Complex Fourier series

We take the formula

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$$

together with the Euler formula

$$e^{inx} = \cos nx + i \sin nx$$

$$e^{-inx} = \cos nx - i \sin nx$$

or

$$\cos nx = \frac{e^{inx} + e^{-inx}}{2}$$

$$\sin nx = \frac{e^{inx} - e^{-inx}}{2i}$$

to write

$$a_n \cos nx + b_n \sin nx = \frac{a_n}{2}\left(e^{inx} + e^{-inx}\right) + \frac{b_n}{2i}\left(e^{inx} - e^{-inx}\right)$$

$$= \frac{1}{2}(a_n - ib_n)\, e^{inx} + \frac{1}{2}(a_n + ib_n)\, e^{-inx}$$

$$= c_n e^{inx} + c_{-n} e^{-inx} \quad \text{where}$$

$$c_n = \frac{1}{2}(a_n - ib_n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)(\cos nx - i\sin nx)\,dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx$$

$$c_{-n} = \frac{1}{2}(a_n + ib_n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)(\cos nx + i\sin nx)\,dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{inx}\,dx.$$

(This works even for $n=0$)

With these formulas and $c_0 = a_0$ we have

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}.$$

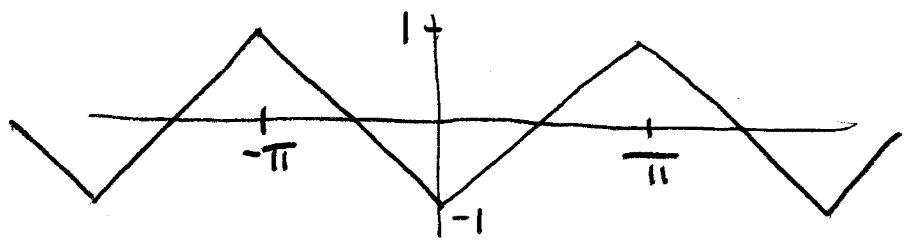
Just as with Fourier series in noncomplex form we can use $[-L, L]$ instead of $[-\pi, \pi]$ and have

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/L} \quad \text{where}$$

$$c_n = \frac{1}{2L} \int_{-L}^{L} f(x) e^{-in\pi x/L}\, dx$$

<u>Ex.</u> Find the complex Fourier series of the 2π periodic function $F(t)$ (sawtooth)

$$F(t) = \begin{cases} -1 - \frac{2t}{\pi} & -\pi \le t \le 0 \\ -1 + \frac{2t}{\pi} & 0 \le t \le \pi \end{cases}$$



$$C_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(t) e^{-int}\, dt$$

$$= \frac{1}{2\pi} \int_{-\pi}^{0} \left(-1 - \frac{2t}{\pi}\right) e^{-int}\, dt + \frac{1}{2\pi} \int_{0}^{\pi} \left(-1 + \frac{2t}{\pi}\right) e^{-int}\, dt$$

$$\int_a^b t e^{-int}\,dt = \frac{it}{n}e^{-int}\Big|_a^b + \frac{1}{n^2}e^{-int}\Big|_a^b$$

$$u = t \qquad dv = e^{-int}dt$$
$$du = dt \qquad v = \frac{i}{n}e^{-int}$$

and also

$$\int_{-\pi}^{\pi} e^{-int}\,dt = 0$$

so $(n \neq 0)$

$$C_n = \frac{1}{2\pi}\left[\frac{-2}{\pi}\left(\frac{i\pi}{n}e^{in\pi} + \frac{1}{n^2} - \frac{1}{n^2}e^{in\pi}\right)\right.$$

$$\left. + \frac{2}{\pi}\left(\frac{i\pi}{n}e^{-in\pi} + \frac{1}{n^2}e^{-in\pi} - \frac{1}{n^2}\right)\right]$$

$$= \frac{1}{\pi^2}\left(-\frac{2}{n^2} + \frac{1}{n^2}\left(e^{in\pi} + e^{-in\pi}\right)\right)$$

$$= \frac{2}{\pi^2 n^2}\left(-1 + \cos n\pi\right) = \frac{2(-1+(-1)^n)}{\pi^2 n^2}.$$

Also

$$c_0 = \frac{1}{2\pi}\int_{-\pi}^{0}\left(-1 - \frac{2t}{\pi}\right)dt + \frac{1}{2\pi}\int_0^{\pi}\left(-1 + \frac{2t}{\pi}\right)dt = 0$$

Thus,
$$F(t) = \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{2(-1+(-1)^n)}{\pi^2 n^2} e^{int}.$$

We can write this in noncomplex form

by taking
$$c_n = \frac{1}{2}(a_n - ib_n)$$
$$c_{-n} = \frac{1}{2}(a_n + ib_n)$$

and solving
$$a_n = c_n + c_{-n}$$
$$-ib_n = c_n - c_{-n} \Rightarrow b_n = i(c_n - c_{-n}).$$

Here,
$$a_n = \frac{2(-1+(-1)^n)}{\pi^2 n^2} + \frac{2((-1)+(-1)^{-n})}{\pi^2(-n)^2} = \frac{4(-1+(-1)^n)}{\pi^2 n^2}$$

$$b_n = i\left(\frac{2(-1)+(-1)^n}{\pi^2 n^2} - \frac{2((-1)+(-1)^{-n})}{\pi^2(-n)^2}\right) = 0$$

Thus,

$$F(t) = \sum_{n=1}^{\infty} \frac{4}{\pi^2 n^2}\left(-1+(-1)^n\right) \cos nt.$$

Note that the complex Fourier coefficients can be derived without reference to real Fourier series. This is due to the fact that the functions $\{e^{inx}\}$ are orthogonal with respect to the complex inner product $(f,g) = \int_{-\pi}^{\pi} f(x)\bar{g}(x)\,dx$.

$$(e^{imx}, e^{inx}) = \int_{-\pi}^{\pi} e^{i(m-n)x}\,dx = \begin{cases} 0 & m \neq n \\ 2\pi & m = n \end{cases}$$

So if $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$, mult. both sides by e^{-ikx} and integrate. Then

$$\int_{-\pi}^{\pi} e^{-ikx} f(x)\,dx = \sum_{n=-\infty}^{\infty} c_n e^{i(n-k)}\,dx = 2\pi c_k.$$