Lesson 30

Forced oscillations / trig polynomials

We consider the nonhomogeneous linear equation

$$y'' + 2y' + 2y = F(t)$$

where $F(t)$ is the saw tooth function of Lesson 29.

Recall that any solution y can be written $y_h + y_p$ where y_h is the general solution to the homogeneous equation, and y_p is a particular solution.

Here we find y_h by solving $r^2 + 2r + 2 = 0$

$$\Rightarrow \quad r = \frac{-2 \pm \sqrt{4-8}}{2} = -1 \pm i$$

so $\quad y_h = c_1 e^{-t} \cos t + c_2 e^{-t} \sin t$

and $\quad y_h \to 0$ as $t \to \infty$.

Thus, to find a steady state solution, we must compute a g_p.

If we use complex Fourier series, we may proceed as follows.

We suppose that g_p is written as a complex Fourier series $g_p = \sum_{n=-\infty}^{\infty} c_n e^{int}$ and substitute, setting terms involving e^{int} equal.

So, first consider a solution $y_n = d_n e^{int}$ to $y_n'' + 2y_n' + 2y_n = e^{int}$. Substituting, we get

$$-n^2 d_n e^{int} + 2in\, d_n e^{int} + 2d_n e^{int} = e^{int}$$

$$\Rightarrow \quad (-n^2 + 2in + 2)\, d_n = 1$$

$$\Rightarrow \quad y_n = \frac{e^{i\omega t}}{-n^2 + 2in + 2}$$

Using this with the Fourier series,

$$F(t) = \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{2\left(-1+(-1)^n\right)}{\pi^2 n^2} e^{int} \quad \text{(sawtooth)}$$

from Lesson 29, we get the steady state solution by multiplying y_n by c_n and adding:

$$\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{2\left(-1+(-1)^n\right)}{\pi^2 n^2\left(-n^2+2in+2\right)} e^{int}$$

This will be done using real Fourier series on pp 7 & 8

The natural approximation in Fourier series is mean square, where the error in approximating f by a function F is

$$E = \int_{-\pi}^{\pi} \left(f(x) - F(x)\right)^2 dx.$$

Suppose that $F(x)$ is a trigonometric polynomial of degree N

$$F(x) = A_0 + \sum_{n=1}^{N} A_n \cos nx + B_n \sin nx.$$

We wish to show that the minimum error occurs when F is taken as the n^{th} partial sum of the Fourier series for f,

$$S_N = a_0 + \sum_{n=1}^{N} a_n \cos nx + b_n \sin nx .$$

As we have done in Lesson 25, write

$$E = \int_{-\pi}^{\pi} f(x)^2 dx - 2\int_{-\pi}^{\pi} f(x)F(x) + \int_{-\pi}^{\pi} F(x)^2 dx$$

$$\int_{-\pi}^{\pi} F(x)^2 dx = \pi\left(2A_0^2 + \sum_{n=1}^{N} A_n^2 + B_n^2\right)$$

$$\int_{-\pi}^{\pi} f(x)F(x)\,dx = \pi\left(2A_0 a_0 + \sum_{n=1}^{N} A_n a_n + B_n b_n\right)$$

Now do the same with F replaced by S_N and call $E^* = \int_{-\pi}^{\pi}\left(f(x) - S_N\right)^2 dx.$

As in Lesson 25 we get

$$E^* = \int_{-\pi}^{\pi} f^2 dx - \pi\left(2a_0^2 + \sum_{n=1}^{N} a_n^2 + b_n^2\right)$$

so that

$$E - E^* = -2\pi\left(2A_0 a_0 + \sum_{n=1}^{N} (A_n a_n + B_n b_n)\right)$$

$$+ \pi\left(2A_0^2 + 2a_0^2 + \sum_{n=1}^{N} A_n^2 + B_n^2 + a_n^2 + b_n^2\right)$$

$$= \pi\left(2(A_0 - a_0)^2 + \sum_{n=1}^{N} (A_n - a_n)^2 + (B_n - b_n)^2\right)$$

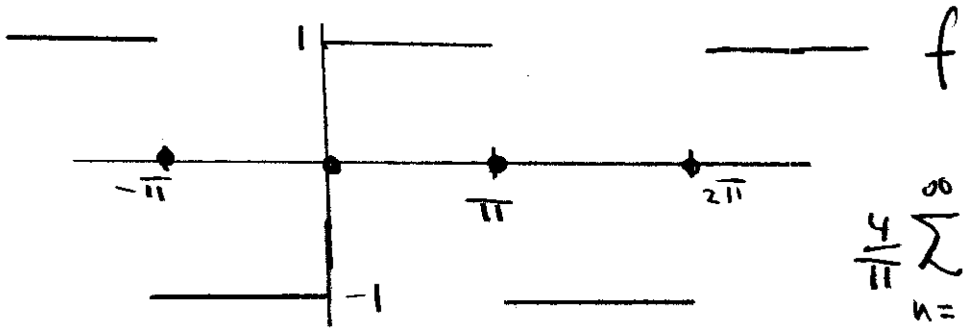
$\geq 0.$

Thus, $E^* \leq E$, and S_N is optimal.

We saw Parseval's identity in Lesson 25 for generalized Fourier series. For trig series this becomes

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)^2 dx = 2a_0^2 + \sum_{n=1}^{\infty} a_n^2 + b_n^2$$

Consider for example the square wave



$$\frac{4}{\pi}\sum_{n=0}^{\infty}\frac{1}{2n+1}\sin(2n+1)x$$

Then, $\displaystyle \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)^2\,dx = \frac{1}{\pi}\cdot 2\pi = 2$ so

$$\frac{16}{\pi^2}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2} = 2$$

ie. $\displaystyle 1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \frac{\pi^2}{8}$

Notice we can also get series from points. Take

$x = \pi/2$ so $\sin(2n+1)\pi/2 = (-1)^n$ and

$$\frac{4}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1} = 1$$

$$\Rightarrow \quad 1 - \tfrac{1}{3} + \tfrac{1}{5} - \dots = \pi/4$$

Example can also be done in real form by substituting $a_n \cos nt + b_n \sin nt$ <u>but it is more work.</u> For the same problem,

$$F(t) = \sum_{n=1}^{\infty} \frac{4}{\pi^2 n^2}\left((-1)+(-1)^n\right)\cos nt$$

$y_n = A_n \cos nt + B_n \sin nt$ gives

$$-n^2 A_n \cos nt - n^2 B_n \sin nt + 2(-nA_n \sin nt + nB_n \cos nt)$$
$$+ 2(A_n \cos nt + B_n \sin nt)$$
$$= \frac{4}{\pi^2 n^2}\left(-1+(-1)^n\right)\cos nt$$

$$-n^2 A_n + 2nB_n + 2A_n = \frac{4}{\pi^2 n^2}\left(-1+(-1)^n\right)$$
$$-n^2 B_n - 2nA_n + 2B_n = 0$$

i.e.

$$(2-n^2)A_n + 2nB_n = \frac{4}{\pi^2 n^2}\left(-1+(-1)^n\right)$$
$$(2-n^2)B_n - 2nA_n = 0$$

$$B_n = \frac{2n}{2-n^2} A_n \qquad \text{so}$$

$$(2-n^2)A_n + 2n \left(\frac{2n}{2-n^2}\right. A_n = \frac{4}{\pi^2 n^2}\left(-1+(-1)^n\right)$$

$$\frac{4+n^4}{2-n^2} A_n = \frac{4}{\pi^2 n^2}\left(-1+(-1)^n\right)$$

$$A_n = \frac{4(2-n^2)\left(-1+(-1)^n\right)}{(4+n^4)\,\pi^2 n^2}$$

$$B_n = \frac{8\left(-1+(-1)^n\right)}{(4+n^4)\pi^2\, n}$$

With these coefficients, the steady state solution is

$$\sum_{n=1}^{\infty} A_n \cos nt + \sum_{n=1}^{\infty} B_n \sin nt$$