Lesson 26

Trigonometric series

The usual Fourier series on $[-\pi, \pi]$ corresponds to the S-L problem

$$y'' + \lambda y = 0 \qquad -\pi \le x \le \pi$$

$$y(-\pi) = y(\pi), \quad y'(-\pi) = y'(\pi),$$

and the eigenfunctions are

$$1, \cos x, \sin x, \cos 2x, \sin 2x, \ldots$$

The Fourier coefficients are

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\,dx$$

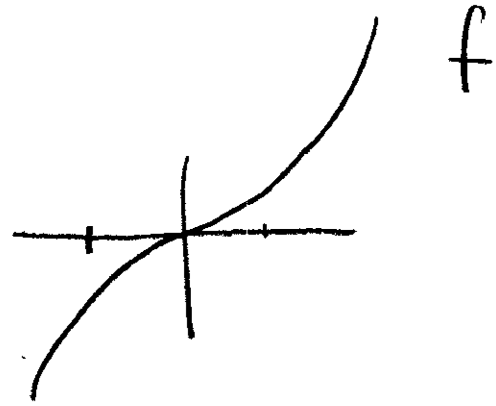
$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\,dx \qquad n = 1, 2, \ldots$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\,dx \qquad " \quad "$$

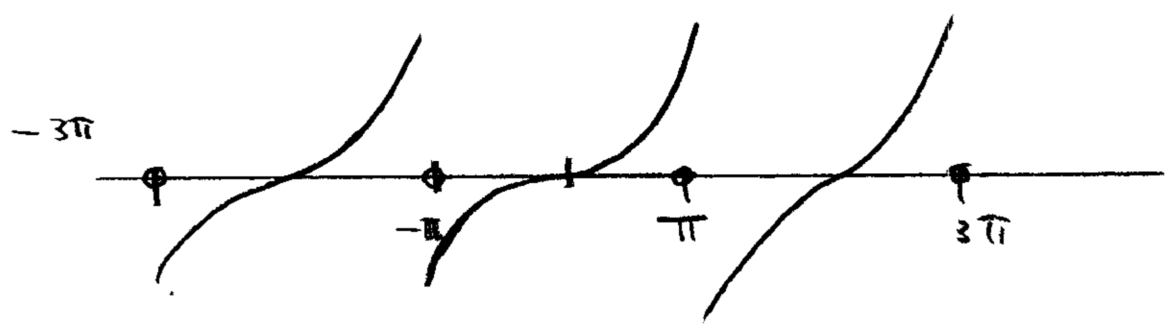
$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

Since each function on the right hand side is periodic, of period 2π, this gives a periodic extension of f outside the interval $[-\pi, \pi]$

Ex. $f(x) = x^3$

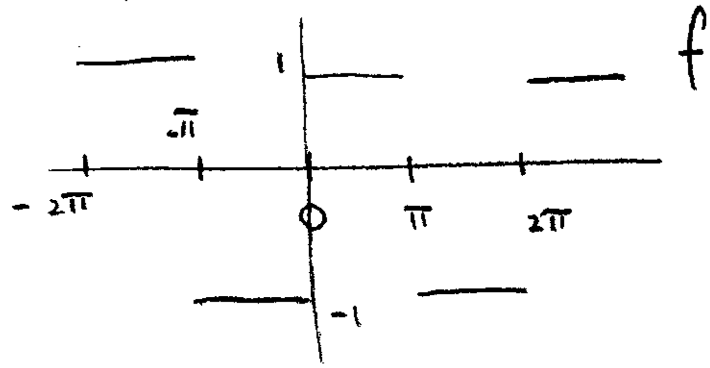


The periodic extension of f from $[-\pi, \pi]$



(discontinuous at $k\pi$ $k = \pm 1, \pm 3, \ldots$)

Ex.



$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{0} -1\, dx + \frac{1}{2\pi}\int_{0}^{\pi} 1\, dx = 0$$

$$a_m = -\frac{1}{\pi}\int_{-\pi}^{0} \cos mx\, dx + \frac{1}{\pi}\int_{0}^{\pi} \cos mx\, dx = 0$$

$$b_m = \frac{-1}{\pi}\int_{-\pi}^{0} \sin mx\, dx + \frac{1}{\pi}\int_{0}^{\pi} \sin mx\, dx =$$

$$\frac{1}{\pi}\left(\frac{\cos mx}{m}\Big|_{-\pi}^{0} - \frac{\cos mx}{m}\Big|_{0}^{\pi} \right)$$

$$= \frac{1}{m\pi}\left(1 - \cos m\pi - \cos m\pi + 1\right)$$

$$= \begin{cases} \frac{4}{m\pi} & m = 1, 3, \ldots \\ 0 & m = 0, 2, 4, \ldots \end{cases}$$

Fourier series $\sum_{n=0}^{\infty} \frac{4}{(2n+1)\pi} \sin(2n+1)x$

We know that for different eigenvalues (n^2)

$1, \sin x, \cos x, \sin 2x, \cos 2x, \ldots$ are orthogonal

because they are eigenfunctions for an S-L

problem with periodic boundary conditions.

This fact can also be verified directly by

the trig identities on page A61. However,

since $\sin nx$ and $\cos nx$ correspond to the same eigenvalue, we must check their orthogonality separately:

$$\int_{-\pi}^{\pi} \sin nx \cos nx \, dx = \frac{1}{2}\int_{-\pi}^{\pi} \sin 2nx \, dx = 0 \quad \checkmark$$

If f is continuous on $[-\pi, \pi]$ (or

piecewise continuous) we can write the

Fourier series for f: $a_0 + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$,

but we have to ask the question as to whether

or not, the Fourier series converges to f.

The set $1, \sin x, \cos x, \sin 2x, \cos 2x, \ldots$
is complete in the set of continuous functions on $[-\pi, \pi]$. Recall that this means that if $S_n = a_0 + a_1 \cos x + b_1 \sin x + \cdots + a_n \cos nx + b_n \sin nx$ is the partial sum of the Fourier series for f, then

$$\| f - S_n \| \to 0 \quad \text{as } n \to \infty,$$

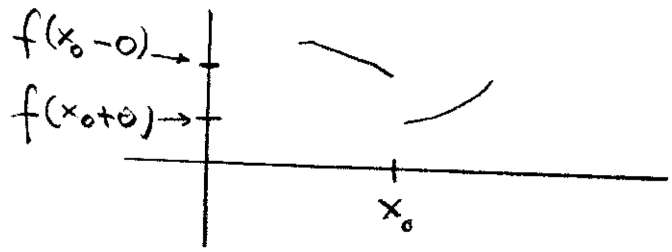
that is $S_n(x)$ converges to $f(x)$ in the <u>mean</u> <u>square</u> <u>sense</u>.

$$\int_{-\pi}^{\pi} \left(f(x) - S_n(x) \right)^2 dx \to 0 \quad \text{as } n \to \infty.$$

<u>This</u> <u>does</u> <u>not</u> <u>say</u> <u>that</u> $S_n(x) \to f(x)$ at each point. The following criterion covers many situations.

We need the notion of one sided limits

$$f(x_0-0)=\lim_{\substack{x\to x_0\\ x<x_0}} f(x)\ ,\quad f(x_0+0)=\lim_{\substack{x\to x_0\\ x>x_0}} f(x)$$



A function is piecewise continuous on [a, b] if it is continuous for all but a finite set of points $x_1, x_2, \dots, x_n$, and has one sided limits at these points.

We also need the notion of one sided derivatives.

$$\lim_{\substack{x\to x_0\\ x<x_0}} \frac{f(x)-f(x_0-0)}{x-x_0}\ ,\quad \lim_{\substack{x\to x_0\\ x>x}} \frac{f(x)-f(x_0+0)}{x-x_0}\ .$$

<u>Theorem</u>. If f is periodic of period 2π in $[-\pi, \pi]$ and has left and right derivatives at each point in $[-\pi, \pi]$, then the Fourier series for f converges to f at each point of continuity, and to $\dfrac{f(x_0+0)+f(x_0-0)}{2}$ at points x_0 of discontinuity.