Lesson 3

Rank and Linear Independence

<u>Notation</u>. $\mathbb{R}^n$ is the set of all n component vectors. Usually we represent these by column vectors.

Suppose $v_1, \dots, v_m$ are vectors in $\mathbb{R}^n$.

A combination

$$c_1 v_1 + c_2 v_2 + \dots + c_m v_m$$

where $c_1, c_2, \dots, c_m$ are numbers, is called a <u>linear combination</u> of $v_1, \dots, v_m$.

If $c_1 = c_2 = \dots = c_m = 0$, then the linear combination is the 0 vector (we write 0 for both the number 0 and the vector $\begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$ so be careful). It can however happen that we get the 0 vector sometimes when some c_j's are not 0.

<u>Ex.</u> $$1\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + (-2)\begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} + 1\begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

When this happens we say that the vectors

are linearly dependent. That is,
if there exist $c_1, \dots, c_n$ not all 0
(note: some may be 0) such that

$$c_1 v_1 + c_2 v_2 + \cdots + c_m v_m = 0$$

eg. If in a list $v_1, \dots, v_m$, the vector V_3 were the 0 vector, the set would be linearly dependent since we could take $c_3 = 1$ and all other c_j's to be 0.

A set $v_1, \dots, v_m$ is linearly independent if the only linear combination

$$c_1 v_1 + c_2 v_2 + \cdots + c_m v_m$$

which gives the zero vector is the trivial one

$$c_1 = c_2 = \cdots = c_m = 0.$$

Ex $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ $\begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$ $\begin{pmatrix} 7 \\ 8 \\ 10 \end{pmatrix}$ linearly independent set.

To see this, let's <u>solve</u> for all c_1, c_2, c_3 such that

$$c_1 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + c_2 \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} + c_3 \begin{pmatrix} 7 \\ 8 \\ 10 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

i.e.

$$\begin{aligned} c_1 + 4c_2 + 7c_3 &= 0 \\ 2c_1 + 5c_2 + 8c_3 &= 0 \\ 3c_1 + 6c_2 + 10c_3 &= 0 \end{aligned}$$

which is a homogeneous linear system.

Using Gaussian elimination (we do not have to write the b column since $b = 0$, and row operations don't change anything)

$$\begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & \cancel{9}\,10 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 4 & 7 \\ 0 & -3 & -6 \\ 0 & -6 & -11 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 4 & 7 \\ 0 & -3 & -6 \\ 0 & 0 & 1 \end{pmatrix}$$

There are no free variables, so back subst. gives only the <u>trivial</u> solution

$$c_1 = 0 \quad c_2 = 0 \quad c_3 = 0.$$

$\mathbb{R}^n$ is one example of a general concept in mathematics known as a <u>vector space</u>. This will be discussed further in Lesson 6. For now we explore the <u>subspaces</u> of $\mathbb{R}^n$. These are subsets which are themselves vector spaces. In order for a subset V to be a vector space it must have <u>closure</u> with respect to addition and multiplication by a scalar. That is,

i) If $v \in V$, then $cv \in V$ for every number c

ii) If $u, v \in V$, then $u+v \in V$.

<u>Ex.</u> The set V of all vectors of the form $\begin{pmatrix} a \\ b \\ 0 \end{pmatrix}$ is a vector space, that is a subspace of $\mathbb{R}^3$.

Check:
$$c\begin{pmatrix} a \\ b \\ 0 \end{pmatrix} = \begin{pmatrix} ca \\ cb \\ 0 \end{pmatrix} \checkmark$$
$$\begin{pmatrix} a \\ b \\ 0 \end{pmatrix} + \begin{pmatrix} c \\ d \\ 0 \end{pmatrix} = \begin{pmatrix} a+c \\ b+d \\ 0 \end{pmatrix} \checkmark$$

Ex. The set V of all vectors $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ with $a \geq 0$ is not a vector space since $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \in V$ but

$$-1\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \\ -3 \end{pmatrix} \notin V.$$

Ex. The set V of all vectors $\begin{pmatrix} a \\ b \end{pmatrix}$ such that $a + 2b = 0$ is a vector space:

✓ $c\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} ca \\ cb \end{pmatrix}$ and $ca + 2cb = c(a+2b) = 0$

✓ $\begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} a+c \\ b+d \end{pmatrix}$ and $a + c + 2(b+d)$
$= a + 2b + c + 2d = 0 + 0 = 0$

However, the set W of vectors $\begin{pmatrix} a \\ b \end{pmatrix}$ such that $a + 2b = 1$ is not a vector space since, for example,

$$0\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \notin W.$$

If V is a vector space, and every vector in V can be written as a linear combination of $v_1, \ldots, v_k$ then $v_1, \ldots, v_k$ is said to <u>span</u> V.

<u>Ex.</u>
$$\begin{pmatrix}1\\0\\0\end{pmatrix} \begin{pmatrix}0\\1\\0\end{pmatrix} \begin{pmatrix}0\\0\\1\end{pmatrix} \quad \text{span } \mathbb{R}^3$$

and so do
$$\begin{pmatrix}1\\2\\3\end{pmatrix} \begin{pmatrix}4\\5\\6\end{pmatrix} \begin{pmatrix}7\\8\\10\end{pmatrix}.$$

<u>In</u> fact
$$\begin{pmatrix}1\\0\\0\end{pmatrix} \begin{pmatrix}0\\1\\0\end{pmatrix} \begin{pmatrix}0\\0\\1\end{pmatrix} \begin{pmatrix}1\\2\\3\end{pmatrix} \quad \text{span } \mathbb{R}^3$$

but we don't need the last vector.

A smallest spanning set for a vector space V is called a <u>basis</u> for V. More precisely, a set $v_1, \ldots, v_k$ is a basis for V if

i) $v_1, \dots, v_k$ span V

ii) $v_1, \dots, v_k$ are linearly independent.

Ex. $\begin{pmatrix}1\\0\\0\end{pmatrix} \begin{pmatrix}0\\1\\0\end{pmatrix} \begin{pmatrix}0\\0\\1\end{pmatrix}$ basis for $\mathbb{R}^3$

$\begin{pmatrix}1\\2\\3\end{pmatrix} \begin{pmatrix}4\\5\\6\end{pmatrix} \begin{pmatrix}7\\8\\10\end{pmatrix}$ " "

$\begin{pmatrix}1\\2\\3\end{pmatrix} \begin{pmatrix}4\\5\\6\end{pmatrix} \begin{pmatrix}7\\8\\9\end{pmatrix}$ <u>not</u> basis for $\mathbb{R}^3$ (we showed earlier they are linearly dependent)

The number of vectors in a basis for a vector space V is called the <u>dimension</u> of V.

<u>Ex</u> $\dim \mathbb{R}^n = n$.

Let A be an $m \times n$ matrix. There are several important vector spaces associated with A

The <u>row space</u> is the set of all linear combinations of the rows. The <u>column space</u> is the set of all linear combinations of the columns.

It is not obvious, but both of these spaces have the same dimension, called the <u>rank</u> of A.

There is a simple algorithm to find the rank of A. Namely, bring A to echelon form by row operations. Then the number of nonzero rows of the echelon form is the rank.

<u>Ex</u> $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$ has rank 2

<u>Ex</u>. $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{pmatrix}$ has rank 3

There is also a simple algorithm to find a basis for the row space and basis for column space

For row space, bring matrix A to echelon form. The nonzero rows form a basis for row space.

The column space is trickier. Bring A to echelon form. Locate columns of echelon form having pivots. Go back to <u>original</u> <u>matrix</u> and take those same columns.

<u>Ex.</u> In Lesson ~~1~~ 2 we found an echelon form for

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \quad \text{to be} \quad \begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{pmatrix}$$

Basis for rowspace $(1 \ \ 2 \ \ 3)$ $(0 \ \ -3 \ \ -6)$

Basis for column space $\begin{pmatrix} 1 \\ 4 \\ 7 \end{pmatrix}$ $\begin{pmatrix} 2 \\ 5 \\ 8 \end{pmatrix}$

rank $A = 2$.

Extra Examples.

Ex. Test to see if the vectors

$$(1\ 2\ 3\ 4) \quad (5\ 6\ 7\ 8) \quad (9\ 10\ 11\ 12)$$

are linearly independent.

Solution: One simple way is to make these the rows of a matrix and perform row operations to bring the matrix to echelon form. If there are no zero rows, then the original set is independent

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -4 & -8 & -12 \\ 0 & -8 & -16 & -24 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -4 & -8 & -12 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The set is dependent.

Ex. Find a basis for the row space and column space of the matrix $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{pmatrix}$

from the previous example

Solution An echelon form is

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -4 & -8 & -12 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

so a basis for row space is

$(1 \ 2 \ 3 \ 4)$ $(0 \ -4 \ -8 \ -12)$

(perhaps $(1 \ 2 \ 3 \ 4)$ $(0 \ 1 \ 2 \ 3)$ looks nicer)

The corner entries for the above echelon form are the 1 and -4 so a basis for the column space is

$$\begin{pmatrix} 1 \\ 5 \\ 9 \end{pmatrix} \quad \begin{pmatrix} 2 \\ 6 \\ 10 \end{pmatrix}$$

Ex. Find a basis for the vector space V of

$v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{pmatrix}$ in $\mathbb{R}^5$ such that

$$v_1 + v_2 = 0$$
$$v_2 + v_3 - 2v_4 = 0.$$

Solution.

Let $v \in V$ and solve the system

$$v_1 + v_2 \quad = 0$$
$$v_2 + v_3 - 2v_4 = 0$$

The augmented matrix to solve for v_1, v_2, v_3, v_4, v_5 is

$$\left(\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & -2 & 0 & 0 \end{array}\right)$$

This is already in echelon form (what good fortune!) so all we have to do is to solve by back substitution. Now, v_3, v_4, v_5 are free. Set $v_3 = r$ $v_4 = s$ $v_5 = t$ and back substitute.

$$v_2 = -r + 2s \qquad v_1 = r - 2s.$$

Thus, any $v \in V$ has the form

$$\begin{pmatrix} r-2s \\ -r+2s \\ r \\ s \\ t \end{pmatrix}$$

This can be written

$$v = r\begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + s\begin{pmatrix} -2 \\ 2 \\ 0 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

so

$$\begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \begin{pmatrix} -2 \\ 2 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

is a basis for V.