Lesson 34

Review

$|f(x)| < Me^{kt}$

$$\mathcal{L}(f)(s) = F(s) = \int_0^\infty f(t)e^{-st}\,dt$$

<u>s shift</u> $F(s-a) = \mathcal{L}(e^{at}f)$

<u>Ex</u>

$$\mathcal{L}(\cosh at) = \frac{s}{s^2-a^2} \qquad s > a > 0$$

$$\mathcal{L}(\sinh at) = \frac{a}{s^2-a^2}$$

$$\mathcal{L}^{-1}\left(\frac{s}{s^2-4s-5}\right) = \mathcal{L}^{-1}\left(\frac{s}{(s-2)^2-9}\right)$$

$$= \mathcal{L}^{-1}\left(\frac{s-2}{(s-2)^2-9} + \frac{2}{(s-2)^2-9}\right)$$

$$= e^{2t}\left(\cosh 3t + \tfrac{2}{3}\sinh 3t\right)$$

t derivatives

$$\mathcal{L}(f^{(n)}) = s^n \mathcal{L}(f) - s^{n-1} f(0) - s^{n-2} f'(0) - \dots - f^{(n-1)}(0).$$

Integrals

$$\mathcal{L}\left(\int_0^t f(\tau)\, d\tau \right) = \frac{1}{s} F(s).$$

t shift

$$\mathcal{L}(f(t-a)\, u(t-a)) = e^{-as} F(s)$$

Ex. $\mathcal{L}\left(t^2 u(t-1) \right)$

$$= \mathcal{L}\left((t+1-1)^2 u(t-1) \right)$$

$$= e^{-s} \mathcal{L}\left((t+1)^2 \right) = e^{-s} \mathcal{L}\left(t^2 + 2t + 1 \right)$$

$$= e^{-s} \left(\frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s} \right).$$

s derivatives

$$\mathcal{L}(t\, f(t)) = -F'(s)$$

Ex. Given that $\mathcal{L}\left(\frac{e^{-1/4t}}{\sqrt{t}}\right) = \frac{\sqrt{\pi}\, e^{-\sqrt{s}}}{\sqrt{s}}$,

find $\mathcal{L}\left(\sqrt{t}\, e^{-1/4t}\right)$.

$$\mathcal{L}\left(\sqrt{t}\, e^{-1/4t}\right) = -\frac{d}{ds} \frac{\sqrt{\pi}\, e^{-\sqrt{s}}}{\sqrt{s}}$$

$$= -\sqrt{\pi} \left(\frac{-\frac{\sqrt{s}}{2\sqrt{s}} e^{-\sqrt{s}} - \frac{e^{-\sqrt{s}}}{2\sqrt{s}}}{s} \right) = \frac{\sqrt{\pi}}{2s} e^{-\sqrt{s}} \left(1 + \frac{1}{\sqrt{s}}\right)$$

(Laplace convolution)

$$f * g(t) = \int_0^t f(\tau)\, g(t-\tau)\, d\tau$$

$$f * g = g * f$$

$$\mathcal{L}(f * g) = \mathcal{L}(f) \cdot \mathcal{L}(g).$$

Ex. $u(t-3) * e^{-2t}$

$$= \int_0^t u(\tau - 3) e^{-2t+2\tau} \, d\tau$$

$$= \begin{cases} 0 & t \leq 3 \\ e^{-2t} \int_3^t e^{2\tau} \, d\tau = \frac{e^{-2t}}{2} e^{2\tau} \Big|_3^t = \frac{e^{-2t}}{2}\left(e^{2t} - e^{6}\right) & t > 3 \end{cases}$$

$$\mathcal{L}\left(u(t-3) * e^{-2t}\right) = \frac{e^{-3s}}{s} \cdot \frac{1}{s+2}$$

$$\mathcal{L}\left\{ \frac{u(t-3)}{2}\left(1 - e^{6-2t}\right)\right\}$$

p periodic functions

$$\mathcal{L}(f) = \frac{1}{1-e^{-ps}} \int_0^p e^{-st} f(t)\, dt$$

See example Lesson 22 p. 7

Dirac δ-function

Ex. $y'' + 2y' + 2y = \delta(t-3) \qquad y(0) = y'(0) = 0$

$$(s^2 + 2s + 2)Y = e^{-3s}$$

$$Y = \frac{e^{-3s}}{(s+1)^2 + 1}$$

$\left(\mathcal{L}^{-1}\left(\frac{1}{(s+1)^2+1}\right) = e^{-t}\sin t \right)$

t shift and s shift $\Rightarrow$

$$y = u(t-3)\, e^{-(t-3)} \sin(t-3).$$

Sturm-Liouville problem

$$(r(x)y')' + (q(x) + \lambda p(x))y = 0$$

$$k_1 y(a) + k_2 y'(a) = 0 \qquad p(x) > 0$$

$$l_1 y(b) + l_2 y'(b) = 0 \qquad a \le x \le b$$

or

$$y(a) = y(b) \qquad y'(a) = y'(b)$$

Ex Find as S-L problem having eigenfunctions $1, \cos(m\pi x/L), \sin(m\pi x/L)$ $m = 1, 2, \ldots$

The S-L equation

$$y'' + \lambda y = 0 \qquad -L \le x \le L$$

has solutions

① $c_1 e^{-\sqrt{|\lambda|}\,x} + c_2 e^{\sqrt{|\lambda|}\,x}$ $\qquad \lambda < 0$

② $c_1 + c_2 x$ $\qquad \lambda = 0$

③ $c_1 \cos\sqrt{\lambda}\,x + c_2 \sin\sqrt{\lambda}\,x$ $\qquad \lambda > 0.$

With periodic endpoint conditions

$$y(-L) = y(L) \quad , \quad y'(-L) = y'(L)$$

① given

$$C_1 e^{\sqrt{|\lambda|} L} + C_2 e^{-\sqrt{\lambda} L} = C_1 e^{-\sqrt{|\lambda|} L} + C_2 e^{\sqrt{|\lambda|} L}$$

$$\Rightarrow \quad C_1 \left(e^{\sqrt{|\lambda|} L} - e^{-\sqrt{|\lambda|} L} \right) = C_2 \left(e^{\sqrt{|\lambda|} L} - e^{-\sqrt{|\lambda|} L} \right)$$

$$\Rightarrow \quad C_1 = C_2$$

and

$$-C_1 \sqrt{|\lambda|} e^{\sqrt{|\lambda|} L} + C_2 \sqrt{|\lambda|} e^{-\sqrt{|\lambda|} L} = -C_1 \sqrt{|\lambda|} e^{-\sqrt{|\lambda|} L} + C_2 \sqrt{|\lambda|} e^{\sqrt{|\lambda|} L}$$

$$\Rightarrow \quad C_1 = -C_2$$

$$\Rightarrow \quad C_1 = C_2 = 0$$

② $\quad C_1 + C_2(-L) = C_1 + C_2 L \Rightarrow C_2 = 0$

$y = C_1$ soln.

③ $C_1 \cos\sqrt{\lambda} L + C_2 \sin\sqrt{\lambda} L = C_1 \cos(-\sqrt{\lambda} L) + C_2 \sin(-\sqrt{\lambda} L)$

$$\Rightarrow 2C_2 \sin\sqrt{\lambda} L = 0$$

$-C_1\sqrt{\lambda}\sin\sqrt{\lambda} L + C_2\sqrt{\lambda}\cos\sqrt{\lambda} L = -C_1\sqrt{\lambda}\sin(-\sqrt{\lambda} L) + C_2\sqrt{\lambda}\cos(-\sqrt{\lambda} L)$

$$\Rightarrow 2C_1\sqrt{\lambda}\sin\sqrt{\lambda} L = 0$$

Thus, $\sqrt{\lambda} = \frac{m\pi}{L}$ and eigenfunctions

$$\cos\frac{m\pi}{L}x, \quad \sin\frac{m\pi}{L}x \qquad m = 1, 2, \ldots$$

Fourier trig series

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}$$

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx$$

for odd function $a_n = 0 \quad n = 0, 1, \dots$
even " $\quad b_n = 0 \quad n = 1, 2, \dots$

Complex form

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i\frac{n\pi x}{L}}$$

$$c_n = \frac{1}{2L}\int_{-L}^{L} f(x) e^{-i\frac{n\pi x}{L}}\, dx$$

Examples lesson 27, 28, 29

Fourier sine Transform

$$\hat{f}_s(w) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin wx\, dx$$

and inversion $\sqrt{\frac{2}{\pi}} \int_0^{\infty} \hat{f}_s(w) \sin wx\, dw$

Fourier cosine transform

$$\hat{f}_c(w) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos wx \, dx$$

and inversion

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{f}_c(w) \cos wx \, dw$$

and complex form

$$\hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-iwx} \, dx$$

and inversion

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(w) e^{iwx} \, dw.$$

<u>Ex.</u> $f(x) = 1 \qquad 0 < x < 2\pi$

$f(x) = 0 \qquad$ elsewhere

$$\hat{f}_s(w) = \sqrt{\frac{2}{\pi}} \int_0^{2\pi} \sin wx \, dx$$

$$= -\sqrt{\frac{2}{\pi}} \, \frac{1}{w} \cos wx \Big|_{x=0}^{x=2\pi}$$

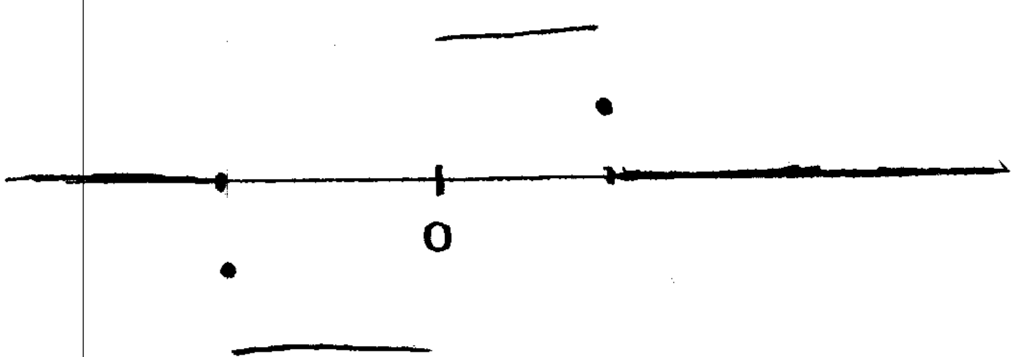
$$= \sqrt{\frac{2}{\pi}} \, \frac{1}{w} \left(1 - \cos 2\pi w\right)$$

Thus,

$$\sqrt{\frac{2}{\pi}} \int_0^{\infty} \sqrt{\frac{2}{\pi}} \, \frac{1}{w} \left(1 - \cos 2\pi w\right) \sin wx \, dw$$

$$= \frac{2}{\pi} \int_0^{\infty} \frac{1 - \cos 2\pi w}{w} \sin wx \, dw = \begin{cases} 1 & 0 < x < 2\pi \\ \frac{1}{2} & x = 2\pi \\ 0 & x > 2\pi \end{cases}$$

and has odd extension



Complex Fourier transform

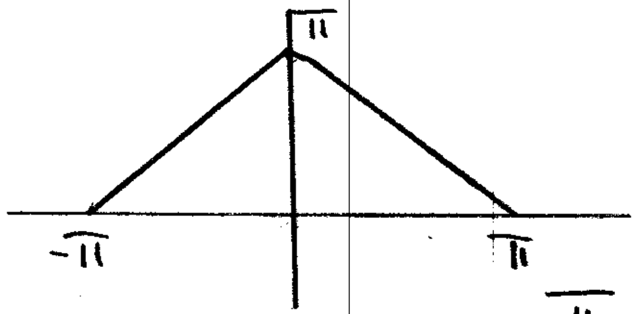
$$\hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} e^{-iwx}\, dx = \frac{-1}{\sqrt{2\pi}\, iw}\, e^{-iwx}\Big|_{x=0}^{x=2\pi}$$

$$= \frac{1}{\sqrt{2\pi}\, iw}\left(1 - e^{-2\pi i w}\right)$$

so

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{1-e^{-2\pi i w}}{w}\, e^{iwx}\, dw = \begin{cases} \frac{1}{2} & x=0 \\ 1 & 0<x<2\pi \\ \frac{1}{2} & x=2\pi \\ 0 & \text{elsewhere} \end{cases}$$

Ex.



$f(x) = \pi - |x|$
even function

$b_n = 0$ $\qquad a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,dx = \frac{1}{\pi}\,\frac{1}{2}\pi^2 = \frac{\pi}{2}$

$n > 0$

$$a_n = \frac{2}{\pi}\int_0^{\pi} (\pi - x)\cos nx\,dx$$

$$\frac{2}{\pi}\cancel{\frac{2}{n}\sin nx\Big|_0^{\pi}} - \frac{2}{\pi}\int_0^{\pi} x\cos nx\,dx)$$

$\left(\begin{array}{ll} u = x & dv = \cos nx\,dx \\ du = dx & v = \frac{\sin nx}{n} \end{array}\right.$

$$= -\frac{2}{\pi}\left(\cancel{\frac{x\sin nx}{n}\Big|_0^{\pi}} - \frac{1}{n}\int_0^{\pi}\sin nx\,dx\right)$$

$$= -\frac{2}{\pi}\left(\frac{1}{n^2}\cos nx\Big|_0^{\pi}\right) = -\frac{2}{\pi n^2}\left((-1)^n - 1\right)$$

$$f(x) = \frac{\pi}{2} - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n - 1}{n^2}\cos nx$$

$$0 = f(\pi) = \frac{\pi}{2} \cancel{+} - \frac{2}{\pi} \sum \frac{(-1)^n - 1}{n^2} \cos n\pi$$

$$\frac{(-1)^n - 1}{n^2} = 0 \quad n \text{ even}$$

$$= -\frac{2}{n^2} \quad n \text{ odd} \leftarrow \cos n\pi = \begin{cases} -1 & n \text{ odd} \\ 1 & n \text{ even} \end{cases}$$

$$0 = \frac{\pi}{2} - \frac{2}{\pi} \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{2}{n^2}$$

$$\Rightarrow \frac{4}{\pi} \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{1}{n^2} = \frac{\pi}{2}$$

$$\Rightarrow \boxed{\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8}}$$

Parseval: $\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)^2\,dx = \sum a_n^2 + b_n^2 + 2a_0^2$

$$\frac{2}{\pi}\int_0^{\pi} (\pi - x)^2\,dx = \frac{-2}{3\pi}(\pi - x)^3 \Big|_0^{\pi} = \frac{2}{3\pi}\pi^3 = \frac{2\pi^2}{3}$$

$$\frac{2\pi^2}{3} = \frac{\pi^2}{2} + \frac{16}{\pi^2}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^4}$$

$$\sum_{n=0}^{\infty}\frac{1}{(2n+1)^4} = \frac{\pi^2}{16}\left(\frac{2\pi^2}{3} - \frac{\pi^2}{2}\right)$$

$$= \frac{\pi^2}{16}\left(\frac{\pi^2}{6}\right) = \frac{\pi^4}{96}$$