Lesson 28

1.

Odd and even functions

$f(x)$ is <u>even</u> if $f(-x) = f(x)$

eg. $\cos x$, x^2, $x \sin x$

$f(x)$ is <u>odd</u> if $f(-x) = -f(x)$

eg. $\sin x$, x, x^3, $x \cos x$

<u>Note</u> product of even · even = even
even · odd = odd
odd · odd = even.

Important fact here is, if f is odd,

$$\int_{-L}^{L} f(x)\,dx = 0$$

since $\displaystyle\int_{-L}^{L} f(x)dx = \int_{-L}^{0} f(x)dx + \int_{0}^{L} f(x)\,dx$

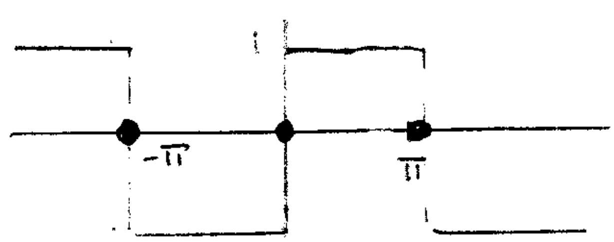
In the first integral on the right, let $x = -t$ so $dx = -dt$. Then,

$$\int_{-L}^{0} f(x)\,dx = \int_{0}^{L} f(-t)\,dt = -\int_{0}^{L} f(t)\,dt$$

so that $$\int_{-L}^{L} f(x)\,dx = -\int_{0}^{L} f(t)\,dt + \int_{0}^{L} f(t)\,dt = 0.$$

This means that if f is even, then all b_n's in its Fourier series are 0, and if f is odd, all a_n's (<u>including</u> a_0) are 0.

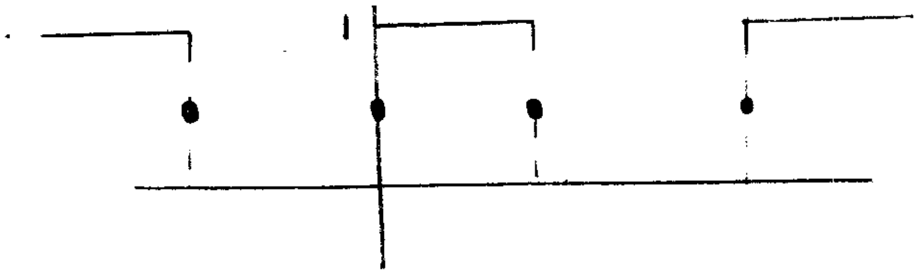
Ex.



square wave
odd

$$\frac{4}{\pi}\sum_{n=0}^{\infty}\frac{1}{2n+1}\sin(2n+1)x$$

but



neither even nor odd

$$\frac{1}{2} + \frac{2}{\pi} \sum_{n=0}^{\infty} \frac{1}{2n+1} \sin(2n+1)x$$

$\uparrow$

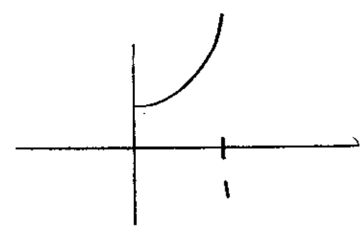
$a_0 \neq 0$

Similarly, if f is even, then

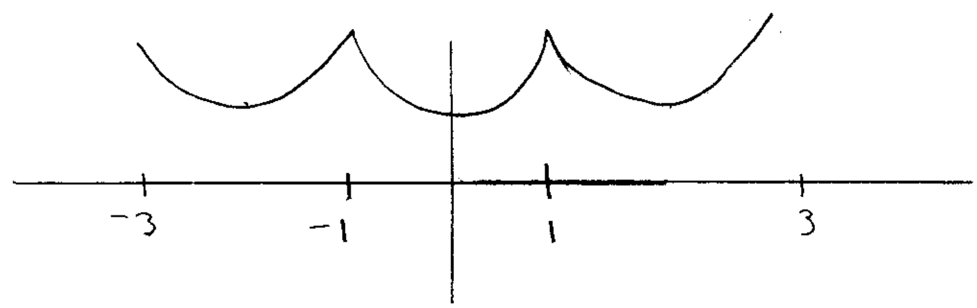
$$\int_{-L}^{L} f(x)\,dx = 2\int_{0}^{L} f(x)\,dx.$$

If f is defined on an interval $[0, L]$, we can make an odd periodic extension by expanding f in a sine series, and an even periodic extension with cosine series.

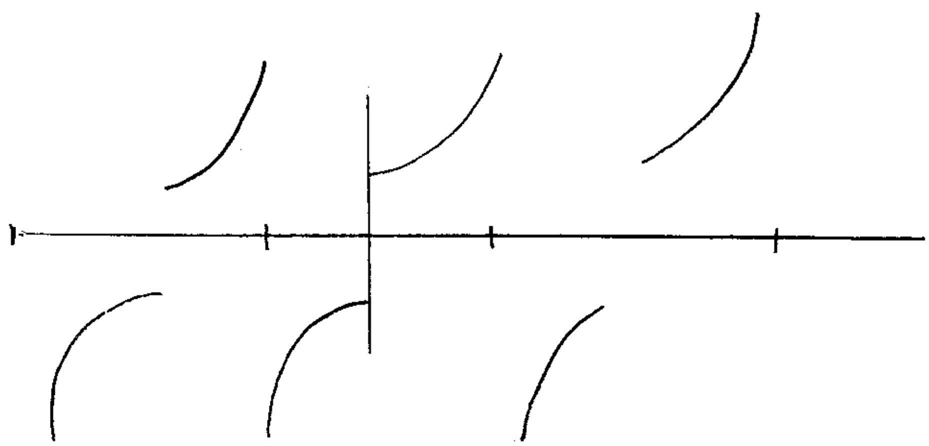
Ex.



$f(x) = x^2 + 1$
$0 \le x \le 1$



even periodic extension



odd periodic extension

For the odd extension of f (Fourier sine series)

$$a_n = 0 \qquad n = 0, 1, 2, \ldots$$

$$b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} \, dx$$

and for the even extension of f (Fourier cosine series)

$$a_0 = \frac{1}{L}\int_0^L f(x)\,dx$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx \qquad n = 1, 2, \ldots$$

$$b_n = 0 \qquad n = 1, 2, \ldots$$

Ex. Find the Fourier sine and cosine series for $f(x) = \sin x \qquad 0 \le x \le \pi$

(1) Cosine series

$$a_0 = \frac{1}{\pi}\int_0^\pi \sin x\,dx = -\frac{\cos x}{\pi}\Big|_0^\pi = \frac{2}{\pi}$$

$$a_1 = \frac{2}{\pi}\int_0^\pi \sin x\cos x\,dx = \frac{1}{\pi}\sin^2 x\Big|_0^\pi = 0$$

$$a_n = \frac{2}{\pi}\int_0^\pi \sin x\cos nx\,dx \qquad (n \ge 2)$$

$$= \frac{1}{\pi}\int_0^\pi \sin(1+n)x + \sin(1-n)x\,dx$$

$$= \frac{1}{\pi}\left(\frac{-1}{1+n} \cos(1+n)x \Big|_0^{\pi} - \frac{1}{1-n} \cos(1-n)x \Big|_0^{\pi} \right)$$

$$= \frac{1}{\pi}\left(\frac{-\cos(1+n)\pi + 1}{1+n} + \frac{-\cos(1-n)\pi + 1}{1-n} \right)$$

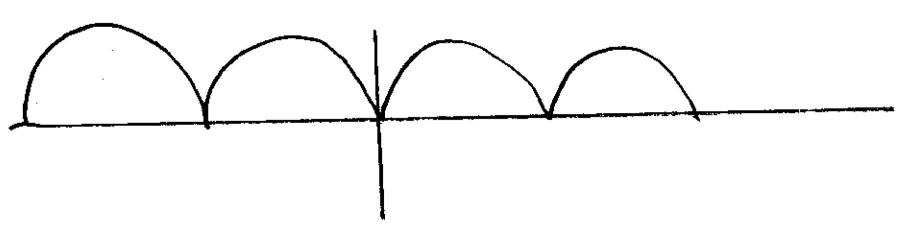
$$= \frac{1}{\pi}\begin{cases} 0 & n \text{ odd} \\ \frac{2}{1+n} + \frac{2}{1-n} & n \text{ even} \end{cases}$$

Thus,

$$a_n = -\frac{4}{\pi(n+1)(n-1)} \qquad n > 1 \text{ even}$$

Fourier series

$$g(x) = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)} \cos 2nx$$



which is the even extension of $\sin x$, the full rectified sine function.

The sine series for $\sin x$ is simply

$$\sin x$$

Notice that the half rectified sine function can be obtained from the full rectified function $g(x)$ $-\infty < x < \infty$ and $f(x) = \sin x$ $-\infty < x < \infty$. We simply take

$$\frac{1}{2} f(x) + \frac{1}{2} g(x).$$

This gives

$$\frac{1}{2} \sin x + \frac{1}{\pi} - \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)} \cos 2nx$$

which checks out with the example of Lesson 27.