Lesson 27

Arbitrary period

Suppose now the interval being used for the Fourier expansion is $[-L, L]$ instead of $[-\pi, \pi]$. Then the orthogonal family becomes

$$1,\ \cos\frac{\pi x}{L},\ \sin\frac{\pi x}{L},\ \cos\frac{2\pi x}{L},\ \sin\frac{2\pi x}{L},\ \dots$$

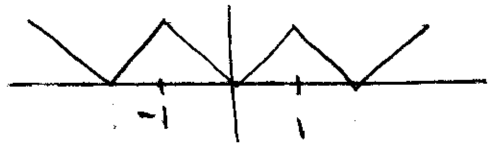
and the Fourier coefficients are

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx \qquad n=1,2,\dots$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx \qquad \text{" "}.$$

Ex. $f(x) = |x| \qquad -1 < x < 1$



$$a_0 = \frac{1}{2}\left(\int_{-1}^{0} -x\,dx + \int_{0}^{1} x\,dx\right)$$

$$= \frac{1}{2}\left(-\frac{1}{2}x^2\Big|_{-1}^{0} + \frac{1}{2}x^2\Big|_{0}^{1}\right) = \frac{1}{2}$$

$$b_n = \int_{-1}^{1} |x| \sin n\pi x\,dx = 0$$ since $|x|\sin n\pi x$ is odd

$$a_n = -\int_{-1}^{0} x\cos n\pi x\,dx + \int_{0}^{1} x\cos n\pi x\,dx$$

$$\int_a^b x\cos n\pi x\,dx = \frac{x}{n\pi}\sin n\pi x\Big|_a^b - \frac{1}{n\pi}\int_a^b \sin n\pi x\,dx$$

$u = x \qquad dv = \cos n\pi x\,dx$

$du = dx \qquad v = \frac{1}{n\pi}\sin n\pi x$

$$a_n = \frac{-1}{n^2\pi^2}\cos n\pi x\Big|_{-1}^{0} + \frac{1}{n^2\pi^2}\cos n\pi x\Big|_{0}^{1}$$

$$= -\frac{1}{n^2\pi^2} + \frac{1}{n^2\pi^2}\cos n\pi + \frac{1}{n^2\pi^2}\cos n\pi - \frac{1}{n^2\pi^2}$$

$$= \frac{2}{n^2\pi^2}\left(\cos n\pi - 1\right)$$

Fourier series

$$\frac{1}{2} + \sum_{n=1}^{\infty} \frac{2\left((-1)^n - 1\right)}{n^2\pi^2}\cos n\pi x$$

$$= \frac{1}{2} - \sum_{n=0}^{\infty} \frac{4}{(2n+1)^2\pi^2}\cos(2n+1)\pi x$$

<u>Ex.</u> Half rectified sine wave

$$f(t) = \begin{cases} 0 & -L < t < 0 \\ \sin \omega t & 0 < t < L \end{cases} \qquad \omega = \frac{\pi}{L}$$

$$a_0 = \frac{\omega}{2\pi}\int_0^{\pi/\omega} \sin\omega t \, dt = \frac{1}{\pi}$$

$$a_n = \frac{\omega}{\pi}\int_0^{\pi/\omega} \sin\omega t \cos n\omega t \, dt$$

$$= \frac{\omega}{2\pi}\int_0^{\pi/\omega} \sin(1+n)\omega t + \sin(1-n)\omega t \, dt$$

where we have used trig. ident. A61.

If $n=1$, we get $a_1 = 0$.

Otherwise,

$$a_n = \frac{\omega}{2\pi}\left(-\frac{\cos(1+n)\omega t}{(1+n)\omega} - \frac{\cos(1-n)\omega t}{(1-n)\omega} \right)\Bigg|_0^{\pi/\omega}$$

$$= \frac{\omega}{2\pi}\left(\frac{-\cos(1+n)\pi + 1}{(1+n)\omega} + \frac{-\cos(1-n)\pi + 1}{(1-n)\omega} \right)$$

$= 0$ if n odd, and for evens $n = 2, 4, \ldots$

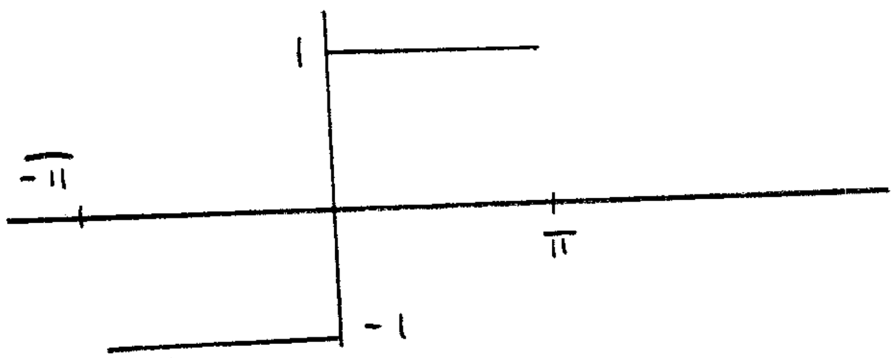
$$a_n = \frac{1}{2\pi}\left(\frac{2}{1+n} + \frac{2}{1-n} \right) = \frac{2}{(1-n)(1+n)\pi}$$

Using A 61 trig identities for b_n,
we find $b_1 = 1/2$ and $b_n = 0$ $n = 2, 3, \ldots$

Thus the Fourier series is

$$\frac{1}{\pi} + \frac{1}{2}\sin\omega t - \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)} \cos 2n\omega t.$$

Ex. Square wave (period 2π)



$$\frac{4}{\pi}\sum_{n=0}^{\infty} \frac{1}{2n+1} \sin(2n+1)x$$