Orthogonal functions

Let $y_0, y_1, \ldots$ be eigenfunctions corresponding to a S-L problem for $a \le x \le b$ with different eigenvalues and boundary value conditions so that y_m and y_n are orthogonal with respect to weight function $r(x)$ for all $m \ne n$. Suppose further that the y_m are normalized

$$1 = \|y_m\|^2 = \int_a^b y_m(x)^2 r(x)\, dx.$$

If $f(x)$ is expanded in terms of these eigenfunctions

$$f(x) = \sum_{m=0}^{\infty} a_m y_m(x),$$

Then

$$a_m = \int_a^b f(x)\, y_m(x)\, r(x)\, dx.$$

The standard trig Fourier series comes from the S-L problem with periodic boundary conditions

$$y'' + \lambda y = 0 \qquad -\pi \le x \le \pi$$

$$y(-\pi) = y(\pi) \qquad y'(-\pi) = y'(\pi).$$

Solving this as in Lesson 24

$$\lambda < 0 \Rightarrow \qquad y = c_1 e^{-\sqrt{|\lambda|}x} + c_2 e^{\sqrt{|\lambda|}x}$$

$$\Rightarrow \qquad c_1 e^{\sqrt{|\lambda|}\pi} + c_2 e^{-\sqrt{|\lambda|}\pi} = c_1 e^{-\sqrt{|\lambda|}\pi} + c_2 e^{\sqrt{|\lambda|}\pi}$$

$$-\sqrt{|\lambda|}c_1 e^{\sqrt{|\lambda|}\pi} + c_2\sqrt{|\lambda|}e^{-\sqrt{|\lambda|}\pi} = -\sqrt{|\lambda|}c_1 e^{-\sqrt{|\lambda|}\pi} + \sqrt{|\lambda|}c_2 e^{\sqrt{|\lambda|}\pi}$$

$\Rightarrow$ with a little algebra $c_1 = c_2 = 0$, ie. no eigenfunctions.

The case $\lambda = 0$ gives an eigenfunction $y = 1$

The only solutions for $\lambda > 0$ are

$$\lambda = n^2 \qquad n = 0, 1, 2, \ldots$$

with eigenfunctions

$$\cos x,\ \sin x,\ \cos 2x,\ \sin 2x,\ \ldots$$

For the function 1, $2\pi = \int_{-\pi}^{\pi} 1\,dx$. Also,

$$\int_{-\pi}^{\pi} \cos^2 mx\,dx = \frac{1}{2}\int_{-\pi}^{\pi} 1 + \cos 2mx\,dx = \pi$$

$$\int_{-\pi}^{\pi} \sin^2 mx\,dx = \frac{1}{2}\int_{-\pi}^{\pi} 1 - \cos 2mx\,dx = \pi,$$

so if

$$f(x) = a_0 + \sum_{m=1}^{\infty} a_m \cos mx + b_m \sin mx$$

Then

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos mx\,dx \qquad m = 1, 2, \ldots$$

$$b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin mx\,dx \qquad m = 1, 2, \ldots$$

Let $y_0, y_1, y_2, \ldots$ be an orthonormal set of functions on an interval $[a,b]$ with respect to a weight function $r(x)$.

The set $\{y_n(x)\}$ is called <u>complete</u> in a space S of functions if for every $f \in S$ and $\varepsilon > 0$ there is a linear combination (finite)

$$a_0 y_0 + \cdots + a_n y_n$$

such that $\| f - a_0 y_0 + \cdots + a_n y_n \| < \varepsilon.$

Writing this out,

$$\int_a^b \left(f(x) - a_0 y_0(x) + \cdots + a_n y_n(x)\right)^2 r(x)\,dx < \varepsilon^2$$

This is called <u>mean square</u> approximation.

We have been taking the a_k's to be the <u>Fourier coefficients</u>

$$a_k = \int_a^b f(x)\, y_k(x)\, r(x)\, dx \qquad (\text{recall } \|y_k\|=1)$$

Let $\quad s_n = \sum_{k=0}^{n} a_k y_k(x).$

Then,

$$\|f - s_n\|^2 = \int_a^b s_n^2\, r\, dx - 2\int_a^b f\, s_n\, r\, dx + \int_a^b f^2 r\, dx$$

$$= \int_a^b \left(\sum_{k=0}^{n} a_k y_k\right)^2 r\, dx - 2\int_a^b f\left(\sum_{k=0}^{n} a_k y_k\right) r\, dx + \|f\|^2$$

$$= \sum a_k^2 - 2\sum a_k^2 + \|f\|^2$$

$$= -\sum_{k=0}^{n} a_k^2 + \|f\|^2.$$

Letting $n \to \infty$, we have Bessel's inequality

$$\sum_{k=0}^{\infty} a_k^2 \leq \|f\|^2$$

If in addition the $\{y_k\}$ are complete, then the left hand side term $\int_a^b (f(x) - S_n(x))^2 dx \to 0$ and we get Parseval's identity

$$\sum_{k=0}^{\infty} a_k^2 = \|f\|^2.$$

(Note: Bessel's inequality is sometimes called Parseval's inequality.)

In Lesson 24 we computed the Fourier coefficients for a Fourier series with Legendre polynomials. In the example of p. 212 of the text some coefficients are computed for $f(x) = \sin x$. We notice that the even coefficients are all 0. To see why that is the case, we examine the exponents of the terms in the series for P_n on p. 180 when n is even. We see these are all even in this case so P_n is even for even n. Then $\sin x\, P_n(x)$ is odd and hence $\int_{-1}^{1} \sin x\, P_n(x)\, dx = 0$