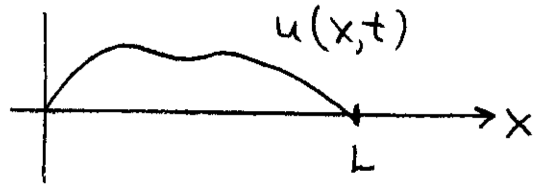
Lesson 38

One dimensional wave equation (cont.)

In lesson 37 we solved the problem of the vibration of a finite string



$$u(0,t) = u(L,t) = 0.$$

In lesson 20 we used Laplace transforms to show that the problem of the half infinite string

$$\frac{\partial^2 w}{\partial t^2} = c^2 \frac{\partial^2 w}{\partial x^2}$$

bdry. cond.'s $\quad w(0,t) = f(t)$

init. cond. $\quad w(x,0) = 0 \qquad \frac{\partial w}{\partial t}(x,0) = 0$

We get (u here being the Heaviside function)

$$w(x,t) = u(t - x/c)\, f(t - x/c).$$

Now consider the <u>doubly</u> <u>infinite</u> <u>string</u>.

The problem is

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

$$u(x,0) = f(x)$$
$$\frac{\partial u}{\partial t}(x,0) = g(x)$$ initial cond's.

<u>No</u> boundary conditions since $-\infty < x < \infty$ so there is no boundary.

This problem is solved by the <u>D'Alembert</u> Solution

$$u(x,t) = \frac{1}{2}\left(f(x+ct) + f(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds.$$

To verify this, consider first

$$u(x,0) = \frac{1}{2}\left(f(x) + f(x)\right) + \cancel{\frac{1}{2c}\int_{x}^{x} g(s)\,ds}^{\,0}$$
$$= f(x) \checkmark$$

To do the differentiation we need chain rule and fundamental Theorem of calculus.

$$\frac{\partial u}{\partial t} = \frac{1}{2}\left(f'(x+ct)\,c - f'(x-ct)\,c \right)$$
$$+ \frac{1}{2c}\left(g(x+ct)\,c + g(x-ct)\,c \right)$$

$$\frac{\partial u}{\partial t}(x,0) = \frac{1}{2}\left(\cancelto{0}{f'(x)\,c - f'(x)\,c}\right)$$
$$+ \frac{1}{2c}\left(g(x)\,c + g(x)\,c \right) = g(x) \quad ✓$$

$$\frac{\partial^2 u}{\partial t^2} = \frac{1}{2}\left(f''(x+ct)\,c^2 + f''(x-ct)\,c^2 \right)$$
$$+ \frac{1}{2c}\left(g'(x+ct)\,c^2 - g'(x+ct)\,c^2 \right)$$

$$= \frac{c^2}{2}\left(f''(x+ct) + f''(x-ct) \right)$$
$$+ \frac{c}{2}\left(g'(x+ct) - g'(x+ct) \right)$$

Finally,

$$\frac{\partial u}{\partial x} = \frac{1}{2}\left(f'(x+ct) + f'(x-ct)\right) + \frac{1}{2c}\left(g(x+ct) - g(x-ct)\right)$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{2}\left(f''(x+ct) + f''(x-ct)\right) + \frac{1}{2c}\left(g'(x+ct) - g'(x-ct)\right)$$

so the PDE is satisfied also.

Suppose f and g are defined on $[0,L]$ and we take their odd periodic extensions f^*, g^*. Then we get the same solution as given by the Fourier series method by using the D'alembert solution with f^*, g^*.

Ex. Semi infinite string

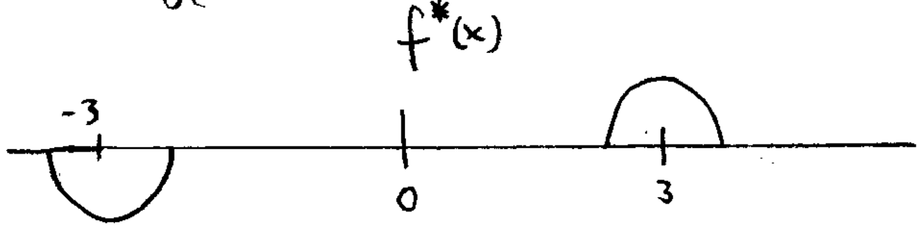
$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \qquad 0 < x < \infty$$

$$u(0,t) = 0$$

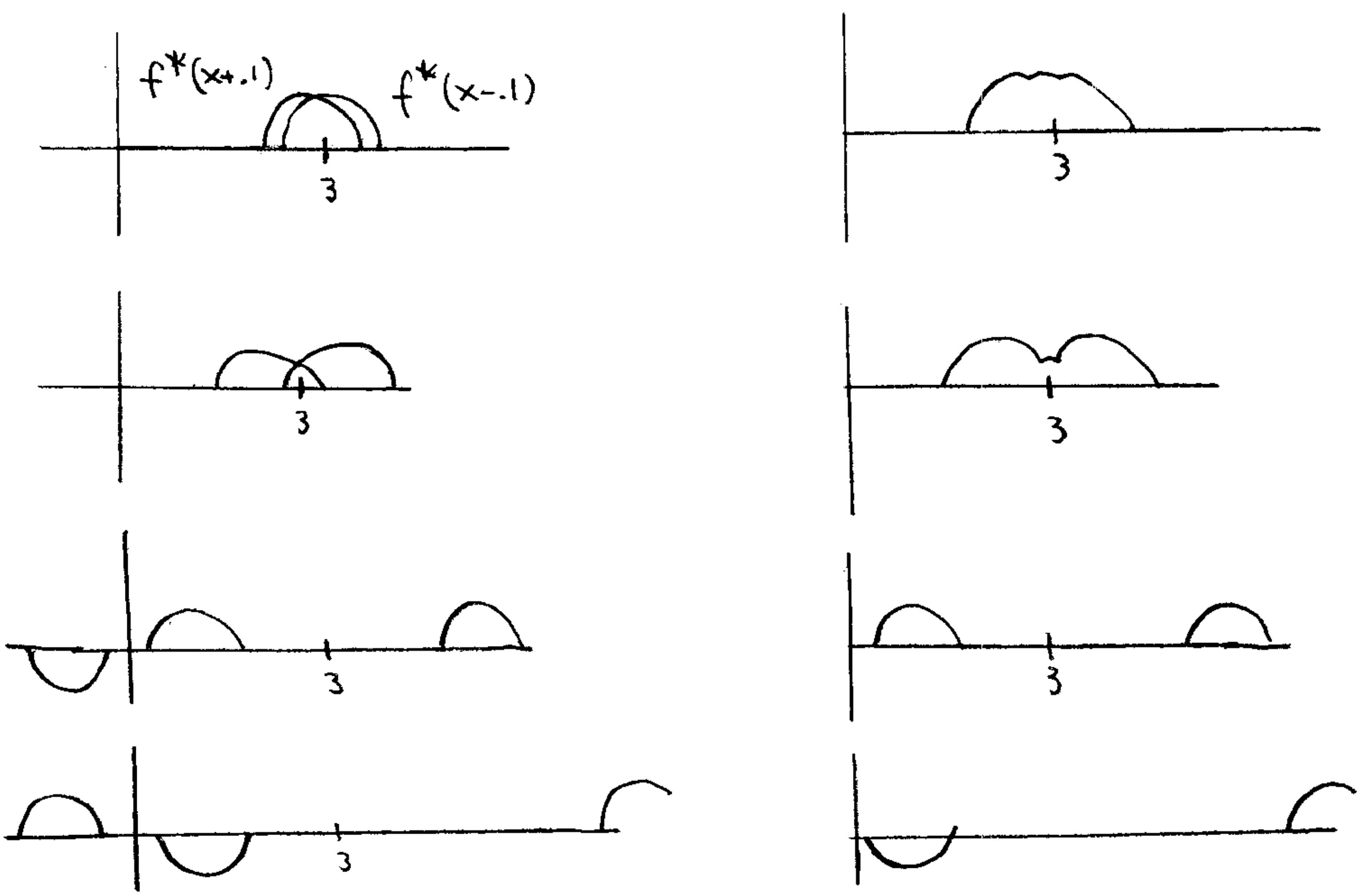
$$u(x,0) = \begin{cases} \cos \pi (x-3) & |x-3| < \frac{\pi}{2} \\ 0 & \text{elsewhere} \end{cases}$$

$$\frac{\partial u}{\partial t}(x,0) = 0$$

f^* odd extension of f



$t = 0$



Ex.
$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$$

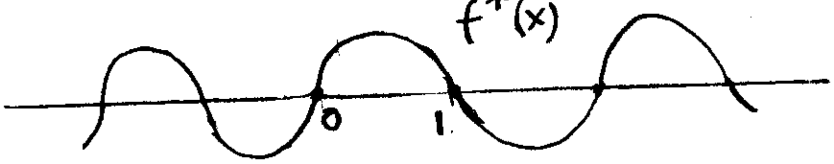
$u(0,t) = u(1,t) = 0$

$u(x,0) = x(1-x) \qquad (= f(x))$

$\partial u/\partial t\,(x,0) = 0$

Let f^* be the odd periodic extension of f

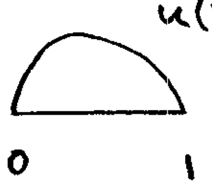
$f^*(x)$



$\frac{1}{2} f^*$

$u(x,t) = \frac{1}{2}\left(f^*(x+t) + f^*(x-t)\right)$

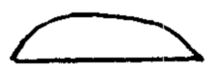
$u(x,t)$



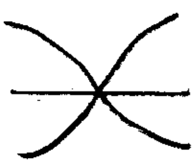
$t=0$

$\frac{1}{2} f^*(x+\frac{1}{4})$

$\frac{1}{2} f^*(x-\frac{1}{4})$

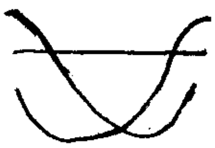


$\frac{1}{2} f^*(x+\frac{1}{2})$



$\frac{1}{2} f^*(x-\frac{1}{2})$

$\frac{1}{2} f^*(x+\frac{3}{4})$



$\frac{1}{2} f^*(x-\frac{3}{4})$

MATLAB PROGRAM for f^*

```
>> f = inline('x.*(1-x).*(x<1).*(x>0)
        - (x-1).*(2-x).*(x<2).*(x>1)
            + (x+1).*(x).*(x>-1).*(x<0)')
>> x= [0:.001:1];
>> x1 = x+.25 ;  x2 = x - .25 ; y= .5*(f(x1) + f(x2));
>> plot(x,y)
```