Lesson 32

Fourier transforms

Let $f(x)$ be continuous for $0<x<\infty$ and absolutely integrable $\left(\int_0^\infty |f(x)|dx < \infty\right)$.

Then, the Fourier cosine transform is

$$\hat{f}_c(w) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos wx \, dx$$

so that, from Lesson 31

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{f}_c(w) \cos wx \, dw$$

for $0<x<\infty$, and the formula gives an even extension to $-\infty<x<0$.

Similarly, the Fourier sine transform is

$$\hat{f}_s(w) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin wx \, dx,$$

and then

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{f}_s(w) \sin wx \, dw$$

(In Lesson 31 we wrote

$$f(x) = \int_0^\infty A(w) \cos wx \, dw \qquad A(w) = \frac{2}{\pi} \int_0^\infty f(v) \cos wv \, dv$$

$$f(x) = \int_0^\infty B(w) \sin wx \, dw \qquad B(w) = \frac{2}{\pi} \int_0^\infty f(v) \sin wv \, dv$$

so we have "split up the $\frac{2}{\pi}$)

We also use the notation

$$\mathcal{F}_c(f) = \hat{f}_c \qquad \mathcal{F}_s(f) = \hat{f}_s$$

From Lesson 31 we then have for example

$$\mathcal{F}_c(e^{-kx}) = \frac{\sqrt{2/\pi}\, k}{k^2 + w^2}.$$

The Fourier cosine and sine transform are linear

$$\mathcal{F}_c(af+bg) = a\,\mathcal{F}_c(f) + b\,\mathcal{F}_c(g)$$

$$\mathcal{F}_s(af+bg) = a\,\mathcal{F}_s(f) + b\,\mathcal{F}_s(g)$$

Theorem. Let f be continuous, absolutely integrable, and $f'(x)$ piecewise continuous on finite intervals, and assume $f(x) \to 0$ as $x \to \infty$. Then,

$$\mathcal{F}_c(f'(x)) = w\,\mathcal{F}_s(f(x)) - \sqrt{\frac{2}{\pi}}\,f(0)$$

$$\mathcal{F}_s(f'(x)) = -w\,\mathcal{F}_c(f(x)).$$

Proof. We prove the first relation.

$$\mathcal{F}_c(f'(x)) = \sqrt{\frac{2}{\pi}} \int_0^\infty f'(x) \cos wx \, dx$$

$$= \sqrt{\frac{2}{\pi}}\left(f(x)\cos wx \Big|_0^{\infty} + w\int_0^{\infty} f(x)\sin wx\,dx \right)$$

$$= -\sqrt{\frac{2}{\pi}}\,f(0) + w\,\mathcal{F}_s(f(x)).$$

From the Theorem we also get

$$\mathcal{F}_c(f''(x)) = -w^2\mathcal{F}_c(f(x)) - \sqrt{\frac{2}{\pi}}\,f'(0)$$

$$\mathcal{F}_s(f''(x)) = -w^2\mathcal{F}_s(f(x)) + \sqrt{\frac{2}{\pi}}\,w f(0).$$

The (complex) Fourier transform.

From Lesson 30,

$$f(x) = \frac{1}{\pi}\int_0^{\infty}\int_{-\infty}^{\infty} f(v)\left(\cos wv\cos wx + \sin wv \sin wx\right)dv\,dw$$

Using sum of angles trig identity,

$$f(x) = \frac{1}{\pi}\int_0^\infty \int_{-\infty}^\infty f(v)\cos(wx - wv)\,dv\,dw$$

$$= \frac{1}{2\pi}\int_{-\infty}^\infty \int_{-\infty}^\infty f(v)\cos(wx-wv)\,dv\,dw$$

Now, $\frac{1}{2\pi}\int_{-\infty}^\infty \int_{-\infty}^\infty f(v)\sin(wx-wv)\,dv\,dw = 0$

by "oddness" so we may add this expression (with i) and use Euler formula to get

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^\infty \int_{-\infty}^\infty f(v)e^{iw(x-v)}\,dv\,dw$$

Adjusting $\frac{1}{2\pi}$ factor as with $\mathcal{F}_c$ and $\mathcal{F}_s$ we

take $\hat{f}(w) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty f(x)e^{-iwx}\,dx$

so that $f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty \hat{f}(w)e^{iwx}\,dw$

Write

$$\mathcal{F}(f) = \hat{f} \quad \text{so that} \quad \mathcal{F}^{-1}(\hat{f}) = f.$$

Ex. $\mathcal{F}\left(e^{-|x|}\right) = ?$

$$\hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{x} e^{-iwx}\,dx + \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-x} e^{-iwx}\,dx$$

$$= \frac{1}{\sqrt{2\pi}} \left(\frac{1}{1-iw}\right) e^{x-iwx}\Big|_{-\infty}^{0} + \frac{1}{\sqrt{2\pi}} \left(\frac{-1}{1+iw}\right) e^{-x-iwx}\Big|_{0}^{\infty}$$

$$= \frac{1}{\sqrt{2\pi}} \left(\frac{1}{1-iw} + \frac{1}{1+iw}\right)$$

$$= \frac{1}{\sqrt{2\pi}} \left(\frac{2}{1+w^2}\right)$$