Lesson 39

One dimensional heat equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

$u(0,t) = 0 \quad u(L,t) = 0$ boundary cond's

$u(x,0) = f(x)$ initial cond.

Separation of variables

$$u = F(x)G(t)$$

$$\frac{\dot{G}}{c^2 G} = \frac{F''}{F} = -k$$

Now, the analysis of

$$F'' + kF = 0$$
$$F(0) = F(L) = 0$$

is precisely the same as it was for the

separation of variables in the wave equation.

$$k = \frac{n^2\pi^2}{L^2} \qquad n = 1, 2, \ldots$$

and the eigenfunctions are

$$\sin \frac{n\pi x}{L} \qquad n = 1, 2, \ldots$$

We therefore must solve

$$\dot{G} = -\left(\frac{cn\pi}{L}\right)^2 G$$

so the corresponding G is of the form $B_n e^{-\left(\frac{cn\pi}{L}\right)^2 t}$

We then arrange B_n's so the building blocks

$$B_n e^{-\left(\frac{cn\pi}{L}\right)^2 t} \sin \frac{n\pi}{L} x$$

can be combined to satisfy initial conditions.

$$u(x,t) = \sum_{n=1}^{\infty} B_n e^{-\frac{c^2 n^2 \pi^2}{L^2} t} \sin \frac{n\pi x}{L}$$

$$f(x) = u(x,0) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L}$$

Thus, to solve the given problem for the one dimensional heat equation, we need only find a Fourier sine series of f.

Suppose now the endpoints are insulated instead of being held at 0. (adiabatic bdry. cond.)

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

$$\frac{\partial u}{\partial x}(0,t) = 0 \qquad \frac{\partial u}{\partial x}(L,t) = 0.$$

$$u(x,0) = f(x).$$

In this case,

$$u = F(x)G(t)$$

and

$$\frac{\dot{G}}{c^2 G} = \frac{F''}{F} = -k$$

gives the S-L problem

$$F'' + kF = 0$$

$$F'(0) = F'(L) = 0.$$

For this we have eigenvalues

$$\frac{n^2\pi^2}{L^2} \quad n = 0, 1, 2, \ldots$$

and eigenfunctions

$$1,\ \cos\frac{\pi x}{L},\ \cos\frac{2\pi x}{L}.$$

We therefore write

$$u(x,t) = \sum_{n=0}^{\infty} A_n e^{-\frac{c^2 n^2 \pi^2}{L^2} t} \cos\frac{n\pi x}{L}$$

so the A_n's are just the coefficients of the Fourier cosine series

$$A_0 = \frac{1}{L}\int_0^L f(x)\,dx\,, \quad A_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx \quad n=1,2,\dots$$

Remarks on boundary conditions. Suppose for the one dimensional wave or heat equation we have

$$u(0,t) = A \quad , \quad u(L,t) = B.$$

Notice that

$$v = A + \frac{B-A}{L}\,x$$

satisfies both the heat and wave equation. Since these equations are linear, if u is the solution with 0 end point condition, then $u+v$ has respective end point values A and B. Thus, if in solving the original problem with $u(0,t) = 0 = u(L,t)$, we replace the $f(x)$ with $f(x) - v(x)$, then $u+v$ works.