Lesson 37

A standard initial-boundary value problem for 1-dimensional wave equation:

$$\frac{\partial^2 u(x,t)}{\partial t^2} = c^2 \frac{\partial^2 u(x,t)}{\partial x^2}$$

$u(0,t) = 0 \qquad u(L,t) = 0$ (boundary cond.'s)

$$\left.\begin{aligned} u(x,0) &= f(x) \\ \frac{\partial u}{\partial t}(x,0) &= g(x) \end{aligned}\right\}$$ (initial cond.'s)

We will solve this problem by the method of <u>separation</u> of <u>variables</u>. This method gives a set of <u>building</u> <u>blocks</u> which are solutions and (since the equation is linear) can be combined in linear combinations to form other solutions.

Our building blocks will be solutions of the form

$$u(x,t) = F(x) \cdot G(t).$$

Remember, we do not expect to solve the initial-boundary value problem with a function like this, but rather to combine solutions of this form.

To find solutions of this form, it is convenient to use $\ddot{G}$, to denote the t derivative and F'' to denote the x derivative.

Then, if u is of this form and satisfies the 1-dimensional wave eqn.,

$$F\ddot{G} = c^2 F'' G$$

$$\Longrightarrow \quad \frac{\ddot{G}}{c^2 G} = \frac{F''}{F}$$

Now the trick is to observe that the left side has no x and the right has no t. We must conclude that these ratios are therefore constant, i.e.

$$\frac{\ddot{G}}{c^2 G} = -k \qquad \frac{F''}{F} = -k \qquad \left(\text{- sign for convenience}\right)$$

We then have 2 familiar S-L problems.

The first one is

$$F'' + kF = 0$$

$$F(0) = 0 \qquad F(L) = 0$$

As in Lesson 24

The eigenvalues and eigenfunctions are

$$k = \frac{n^2\pi^2}{L^2} \qquad n = 1, 2, \ldots$$

$$\sin \frac{n\pi}{L} x \qquad n = 1, 2, \ldots$$

For each such $n = 1, 2, \ldots$, we then have to find solution to

$$\ddot{G} + \frac{c^2 n^2 \pi^2}{L^2} G = 0$$

so the corresponding solutions G are

$$B_n \cos \frac{cn\pi}{L} t + B_n^* \sin \frac{cn\pi}{L} t .$$

Thus, we get our building blocks

$$\left(B_n \cos \frac{cn\pi}{L} t + B_n^* \sin \frac{cn\pi}{L} t \right) \sin \frac{n\pi x}{L} .$$

Now, each building block satisfies the boundary conditions since $\sin \frac{n\pi x}{L} = 0$ when $x = 0$ and $x = L$.

Finally, we want to put them together so that

$$u(x,t) = \sum_{n=1}^{\infty} \left(B_n \cos \frac{cn\pi t}{L} + B_n^* \sin \frac{cn\pi t}{L} \right) \sin \frac{n\pi x}{L}$$

also satisfies the initial condition.

For this we need

$$u(x,0) = \sum_{n=0}^{\infty} B_n \sin \frac{n\pi x}{L} = f(x)$$

$$\frac{\partial u}{\partial t}(x,0) = \sum_{n=0}^{\infty} B_n^* \frac{cn\pi}{L} \sin \frac{n\pi x}{L} = g(x)$$

and we therefore have 2 Fourier Series problems:

$$B_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} \, dx$$

$$B_n^* = \frac{2}{cn\pi} \int_0^L g(x) \sin \frac{n\pi x}{L} \, dx.$$

Ex. Find solutions to Laplace's eqn

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{by separation of variables}$$

$$u(x,y) = F(x)\,G(y)$$

$$\Rightarrow \quad F''G + FG'' = 0$$

$$\Rightarrow \quad \frac{F''}{F} = -\frac{G''}{G} \equiv k$$

(we are not solving boundary value problem)

$$F'' - kF = 0$$

$$G'' + kG = 0$$

① $k<0$

$$\left(C_1 \cos\sqrt{|k|}\,x + C_2 \sin\sqrt{|k|}\,x\right)\left(D_1 e^{-\sqrt{|k|}\,y} + D_2 e^{\sqrt{|k|}\,y}\right)$$

② $k=0$ $\quad (A+Bx)(C+Dy)$

③ $k>0$

$$\left(C_1 e^{-\sqrt{k}\,x} + C_2 e^{\sqrt{k}\,x}\right)\left(D_1 \cos\sqrt{k}\,y + D_2 \sin\sqrt{k}\,y\right)$$

Ex.

$$u_x + u_y = 0$$

$$u(x,y) = F(x)\,G(y)$$

$$F'G + FG' = 0$$

$$\Rightarrow \quad \frac{F'}{F} = -\frac{G'}{G} = k$$

$$\Rightarrow \quad F(x) = c_1 e^{kx}$$

$$G(y) = c_2 e^{-ky}$$

$$u(x,y) = C\,e^{k(x-y)}$$