Lesson 23

Laplace Transforms and Systems

Consider the linear system

$$\frac{dx}{dt} = Ax + r(t) \qquad x(0) = x_0$$

$$x = \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix}, \quad A = \begin{pmatrix} a_{11} \cdots a_{1n} \\ \vdots \\ a_{n1} \cdots a_{nn} \end{pmatrix} \quad a_{ij}\text{'s constant}$$

$$r = \begin{pmatrix} r_1(t) \\ \vdots \\ r_n(t) \end{pmatrix}, \quad x_0 = \begin{pmatrix} x_0^{(1)} \\ \vdots \\ x_0^{(n)} \end{pmatrix} \text{ initial conditions}$$

We can take Laplace transforms component-wise

$$X = \mathcal{L}(x) = \begin{pmatrix} \mathcal{L}(x_1) \\ \vdots \\ \mathcal{L}(x_n) \end{pmatrix} \qquad R = \begin{pmatrix} \mathcal{L}(r_1) \\ \vdots \\ \mathcal{L}(r_n) \end{pmatrix}$$

to obtain

$$sX(s) - x_0 = AX(s) + R(s)$$

$$(sI - A)X(s) = x_0 + R(s)$$

$$X(s) = (sI - A)^{-1}(x_0 + R(s))$$

$$x(t) = \mathcal{L}^{-1}\left((sI - A)^{-1}(x_0 + R(s))\right)$$

Ex.

$$\frac{dx_1}{dt} = 2x_1 - x_2 + \sin t \qquad x_1(0) = 1$$

$$\frac{dx_2}{dt} = -2x_1 + x_2 + 1 \qquad x_2(0) = -1$$

$$\frac{dx}{dt} = \begin{pmatrix} 2 & -1 \\ -2 & 1 \end{pmatrix} x + \begin{pmatrix} \sin t \\ 1 \end{pmatrix}.$$

$$X = \left(s\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 2 & -1 \\ -2 & 1 \end{pmatrix} \right)^{-1} \left(\begin{pmatrix} 1 \\ -1 \end{pmatrix} + \begin{matrix} \frac{1}{s^2+1} \\ \frac{1}{s} \end{matrix} \right)$$

$$= \begin{pmatrix} s-2 & 1 \\ 2 & s-1 \end{pmatrix}^{-1} \begin{pmatrix} \frac{s^2+2}{s^2+1} \\ \frac{1-s}{s} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{s-1}{s(s-3)} & \frac{-1}{s(s-3)} \\ \frac{-2}{s(s-3)} & \frac{s-2}{s(s-3)} \end{pmatrix} \begin{pmatrix} \frac{s^2+2}{s^2+1} \\ \frac{1-s}{s} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{(s-1)(s^3+s^2+2s+1)}{s^2(s-3)(s^2+1)} \\ -\frac{(s^4-s^3+3s^2+s+2)}{s^2(s-3)(s^2+1)} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{4}{9}\frac{1}{s} + \frac{1}{3}\frac{1}{s^2} - \frac{1}{s}\frac{1}{s^2+1} - \frac{2}{s}\frac{s}{s^2+1} + \frac{43}{45}\frac{1}{s-3} \\ \frac{5}{9}\frac{1}{s} + \frac{2}{3}\frac{1}{s^2} + \frac{1}{s}\frac{1}{s^2+1} - \frac{3}{s}\frac{s}{s^2+1} - \frac{43}{45}\frac{1}{s-3} \end{pmatrix}$$

$$x = \begin{pmatrix} \frac{4}{9} + \frac{1}{3}t - \frac{1}{s}\sin t - \frac{2}{s}\cos t + \frac{43}{45}e^{3t} \\ \frac{5}{9} + \frac{2}{3}t + \frac{1}{s}\sin t - \frac{3}{s}\cos t - \frac{43}{45}e^{3t} \end{pmatrix}$$