Lesson 2

# Linear Systems and Gaussian elimination

System of m equations in n unknowns:

$$a_{11}x_1 + \cdots + a_{1n}x_1 = b_1$$

$$a_{21}x_1 + \cdots + a_{2n}x_n = b_2$$

$$\vdots$$

$$a_{m1}x_1 + \cdots + a_{mn}x_n = b_m$$

System is <u>overdetermined</u> if $m>n$, <u>determined</u> if $m=n$, <u>underdetermined</u> if $m<n$. In matrix form write

$$Ax = b \qquad \text{where}$$

$$A = (a_{ij}) \qquad x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \qquad b = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$$

The <u>augmented</u> matrix $\tilde{A}$ is

$$\tilde{A} = \begin{pmatrix} a_{11} & \cdots & a_{1n} & b_1 \\ a_{21} & \cdots & a_{2n} & b_2 \\ & \vdots & & \\ a_{m1} & \cdots & a_{mn} & b_m \end{pmatrix}$$

Elementary Row operations

There are 3 row operations

1. Switch two rows.

2. Multiply one row by nonzero number

3. Add multiple of a row to another row.

A is row equivalent to B if A can be obtained from B by row operations. Since row operations can be reversed, then if A is row equivalent to B it is also true that B is row equivalent to A. Write $A \sim B$ or $B \sim A$.

The important fact is that if $(A|b)$ and $(C|d)$ are augmented matrices for two systems of equations and $(A|b) \sim (C|d)$, then the two systems have precisely the same solutions.

Gaussian elimination is the process of using elementary row operations on the augmented matrix to solve a system.

If $b = 0$, the system is homogeneous and there is no need to include it in the augmented matrix since it doesn't change.

Ex.

$$x_1 + 2x_2 + 3x_3 = 1$$

$$4x_1 + 5x_2 + 6x_3 = 0$$

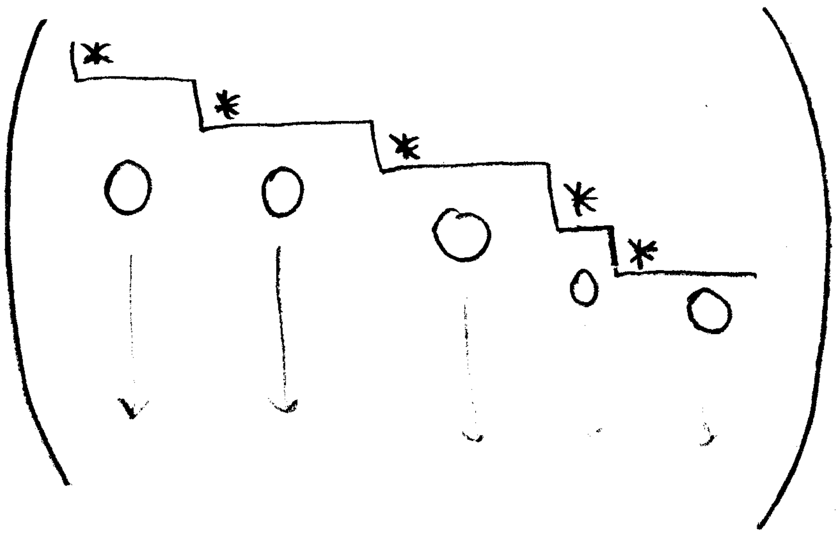
$$7x_1 + 8x_2 + 9x_3 = 1$$

$$\begin{pmatrix} 1 & 2 & 3 & 1 \\ 4 & 5 & 6 & 0 \\ 7 & 8 & 9 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 3 & 1 \\ 0 & -3 & -6 & -4 \\ 0 & -6 & -12 & -6 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 2 & 3 & 1 \\ 0 & -3 & -6 & -4 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

$\therefore$ System has <u>no</u> solutions.

In this algorithm we have worked the augmented matrix down to echelon form. In this form there are corner entries (pivots) below which we have made zeros systematically.



If there is a solution, it can be found by back substituting.

Ex. Change the 9 to 10 in previous example:

$$\begin{pmatrix} 1 & 2 & 3 & 1 \\ 4 & 5 & 6 & \cancel{4}\,0 \\ 7 & 8 & 10 & \cancel{2}\,1 \end{pmatrix}$$

The same sequence of row operations yields the echelon form

$$\begin{pmatrix} 1 & 2 & 3 & 1 \\ 0 & -3 & -6 & -4 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

Since the original system has same solution we get

$$x_3 = 2$$

$$\Rightarrow \quad -3x_2 = 6 \cdot 2 - 4 \Rightarrow x_2 = -8/3$$

and

$$x_1 = -2\left(-8/3\right) - 3(2) + 1 = \frac{16}{3} - \frac{18}{3} + \frac{3}{3}$$

$$= 1/3$$

Finally, let's change to 10 back to 9 and change the 1 in the lower right corner to $-1$.

Then,

$$\begin{pmatrix} 1 & 2 & 3 & 1 \\ 4 & 5 & 6 & 0 \\ 7 & 8 & 9 & -1 \end{pmatrix} \quad \text{becomes}$$

$$\begin{pmatrix} 1 & 2 & 3 & 1 \\ 0 & -3 & -6 & -4 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{consistent}$$

This solution has infinitely many solutions which can be found by back substitution after assigning letters to the non corner entries which are called free variables.

Here $x_3$ is free.

Let $x_3 = t$

Then, $-3x_2 = 6t - 4$ $\qquad x_2 = -2t + 4/3$

and

$$x_1 = -2\left(-2t + 4/3\right) - 3t + 1$$