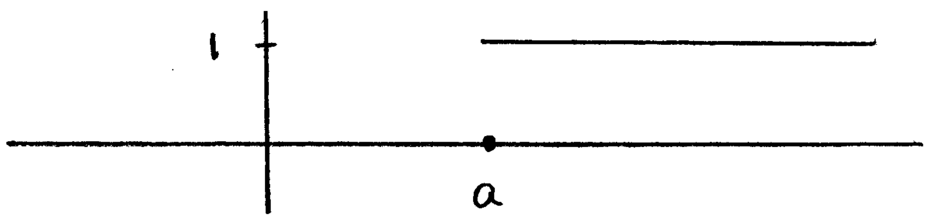
t shifting and δ function.

The unit step function $u(t)$ is given by

$$u(t) = \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases}$$

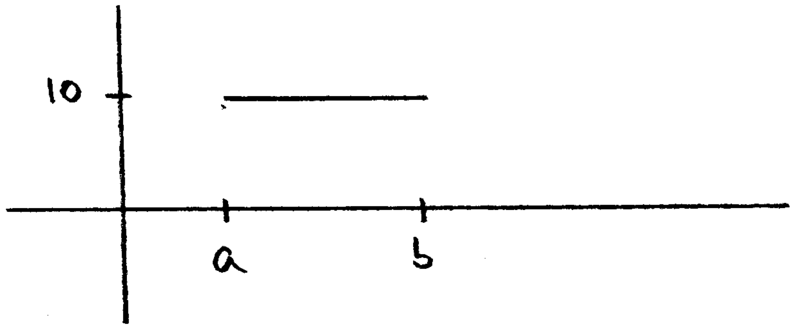
so that $u(t-a)$ has graph



The function $u(t)$ is called the Heaviside function and $u(t-a)$ is often denoted by $H(t-a)$ or $H_a(t)$.

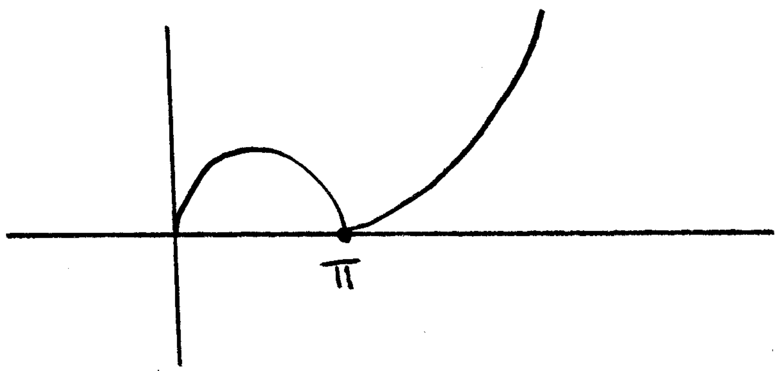
It can be used as a building block for common signals.

Ex.



$$10\,u(t-a) - 10\,u(t-b)$$

Ex.



$$(u(t) - u(t-\pi))\sin t + u(t-\pi)(t-\pi)^2$$

$$\mathcal{L}(u(t-a)) = \int_a^\infty e^{-st}\,dt = e^{-as}/s.$$

This illustrates our t-shift theorem.

Recall we have an s-shift theorem

$$\mathcal{L}(e^{at}f(t)) = F(s-a)$$

Our t-shift theorem says

$$\mathcal{L}\left(u(t-a)f(t-a)\right) = e^{-as}F(s).$$

In the above $f(t)=1$ so $F(s)=1/s$, and

$$\mathcal{L}\left(u(t-a)\right) = e^{-as}/s.$$

To prove the t-shift theorem,

$$\mathcal{L}\left(u(t-a)f(t-a)\right) = \int_a^\infty e^{-st} f(t-a)\,dt$$

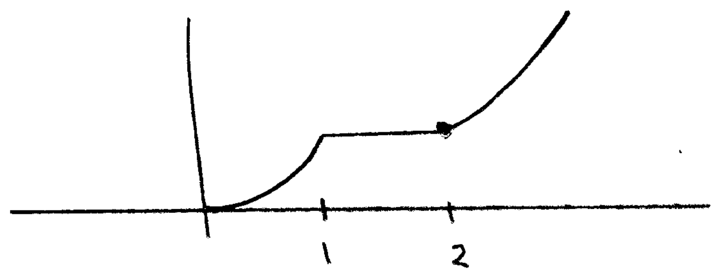
Let $\tau = t-a$; then we have $d\tau = dt$, and

$$\int_0^\infty e^{-s(\tau+a)} f(\tau)\,d\tau = e^{-as}\int_0^\infty e^{-s\tau} f(\tau)\,d\tau$$

$$= e^{-as}F(s).$$

Ex.

$$f(t) = \begin{cases} t^2 & 0 < t < 1 \\ 1 & 1 < t < 2 \\ e^{t-2} & 2 < t < \infty \end{cases}$$

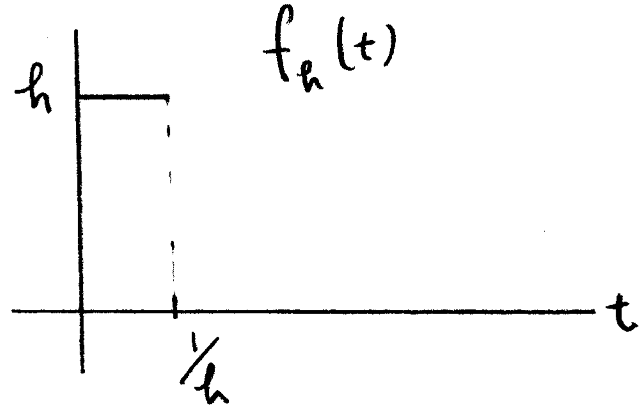


$$\mathcal{L}(f) = \mathcal{L}\Big(t^2 - u(t-1)t^2 + u(t-1) - u(t-2)$$
$$+ u(t-2)e^{t-2}\Big)$$

$$= \frac{2}{s^3} - e^{-s}\mathcal{L}((t+1)^2) + \frac{e^{-s}}{s} - \frac{e^{-2s}}{s}$$
$$+ \frac{e^{-2s}}{s-1}$$

$$\mathcal{L}((t+1)^2) = \frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s}$$

The "impulse function" $\delta(t)$ is not a function in a precise way. We consider a short impulse with unit area.



Now, f_h has no limit function (it is ∞ at 0 and 0 for $t>0$, but we may compute the Laplace transforms

$$\mathcal{L}(f_h(t)) = \int_0^{1/h} h e^{-st}\,dt = -\frac{h}{s}\left(e^{-s/h} - 1\right).$$

By l'Hôpital's rule we may compute the limit as $h \to \infty$

$$\lim_{h\to\infty} -\frac{h}{s}\left(e^{-s/h}-1\right) = \lim_{h\to\infty} -\frac{e^{-s/h}-1}{1/h}\cdot\frac{1}{s}$$

$$= \lim_{h\to\infty} -\frac{\frac{s}{h^2}e^{-s/h}}{-\frac{1}{h^2}}\cdot\frac{1}{s} = 1.$$

Thus, the limit of the Laplace transform is 1 for all s. We then call the impulse function the Dirac delta function $\delta(t)$ and have

$$\mathcal{L}(\delta(t)) = 1.$$

By a t shift we get

$$\mathcal{L}(\delta(t-a)) = e^{-as}$$

Ex $y'' - 4y' + 13y = 4\delta(t-3)$

$y(0) = y'(0) = 0$

$$s^2 Y - 4sY + 13Y = 4e^{-3s}$$

$$Y = \frac{4e^{-3s}}{s^2 - 4s + 13} = \frac{4e^{-3s}}{(s-2)^2 + 9}$$

Now $\mathcal{L}^{-1}\left(\frac{1}{(s-2)^2+9}\right) = \frac{1}{3} e^{2t} \sin 3t$

so

$$y = \frac{4}{3} e^{2(t-3)} \sin 3(t-3)\; u(t-3).$$

Ex. $\mathcal{L}\left((t^2 + \sin t)u(t-2)\right)$

$$= e^{-2s} \mathcal{L}\left((t+2)^2\right) + e^{-2s} \mathcal{L}(\sin(t+2))$$

$$= e^{-2s}\left(\frac{2}{s^3} + \frac{4}{s^2} + \frac{4}{s} + \frac{(\cos 2)}{s^2+1} + \frac{(\sin 2)s}{s^2+1}\right)$$