Lesson 21

Convolutions and integral equations.

Recall the transfer function $Q(s)$ associated with an equation

$$y'' + ay' + by = r(t)$$

is

$$Q(s) = \frac{1}{s^2 + as + b}.$$

so that

$$(s^2 + as + b)Y = (s+a)y(0) + y'(0) + R$$

becomes

$$Y(s) = \left[(s+a)y(0) + y'(0)\right]Q(s) + Q(s)R(s).$$

and when $y(0) = y'(0) = 0$ this becomes

$$Y(s) = R(s)\,Q(s).$$

This shows the need to deal with products of Laplace transforms. This is done using the <u>convolution</u> $f*g\,(t)$ of functions.

$$f*g\,(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau$$

<u>Ex.</u> $f(t) = t \qquad g(t) = e^t$

$$f*g = \int_0^t \tau e^{t-\tau}\,d\tau = e^t\int_0^t \tau e^{-\tau}\,d\tau$$

$u = \tau \qquad dv = e^{-\tau}d\tau$

$du = d\tau \qquad v = -e^{-\tau}$

$$f * g = e^t \left(-\tau e^{-\tau} \Big|_0^t + \int_0^t e^{-\tau}\, d\tau \right)$$

$$= e^t \left(-t e^{-t} - e^{-\tau} \Big|_0^t \right)$$

$$= -t - 1 + e^t$$

Convolutions are commutative: $\boxed{f * g = g * f}$

In the example, we could have computed

$$\int_0^t (t-\tau) e^{\tau}\, d\tau = t \int_0^t e^{\tau}\, d\tau - \int \tau e^{\tau}\, d\tau$$

$$= t\left(e^t - 1\right) - \left(\tau e^{\tau} \Big|_0^t - \int_0^t e^{\tau}\, d\tau \right)$$

$$= t\left(e^t - 1\right) - t e^t + e^t - 1 = -t + e^t - 1$$

Convolution Theorem

$$\mathcal{L}(f) = F \quad \mathcal{L}(g) = G \Rightarrow \mathcal{L}(f * g) = F \cdot G.$$

In the previous example

$$\mathcal{L}(f) = 1/s^2 \qquad \mathcal{L}(g) = \frac{1}{s-1}$$

$$\mathcal{L}(f*g) = \mathcal{L}\left(-t-1+e^t\right) = -\frac{1}{s^2} - \frac{1}{s} + \frac{1}{s-1}.$$

$$= \frac{s^2}{s^2(s-1)} - \frac{s(s-1)}{s^2(s-1)} - \frac{s-1}{s^2(s-1)} = \frac{1}{s^2(s-1)}$$

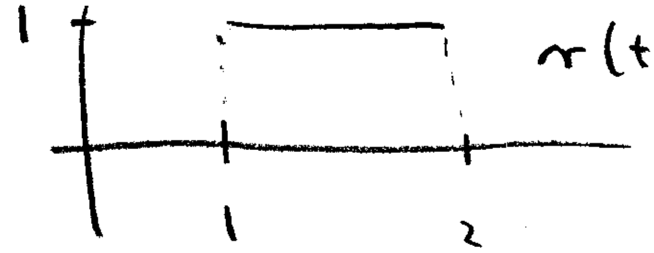
and $\qquad F(s)\,G(s) = \frac{1}{s^2}\,\frac{1}{s-1}$ ✓

Returning to the transfer function relation $Y = RQ$ we get

$$y = \int_0^t r(\tau)\, q(t-\tau)\, d\tau \qquad q = \mathcal{L}^{-1}(Q).$$

Ex.

$$y'' + 3y' + 2y = r(t) \qquad y(0) = y'(0) = 0$$



$r(t)$ square wave

$$Q(s) = \frac{1}{s^2 + 3s + 2} = \frac{1}{s+1} - \frac{1}{s+2}$$

$$\Rightarrow \quad q(t) = e^{-t} - e^{-2t}$$

Thus

$$y = \int_0^t \left(e^{-(t-\tau)} - e^{-2(t-\tau)} \right) r(\tau)\, d\tau$$

① for $0 < t < 1$ $\quad r(\tau) = 0$ so $\quad y(t) = 0$

② for $1 < t < 2$

$$y = \int_1^t \left(e^{-(t-\tau)} - e^{-2(t-\tau)} \right) d\tau$$

$$= \frac{1}{2} - e^{-(t-1)} + \frac{1}{2} e^{-2(t-1)}$$

for $t > 2$

$$y = \int_1^2 \left(e^{-(t-\tau)} - e^{-2(t-\tau)}\right) d\tau$$

$$= e^{-(t-2)} - e^{-(t-1)} - \frac{1}{2}\left(e^{-2(t-2)} - e^{-2(t-1)}\right)$$

Solving integral equations by convolution.

Ex. $y(t) = t + \int_0^t y(\tau)\sin(t-\tau)\,d\tau$

We first must describe this involving a convolution

$$y = t + y * \sin t.$$

Then, if $Y = \mathcal{L}(y)$ we have

$$Y = \frac{1}{s^2} + Y \cdot \frac{1}{s^2+1} \qquad Y\left(1 - \frac{1}{s^2+1}\right) = \frac{1}{s^2}$$

$$\Rightarrow \quad Y = \frac{s^2+1}{s^4} = \frac{1}{s^2} + \frac{1}{s^4}.$$

Inverting, we have

$$y = t + \frac{1}{6}t^3.$$

Ex. $\mathcal{L}^{-1}\left(\frac{1}{(s^2+1)^2}\right) = ?$

(We have already used this in Lesson 20)

Using convolution,

$$\frac{1}{(s^2+1)^2} = \frac{1}{(s^2+1)} \cdot \frac{1}{(s^2+1)},$$

so the answer is

$$\sin t * \sin t = \int_0^t \sin\tau \sin(t-\tau)\,d\tau$$

$$\left(\sin x \sin y = \frac{1}{2}\left(-\cos(x+y) + \cos(x-y)\right)\right)$$

$$= \frac{1}{2}\int_0^t -\cos t + \cos(2\tau - t)\,d\tau$$

$$= -\frac{1}{2}t\cos t + \frac{1}{4}\sin(2\tau - t)\Big|_0^t$$

$$= -\frac{1}{2}t\cos t + \frac{1}{4}\sin t - \frac{1}{4}\sin(-t)$$

$$= -\frac{1}{2}t\cos t + \frac{1}{2}\sin t.$$