Lesson 20. More Partial Fractions

We will use formulas $\mathcal{L}^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\} = \frac{t}{2a}\sin at$ and also $\mathcal{L}^{-1}\left\{\frac{1}{(s^2+a^2)^2}\right\} = \frac{1}{2a^3}(\sin at - at\cos at)$ which will be derived in 2 different ways in Lessons 21 & 22.

Let $$F(s) = \frac{s^3+2s+1}{(s^2-2s+5)^2}$$

Now, s^2-2s+5 irreducible: $b^2-4ac = 4-20 < 0$

$$\frac{x_1+x_2 s}{s^2-2s+5} + \frac{x_3+x_4 s}{(s^2-2s+5)^2} = F(s)$$

$$(x_1+x_2 s)(s^2-2s+5) + x_3+x_4 s = s^3+2s+1$$

$$x_2 s^3 + (x_1-2x_2)s^2 + (-2x_1+5x_2+x_4)s + 5x_1+x_3 = s^3+2s+1$$

$x_2 = 1$

$x_1 - 2x_2 = 0$

$-2x_1+5x_2+x_4 = 2$

$5x_1 + x_3 = 1$

$x_1 = 2$

$x_2 = 1$

$x_3 = -9$

$x_4 = 1$

$$F(s) = \frac{2+s}{s^2-2s+5} + \frac{-9+s}{(s^2-2s+5)^2}$$

$$= \frac{2+s}{(s-1)^2+4} + \frac{-9+s}{((s-1)^2+4)^2}$$

$$= \frac{3}{(s-1)^2+4} + \frac{s-1}{(s-1)^2+4}$$

$$- \frac{8}{((s-1)^2+4)^2} + \frac{s-1}{((s-1)^2+4)^2}$$

$$f(t) = \frac{3}{2} e^t \sin 2t + e^t \cos 2t$$

$$- \frac{1}{2} e^t (\sin 2t - 2t \cos 2t)$$

$$+ \frac{1}{4} e^t \, t \sin 2t$$

Linear factors are of course easier

Ex.

$$\frac{s-1}{(s-3)^2} = \frac{A}{s-3} + \frac{B}{(s-3)^2}$$

$$A(s-3) + B = s-1$$

$$A = 1$$

$$-3A + B = -1 \Rightarrow B = 2$$

so we get

$$\frac{1}{s-3} + \frac{2}{(s-3)^2}$$

There is a formula for

$$\frac{A_m}{(s-a)^m} + \frac{A_{m-1}}{(s-a)^{m-1}} + \cdots + \frac{A_1}{s-a}.$$

The formula can also be used when a is complex (and the irreducible quadratics are factored into linear factors with complex roots)

$$A_k = \frac{1}{(m-k)!} \lim_{s \to a} \frac{d^{m-k}}{ds^{m-k}} \left((s-a)^m F(s) \right).$$

When $m=k$, $(m-k)! = 1$ $\quad (0! = 1)$

and the $\frac{d^{m-k}}{ds^{m-k}}$ disappears.

To prove this relation, write

$$F(s) = \frac{A_m}{(s-a)^m} + \cdots \frac{A_1}{s-a} + \underbrace{\text{other fractions}}_{Q(s)}$$

so that

$$(s-a)^m F(s) = A_m + A_{m-1}(s-a) + \cdots + A_k (s-a)^{m-k} + \ldots A_1 (s-a)^{m-1} + (s-a)^m Q(s)$$

$$\frac{d^{m-k}}{ds^{m-k}} (s-a)^m F(s) = (m-k)!\, A_k + (s-a) R(s),$$

ie. the right hand side is $(m-k)!\, A_k$ + terms, all of which have at least one factor of $(s-a)$.

Applying this to previous example

$$F(s) = \frac{s-1}{(s-3)^2} = \frac{A}{s-3} + \frac{B}{(s-3)^2}$$

$$B = \lim_{s \to 3} (s-3)^2 F(s) = \lim_{s \to 3} (s-1) = 2$$

$$A = \lim_{s \to 3} \frac{d}{ds} (s-3)^2 F(s) = \lim_{s \to 3} \frac{d}{ds} (s-1) =$$

$$\lim_{s \to 3} 1 = 1$$

<u>Periodic functions</u> $f(t+T) = f(t)$.

In order to derive a formula for $\mathcal{L}(f)$ with periodic f, we want to recall the <u>geometric series</u>.

Let r be a number $|r|<1$, and consider

$$S_n = 1 + r + r^2 + \dots + r^n.$$

Then $(1-r)\,S_n = 1 - r^{n+1}$ so that

$$S_n = \frac{1-r^{n+1}}{1-r}.$$

Since $|r|<1$, $S_n \longrightarrow \frac{1}{1-r}$ as $n \longrightarrow \infty$.

This shows that

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r} \qquad (|r|<1).$$

Suppose now f has period T, and write

$$\mathcal{L}(f) = \int_0^\infty e^{-st} f(t)\,dt = \int_0^T e^{-st} f(t)\,dt + \int_T^{2T} e^{-st} f(t)\,dt$$

$$+ \cdots \int_{nT}^{(n+1)T} e^{-st} f(t)\,dt + \cdots$$

$$= \sum_{n=0}^{\infty} \int_{nT}^{(n+1)T} e^{-st} f(t)\,dt$$

Now, let $\tau = t - nT$. Then, $d\tau = dt$ and

$$\int_{nT}^{(n+1)T} e^{-st} f(t)\,dt = \int_0^T e^{-s(\tau + nT)} f(\tau + nT)\,d\tau$$

$$= \int_0^T e^{-s(\tau+nT)} f(\tau)\,d\tau = e^{-nsT} \int_0^T e^{-s\tau} f(\tau)\,d\tau$$

so that

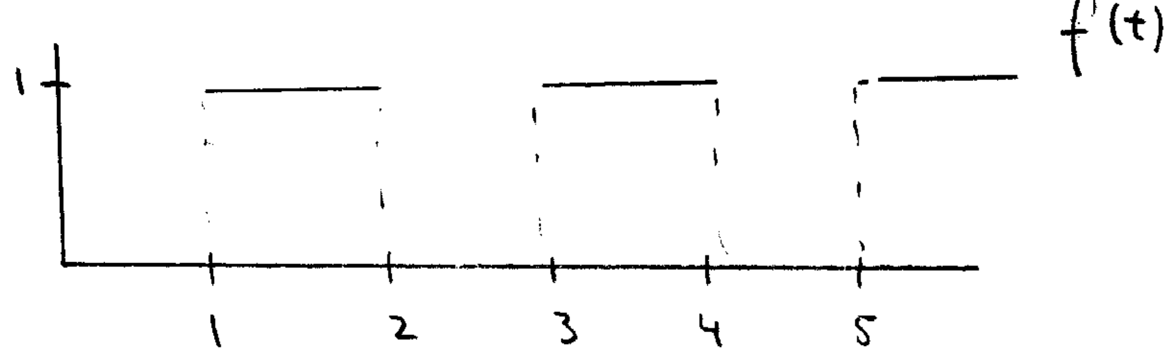
$$\mathcal{L}(f) = \sum_{n=0}^{\infty} e^{-nsT} \int_0^T e^{-s\tau} f(\tau)\,d\tau$$

$$= \int_0^T e^{-s\tau} f(\tau)\,d\tau \sum_{n=0}^{\infty} \left(e^{-sT}\right)^n.$$

With $r = e^{-sT}$ in the geometric series, we get

$$\mathcal{L}(f) = \frac{1}{1-e^{sT}} \int_0^T e^{-s\tau} f(\tau)\,d\tau$$

Ex.



$$F(s) = \frac{1}{1-e^{-2s}} \int_1^2 e^{-st}\,dt = \frac{1}{1-e^{-2s}} \left(-\frac{1}{s} e^{-st} \Big|_1^2 \right)$$

$$\frac{1}{s(1-e^{-2s})} \left(e^{-s} - e^{-2s}\right) = \frac{e^{-s}}{s(1-e^{-2s})} \left(1-e^{-s}\right)$$

$$= \frac{e^{-s}(1-e^{-s})}{s(1-e^{-s})(1+e^{-s})} = \frac{e^{-s}}{s(1+e^{-s})} = \frac{1}{s(1+e^{s})}$$