Lesson 31

Fourier Integrals

Suppose $f(x)$ is defined for $-\infty < x < \infty$, and is expanded in a Fourier series on $[-L, L]$,

$$f_L(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos w_n x + b_n \sin w_n x$$

$$w_n = \frac{n\pi}{L}.$$

Then, f_L is $2L$ periodic and $f_L(x) = f(x)$ when $-L < x < L$. (Assuming e.g. that f has continuous derivative).

Writing the formulas for a_n's and b_n's we get

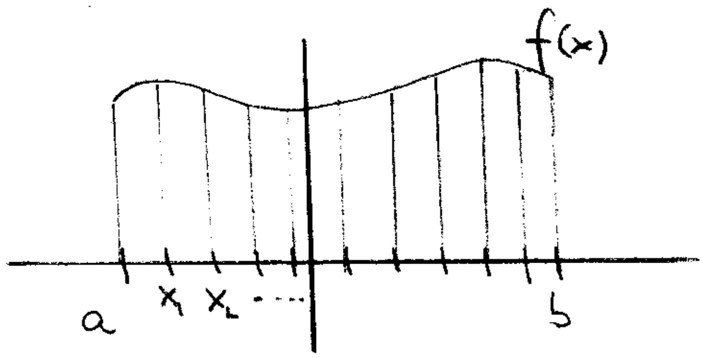
$$f_L(x) = \frac{1}{2L}\int_{-L}^{L} f(v)\,dv + \frac{1}{L}\sum_{n=1}^{\infty}\left(\cos w_n x \int_{-L}^{L} f(v)\cos w_n v\,dv\right.$$

$$\left. + \sin w_n x \int_{-L}^{L} f(v)\sin w_n v\,dv\right)$$

To see how this evolves, we recall the definition of the integral from elementary calculus

$$\int_a^b f(x)\,dx = \lim_{\Delta x_i \to 0} \sum_{i=1}^{m} f(\hat{x}_i)\,\Delta x_i$$

(Riemann sum)



$$a = x_0 < x_1 < \dots < x_m = b$$

$$\Delta x_i = x_i - x_{i-1}$$

$$x_{i-1} \le \hat{x}_i \le x_i$$

In the expression for f_L we take

$$\Delta w = w_{n+1} - w_n = \frac{(n+1)\pi}{L} - \frac{n\pi}{L} = \pi/L.$$

As we make the expression for f_L look like a Riemann sum, there are some technical differences. ① Riemann sum is finite, ② the interval of integration is a finite interval $[a,b]$ whereas we want $L \to \infty$.

$$f_L(x) = \frac{1}{2L}\int_{-L}^{L} f(v)\,dv + \frac{1}{\pi}\sum_{n=1}^{\infty}\left(\cos w_n x\,\Delta w\int_{-L}^{L} f(v)\cos w_n v\,dv\right.$$

$$\left. + \sin w_n x\,\Delta w\int_{-L}^{L} f(v)\sin w_n v\,dv\right)$$

If f is a continuous (piecewise continuous) function such that $\int_{-\infty}^{\infty}|f(x)|\,dx < \infty$, then by letting $L \to \infty$ we get the <u>Fourier</u> <u>integral</u>

$$f(x) = \frac{1}{\pi}\int_{0}^{\infty}\left(\cos wx\int_{-\infty}^{\infty} f(v)\cos wv\,dv\right.$$

$$\left. + \sin wx\int_{-\infty}^{\infty} f(v)\sin wv\,dv\right)dw$$

$$= \int_0^\infty \left(A(w)\cos wx + B(w)\sin wx\right) dw$$

where

$$A(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos wv\, dv$$

$$B(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin wv\, dv$$

As with Fourier series, we can take odd and even extensions of f which is defined for $0 < x < \infty$. These give <u>Fourier sine</u> and <u>cosine integrals</u>.

For the odd extension $A(w) = 0$ and

$$B(w) = \frac{2}{\pi}\int_0^\infty f(v)\sin wv\, dv,$$

and for even extension $B(w) = 0$ and

$$A(w) = \frac{2}{\pi}\int_0^\infty f(v)\cos wv\, dv.$$

Ex Find Fourier cosine and sine integrals of $f(x) = e^{-kx}$ $\quad (x>0,\ k>0)$

For cosine integral,

$$A(w) = \frac{2}{\pi}\int_0^\infty e^{-kv}\cos wv\,dv.$$

This looks familiar:

$$\int_0^\infty e^{-st}\cos wt\,dt = \mathcal{L}(\cos wt) = \frac{s}{s^2+w^2}$$

$$A(w) = \frac{2}{\pi}\mathcal{L}(\cos wt)(k) = \frac{2}{\pi}\,\frac{k}{k^2+w^2}$$

Thus,

$$e^{-kx} = \frac{2k}{\pi}\int_0^\infty \frac{\cos wx}{k^2+w^2}\,dw. \quad (x>0)$$

Similarly,

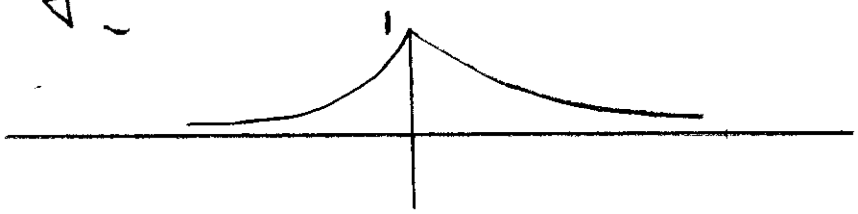
The sine integral has

$$B(w) = \frac{2}{\pi}\int_0^{\infty} e^{-kv} \sin wv \, dv = \frac{2}{\pi}\mathcal{L}(\sin wt)(k)$$

$$= \frac{2}{\pi} \frac{w}{k^2+w^2}$$

$$\Rightarrow \quad e^{-kx} = \frac{2}{\pi}\int_0^{\infty} \frac{w \sin wx}{k^2+w^2}\, dw \qquad (x>0)$$

Graph of cosine integral for $-\infty < x < \infty$



Graph of sine integral for $-\infty < x < \infty$