Lesson 33

Fourier transforms (cont.)

Ex. $f(x) = e^{-x^2}$

x^2+bx+c

$= (x+\frac{b}{2})^2 - \frac{b^2}{4} + c$

$$\hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2 - iwx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(x+\frac{iw}{2})^2 + (\frac{iw}{2})^2}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{w^2}{4}} \int_{-\infty}^{\infty} e^{-(x+\frac{iw}{2})^2}\, dx$$

With the substitution $v = x + \frac{iw}{2}$

$*$

this becomes

$$\frac{1}{\sqrt{2\pi}} e^{-\frac{w^2}{4}} \int_{-\infty}^{\infty} e^{-v^2}\, dv = \frac{1}{\sqrt{2\pi}} e^{-\frac{w^2}{4}} \sqrt{\pi}$$

Thus, $\hat{f}(w) = \frac{1}{\sqrt{2}} e^{-w^2/4}$

$*$ This uses complex analysis.

The Fourier transform is linear:

$$\mathcal{F}(af+bg) = a\mathcal{F}(f) + b\mathcal{F}(g).$$

Also, if $f(x) \to 0$ as $x \to +/- \infty$,

$$\mathcal{F}(f'(x)) = iw\,\mathcal{F}(f(x))$$

$$\left(\Rightarrow \mathcal{F}(f''(x)) = -w^2 \mathcal{F}(f(x)) \right)$$

To see this

$$\mathcal{F}(f'(x)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f'(x) e^{-iwx}\,dx$$

$$= \frac{1}{\sqrt{2\pi}} \left(f(x) e^{-iwx} \Big|_{-\infty}^{\infty} - (-iw) \int_{-\infty}^{\infty} f(x) e^{-iwx}\,dx \right)$$

$$= iw\,\mathcal{F}(f(x)).$$

$\underline{Ex}$. $\mathcal{F}(xe^{-x^2}) = \mathcal{F}\left(-\frac{1}{2}\frac{d}{dx} e^{-x^2}\right)$

$$= -\frac{iw}{2\sqrt{2}} e^{-w^2/4}$$

The convolution for Fourier transforms is defined differently than it is for Laplace transforms. (To avoid confusion we'll call this the <u>Fourier</u> <u>convolution</u>)

$$f*g(x) = \int_{-\infty}^{\infty} f(p)\, g(x-p)\, dp \,,$$

and as with the convolution with Laplace transforms $f*g = g*f$.

<u>Convolution Theorem</u>. Let f and g be piecewise continuous, bounded, and absolutely integrable.

Then $$\mathcal{F}(f*g) = \sqrt{2\pi}\, \mathcal{F}(f) \cdot \mathcal{F}(g).$$

An important consequence of this, which is used in the study of the heat equation, comes from taking inverse Fourier transform of both sides of this identity. We get

$$f * g = \int_{-\infty}^{\infty} \hat{f}(w)\hat{g}(w)e^{iwx}\,dw.$$

To prove the convolution theorem, write

$$\mathcal{F}(f*g) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(p)g(x-p)e^{-iwx}\,dp\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(p)g(x-p)e^{-iwx}\,dx\,dp$$

$$(q = x-p)$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(p)g(q)e^{-iw(p+q)}\,dq\,dp$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(p)e^{-iwp}\,dp\int_{-\infty}^{\infty} g(q)e^{-iqw}\,dq$$

$$= \sqrt{2\pi}\,\hat{f}(w)\,\hat{g}(w).$$