Lesson 24

Sturm-Liouville problems

We need this theory for the foundation of expansions in terms of orthogonal functions and in particular, Fourier series (we observed earlier that the trig function $\{\sin nx\}$, $\{\cos nx\}$ are orthogonal with the inner product $(f,g) = \int_0^{2\pi} f(x)g(x)\,dx$.

The Sturm-Liouville equation is

$$(p(x)y')' + (q(x) + \lambda r(x))y = 0$$

when $p(x)$, $p'(x)$, $q(x)$, $r(x)$ are continuous, $r(x) > 0$.

Ex. $y'' + \lambda y = 0$ is a simple S-L eqn. with $r(x) \equiv 1$ $q(x) \equiv 0$ $p(x) \equiv 1$, and has general solution

$$y = \begin{cases} c_1 \cos\sqrt{\lambda}\,x + c_2 \sin\sqrt{\lambda}\,x & \lambda > 0 \\ c_1 + c_2 x & \lambda = 0 \\ c_1 e^{-\sqrt{|\lambda|}\,x} + c_2 e^{\sqrt{|\lambda|}\,x} & \lambda < 0. \end{cases}$$

<u>Ex</u>. $(1-x^2)y'' - 2xy' + n(n+1)y = 0$
is <u>Legendre's</u> eqn., and is S-L eqn.

$$\left((1-x^2)y'\right)' + \lambda y = 0 \qquad \lambda = n(n+1)$$

<u>Ex</u> <u>Bessel's</u> eqn.

$$x^2y'' + xy' + (k^2x^2 - n^2)y = 0$$

can be written

$$(xy')' + \left(\frac{-n^2}{x} + \lambda x\right)y = 0. \qquad \lambda = k^2.$$

The inner product associated with the S-L equation

$$(p(x)y')' + (q(x) + \lambda r(x))y = 0$$

on an interval $a \le x \le b$ is

$$(f, g) = \int_a^b f(x)g(x)\, r(x)\, dx .$$

The S-L problem is an S-L equation on an interval $a \le x \le b$ along with a boundary value problem

$$k_1 y(a) + k_2 y'(a) = 0 \qquad k_1, k_2 \text{ not both } 0$$
$$l_1 y(b) + l_2 y'(b) = 0 \qquad l_1, l_2 \text{ " "}$$

Ex. Consider the S-L problem

$$y'' + \lambda y = 0 \qquad 0 \le x \le 1$$
$$y(0) = 0 \qquad (k_1 = 1 \;\; k_2 = 0)$$
$$y'(1) = 0 \qquad (l_1 = 0 \;\; l_2 = 1)$$

Now, if $\lambda < 0$,

$$0 = c_1 e^{-\sqrt{|\lambda|}\cdot 0} + c_2 e^{\sqrt{|\lambda|}\, 0} = c_1 + c_2$$

$$0 = -\sqrt{|\lambda|}\, c_1 e^{-\sqrt{|\lambda|}\cdot 1} + \sqrt{|\lambda|}\, c_2 e^{\sqrt{|\lambda|}\, 1}$$

$$= \sqrt{|\lambda|}\, c_2 e^{-\sqrt{|\lambda|}} + \sqrt{|\lambda|}\, c_2 e^{\sqrt{|\lambda|}} = \sqrt{|\lambda|}\, c_2 \left(e^{-\sqrt{|\lambda|}} + e^{\sqrt{|\lambda|}}\right)$$

$$\Rightarrow \quad c_2 = 0.$$

Similarly, $c_1 = 0$. Thus, the only solution if $\lambda < 0$ is $y \equiv 0$.

If $\lambda = 0$, $y = c_1 + c_2 x \Rightarrow$

$$c_1 = 0 \ \ \& \ \ c_2 = 0 \Rightarrow y \equiv 0.$$

Finally when $\lambda > 0$, $y = c_1 \cos\sqrt{\lambda}\, x + c_2 \sin\sqrt{\lambda}\, x$

$$c_1 \cos 0 + c_2 \sin 0 = 0 \Rightarrow c_1 = 0$$

$$\left. \cancel{-\sqrt{\lambda}\, c_1 \sin\sqrt{\lambda}\, x}_{\,0} + \sqrt{\lambda}\, c_2 \cos\sqrt{\lambda}\, x \right|_{x=1} = 0$$

$\Rightarrow \sqrt{\lambda} = \pi/2 + k\pi \qquad k = 0, 1, 2, \ldots$

When there is a nontrivial solution, it is called an <u>eigenfunction</u>, and the corresponding λ are <u>eigenvalues</u>. Here, eigenvalues are $\left\{\left(\frac{\pi}{2} + k\pi\right)^2\right\}$ $k = 0, 1, 2, \ldots$
eigenfunction $\left\{ \sin\left(\frac{\pi}{2} + k\pi\right)x \right\}$ $\quad k = 0, 1, \ldots$

For a general S-L problem on $[a, b]$ eigenfunctions $y_m(x)$, $y_n(x)$ corresponding to different eigenvalues λ_m, λ_n are orthogonal. (with the weight function $r(x)$).

If $p(a) = 0$ then the first boundary value condition can be dropped, and

if $p(b)=0$ the second can be dropped.
If $p(a)=p(b)$ the boundary conditions can be replaced by periodic boundary conditions $y(a)=y(b)$, $y'(a)=y'(b)$.

Proof. Let's just do the first part.

Now,
$$(p y_m')' + (q+\lambda_m r) y_m = 0$$
$$(p y_n')' + (q+\lambda_n r) y_n = 0$$

$$\Rightarrow \quad (\lambda_m - \lambda_n) r y_m y_n = y_n (p y_m')' - y_m (p y_n')'$$
$$= (p y_n' y_m - p y_m' y_n)'$$

$\Rightarrow$
$$(\lambda_m - \lambda_n) \int_a^b y_m(x) y_n(x) r(x)\, dx$$
$$= p(x)\big(y_n'(x) y_m(x) - y_m'(x) y_n(x)\big)\Big|_a^b$$

$$= p(b)\left(y_n'(b)\,y_m(b) - y_m'(b)\,y_n(b)\right)$$
$$- p(a)\left(y_n'(a)\,y_m(a) - y_m'(a)\,y_n(a)\right)$$

Suppose eg. $k_1 \neq 0$

$$y_n(a) \quad k_1 y_m(a) + k_2 y_m'(a) = 0$$
$$y_m(a) \quad k_1 y_n(a) + k_2 y_n'(a) = 0$$
$$\Rightarrow \quad k_2\left(y_m'(a)\,y_n(a) - y_n'(a)\,y_m(a)\right) = 0$$

which makes second term 0. Using l_1 or l_2 we get the first expression also to be 0.

Ex Legendre polynomials $\quad -1 \leq x \leq 1$

$$\left((1-x^2)y'\right)' + \lambda y = 0 \qquad \lambda = n(n+1),\ n=0,1$$

The boundary conditions can both be dropped since $p(1) = p(-1) = 0$.

The formulas for eigenfunctions are given on page 180 and are derived from recursion formulas derived from substitution of power series into the equation:

$$P_0(x) = 1$$

$$P_1(x) = x$$

$$P_2(x) = \frac{1}{2}(3x^2 - 1)$$

$$P_3(x) = \frac{1}{2}(5x^3 - 3x)$$

$$\vdots$$

We can use (11) on 180 to get an expansion of a given function $f(x)$ $-1 \leq x \leq 1$ in terms of these. For this we formally write

$$f(x) = c_0 P_0(x) + c_1 P_1(x) + c_2 P_2(x) + \cdots$$

and solve for $c_0, c_1, \ldots$ using the fact that

$(p_m, p_n) = 0$ if $m \neq n$, and we also need the formula

$$\|p_m\| = \sqrt{\int_{-1}^{1} p_m^2(x)\,dx} = \sqrt{\frac{2}{2m+1}}$$

Then, by multiplying $f(x)$ by $p_m(x)$ and integrating we get

$$a_m = \frac{2m+1}{2}\int_{-1}^{1} f(x)\,p_m(x)\,dx$$

If $f(x) = x^2$,

we can compute

$$c_0 = \frac{1}{2}\int_{-1}^{1} x^2\,dx = \frac{1}{6}\,x^3\Big|_{-1}^{1} = \frac{1}{3}$$

$$c_1 = \frac{3}{2}\int_{-1}^{1} x^3\,dx = \frac{3}{8}\,x^4\Big|_{-1}^{1} = 0$$

$$C_2 = \frac{5}{2}\int_{-1}^{1} \frac{1}{2}(3x^4 - x^2)\,dx = \frac{5}{4}\left(\frac{3}{5}x^5 - \frac{1}{3}x^3\right)\Big|_{-1}^{1}$$

$$\frac{5}{4}\left(\frac{6}{5} - \frac{2}{3}\right) = \frac{5}{4}\left(\frac{8}{15}\right) = \frac{2}{3}$$

$$f(x) = \frac{1}{3}\cdot p_0 + 0\cdot p_1 + \frac{2}{3}\, p_2$$

$$= \frac{1}{3}\cdot 1 + 0\cdot x + \frac{2}{3}\cdot\frac{1}{2}\left(3x^2 - 1\right) = x^2$$

If p_n as above represents the n^{th} Legendre polynomial, then consider the function $q_n(\theta) = p_n(\sin\theta)$, $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$. Then,

$$\int_{-\pi/2}^{\pi/2} q_m(\theta)\, q_n(\theta) \cos\theta \, d\theta = \int_{-\pi/2}^{\pi/2} p_m(\sin\theta)\, p_n(\sin\theta) \cos\theta \, d\theta$$

$t = \sin\theta \quad dt = \cos\theta\, d\theta$

$$= \int_{-1}^{1} p_m(t)\, p_n(t)\, dt = 0$$ if $m \neq n$ since p_m, p_n are orthogonal on $[-1, 1]$