More calculus of Laplace Transforms

We have

$$\mathcal{L}(f') = s\,F(s) - f(0),$$

That is we have a formula for Laplace transform of f'. Now we ask the question, what is the inverse Laplace transform of $F'(s)$?

Simply

$$\mathcal{L}^{-1}\big(F'(s)\big) = -t\,f(t).$$

To verify this we must differentiate under the integral sign.

$$F(s) = \int_0^\infty e^{-st} f(t)\,dt$$

$$F'(s) = -\int_0^\infty t\,e^{-st} f(t)\,dt$$

This enables us to expand our library:

i) $\mathcal{L}\left(\frac{1}{2\beta^3}(\sin\beta t - \beta t\cos\beta t)\right) = \frac{1}{(s^2+\beta^2)^2}$

ii) $\mathcal{L}\left(\frac{t}{2\beta}\sin\beta t\right) = \frac{s}{(s^2+\beta^2)^2}$

iii) $\mathcal{L}\left(\frac{1}{2\beta}(\sin\beta t + \beta t\cos\beta t)\right) = \frac{s^2}{(s^2+\beta^2)^2}$

To get these formulas, apply F' formula to $\sin\beta t$:

$$\mathcal{L}(t\sin\beta t) = -\frac{d}{ds}\frac{\beta}{s^2+\beta^2} = \frac{2\beta s}{(s^2+\beta^2)^2}$$

which gives ii).

Similarly,

$$\mathcal{L}(t\cos\beta t) = -\frac{d}{ds}\frac{s}{s^2+\beta^2} = -\frac{s^2+\beta^2-2s^2}{(s^2+\beta^2)^2}$$

$$= \frac{s^2-\beta^2}{(s^2+\beta^2)^2}$$

Then, $\mathcal{L}\left(t\cos\beta t + \frac{1}{\beta}\sin\beta t\right)$

$$= \frac{s^2-\beta^2}{(s^2+\beta^2)^2} + \frac{s^2+\beta^2}{(s^2+\beta^2)^2} = \frac{2s^2}{(s^2+\beta^2)^2}$$

which gives iii) and

$$\mathcal{L}\left(t\cos\beta t - \frac{1}{\beta}\sin\beta t\right) = \frac{s^2-\beta^2}{(s^2+\beta^2)^2} - \frac{s^2+\beta^2}{(s^2+\beta^2)^2}$$

$$= -\frac{2\beta^2}{(s^2+\beta^2)^2}$$

which gives i)

With some care, the formula involving F' can be interpreted as a formula involving $\int F$.

Consider $\frac{f(t)}{t}$ instead of $f(t)$.

If $f(0) \neq 0$, then its integral at 0 blows up, so the formula involving $\int F$ will require that $f(0)=0$, and in fact that $f(t)/t$ is integrable at 0.

We write

$$\int_s^\infty F(\tilde{s})\,d\tilde{s} = \int_s^\infty \int_0^\infty e^{-\tilde{s}t} f(t)\,dt\,d\tilde{s}$$

$$= \int_0^\infty f(t) \int_s^\infty e^{-\tilde{s}t}\,d\tilde{s}\,dt = -\int_0^\infty f(t)\,\frac{1}{t}\,e^{-\tilde{s}t}\Big|_{\tilde{s}=s}^{\tilde{s}\to\infty}\,dt$$

$$= \int_0^\infty \frac{f(t)}{t}\,e^{-st}\,dt = \mathcal{L}\left(f(t)/t\right).$$

That is,

$$\mathcal{L}\left(f(t)/t \right) = \int_s^{\infty} F(\tilde{s})\, d\tilde{s}.$$

Ex.

$$\mathcal{L}\left(\frac{1}{t}\left(1-\cosh t\right)\right) = \int_s^{\infty} \frac{1}{\tilde{s}} - \frac{\tilde{s}}{\tilde{s}^2-1}\, d\tilde{s}$$

$$= \ln \tilde{s} - \frac{1}{2}\ln(\tilde{s}^2-1)\Big|_s^{\infty} = \ln \frac{\tilde{s}}{\sqrt{\tilde{s}^2-1}}\Big|_s^{\infty}$$

$$= -\ln \frac{s}{\sqrt{s^2-1}}$$

Multiplying 2 on both sides and simplifying,

$$\mathcal{L}\left(\frac{2}{t}\left(1-\cosh t\right)\right) = \ln \frac{s^2-1}{s^2} = \ln\left(1-\frac{1}{s^2}\right)$$

Ex. $\mathcal{L}^{-1}\left(\arctan(1/s)\right) = ?$

$$F(s) = \arctan(1/s) = \int_s^\infty \frac{d\tilde{s}}{\tilde{s}^2+1}$$

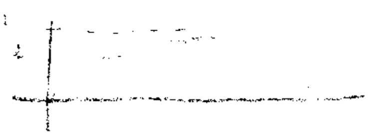
$$\left(\frac{d}{dx} \arctan \frac{1}{x} = - \frac{1}{1+x^2} \right)$$

Thus,

$$\mathcal{L}^{-1}\left(\arctan(1/s)\right) = \frac{1}{t}\mathcal{L}^{-1}\left(\frac{1}{s^2+1}\right)$$

$$= \frac{\sin t}{t}.$$

arctan s



$$\mathcal{L}\left\{t \frac{\sin t}{t}\right\} = \frac{1}{s^2+1} = -F'(s)$$

$$F(s) = -\arctan s + C \qquad C = \frac{\pi}{2}$$

Solving partial differential eqn with Laplace transform

Later on we shall study partial differential equations. Sometimes the Laplace transform is useful.

Ex one dimensional wave equation

$$\frac{\partial^2 w}{\partial t^2} = c^2 \frac{\partial^2 w}{\partial x^2}$$

boundary conditions $\quad w(0,t) = f(t) \qquad \lim_{x\to\infty} w(x,t) = 0$

initial condition $\quad w(x,0) = 0$

$$\frac{\partial w}{\partial t}(x,0) = 0.$$

Take Laplace transform w.r.t. t.

$$\mathcal{L}\left(\frac{\partial^2 w}{\partial t^2}\right) = s^2 \mathcal{L}(w(x,t)) - s\, w(x,0) - \frac{\partial w}{\partial t}(x,0) = c^2 \mathcal{L}\left(\frac{\partial^2 w}{\partial x^2}\right).$$

so, using the initial conditions,

$$\mathcal{L}\left(\frac{\partial^2 w}{\partial t^2}\right) = s^2 \mathcal{L}(w) = c^2 \mathcal{L}\left(\frac{\partial^2 w}{\partial x^2}\right).$$

$$= c^2 \int_0^\infty e^{-st} \frac{\partial^2 w}{\partial x^2}\, dt = c^2 \frac{\partial^2}{\partial x^2} \int_0^\infty e^{-st} w(x,t)\, dt = c^2 \frac{\partial^2}{\partial x^2} \mathcal{L}(w).$$

Let $W(x,s) = \mathcal{L}(w)$. Then we have

$$s^2 W = c^2 \frac{\partial^2 W}{\partial x^2} \qquad \text{for each } s.$$

We can solve

$$\frac{\partial^2 W}{\partial x^2} - \frac{s^2}{c^2} W = 0 \qquad \text{for each } s:$$

$$W(x,s) = A(s) e^{sx/c} + B(s) e^{-sx/c}.$$

Using the boundary conditions, with $F(s) = \mathcal{L}(f)$,

$$W(0,s) = \mathcal{L}(w(0,t)) = \mathcal{L}(f(t)) = F(s).$$

Also,

$$\lim_{x\to\infty} W(x,s) = \lim_{x\to\infty} \int_0^\infty e^{-st} w(x,t)\,dt = \int_0^\infty e^{-st} \lim_{x\to\infty} w(x,t)\,dt$$

$$= 0 \quad \Rightarrow \quad A(s) = 0.$$

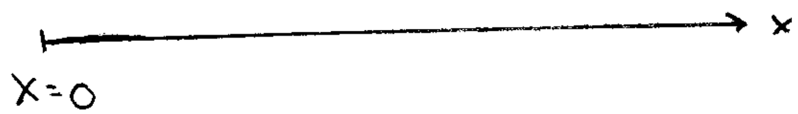
Thus, $B(s) = F(s)$ so hence $W(x,s) = F(s)\, e^{-sx/c}$.

Using the s shift theorem,

$$w(x,t) = f(t - x/c)\, u(t - x/c).$$

"Snapshots" if $f(t) = u(\pi - t)\sin t$

$(t=0)$



$t = \pi/2$

$t = 3$