3+3s}{(s+1)^2+4} = \frac{3(s+1)}{(s+1)^2+4} - \frac{6}{(s+1)^2+4}$$

We thus obtain

$$1 + t + 3e^{-t}\cos 2t - 3e^{-t}\sin 2t.$$

We can add to our library of Laplace Transforms by using the differentiation formula. From the differentiation formula, we get the integration formula.

$$\boxed{\mathcal{L}\left(\int_0^t f(\tau)\,d\tau\right) = \frac{1}{s}\mathcal{L}(f)}.$$

(just take $g(t) = \int_0^t f(\tau)\,d\tau$; then $g(0) = 0$ and this becomes $\mathcal{L}(g') = s\,\mathcal{L}(g)$.)

Now consider

$$\frac{1}{s}\;\frac{a}{s^2+a^2} = \frac{1}{s}\mathcal{L}(\sin at).$$

Then

$$\mathcal{L}^{-1}\left(\frac{1}{s}\;\frac{a}{s^2+a^2}\right) = \int_0^t \sin a\tau\,d\tau$$

$$= -\frac{1}{a}\cos a\tau\Big|_0^t = -\frac{1}{a}\left(\cos at - 1\right).$$

We can write this

$$\mathcal{L}^{-1}\left(\frac{1}{s(s^2+a^2)}\right) = \frac{1}{a^2}\left(1-\cos at\right).$$

Similarly,

$$\mathcal{L}^{-1}\left(\frac{1}{s^2(s^2+a^2)}\right) = \mathcal{L}^{-1}\left(\frac{1}{s}\,\frac{1}{s(s^2+a^2)}\right)$$

$$= \frac{1}{a^2}\int_0^t (1-\cos a\tau \, d\tau) = \frac{1}{a^2}\left(t - \frac{\sin at}{a}\right)$$

If we wish to solve a differential equation where the initial conditions are specified at a point other than $t=0$ we make a change of variables.

Ex. (Text)

$$y'' + y = 2t \qquad y(\pi/4) = \pi/2$$

$$y'(\pi/4) = 2 - \sqrt{2}$$

$$t = \tilde{t} + \pi/4$$

$$\tilde{y}'' + \tilde{y} = 2(\tilde{t} + \pi/4) \qquad \tilde{y}(0) = \pi/2 \qquad \tilde{y}'(0) = 2 - \sqrt{2}$$

$$s^2 \tilde{Y} - s\,\pi/2 - (2 - \sqrt{2}) + \tilde{Y} = \frac{2}{s^2} + \frac{\pi}{2}\frac{1}{s}$$

$$\vdots$$

$$\tilde{y}(\tilde{t}) = 2\tilde{t} + \frac{1}{2}\pi - \sqrt{2}\sin\tilde{t} \qquad \tilde{t} = t - \pi/4$$

$$y(t) = 2t - \sqrt{2}\sin(t - \pi/4)$$

$$= 2t - \sin t + \cos t$$