Lesson 4

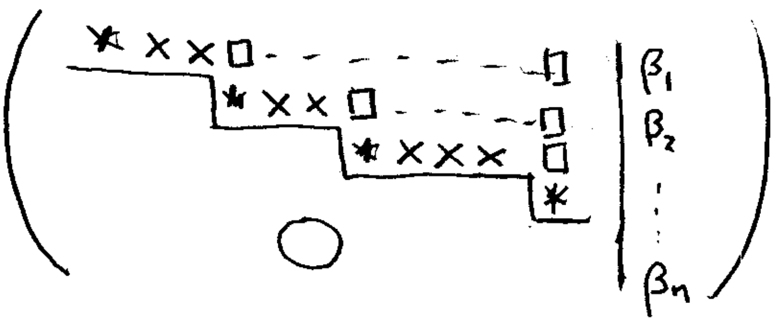
Solutions to Linear Systems

We have seen how Gaussian elimination can be used to solve a linear system with m equations and n unknowns:

$$Ax = b \qquad A \quad m \times n \text{ matrix.}$$

First we form the augmented matrix, then, using row operations bring to echelon form:

Schematic:



$*$ corner variable position

$\times$ free variable position

$\square$ junk

① If there is a $\beta_j \neq 0$ in a row of zeros in the echelon form of A, the system has no solution. (The system is inconsistent.)

(2) If there is no $\beta_j \neq 0$ in a row of zeros of the echelon form of A (such as when A has no row of zeros) then the system is <u>consistent</u> and there are two possibilities:

i) No free variables $\Rightarrow$ unique solution

ii) free variables $\Rightarrow$ infinitely many solutions.

If the system were homogeneous ($b = 0$) then (1) above cannot happen, so we either have a unique solution (the trivial solution in this case) or infinitely many solutions.

Consider now just the homogeneous case

$$Ax = 0. \qquad A \ m \times n \text{ matrix}$$

Then the set $V \subseteq \mathbb{R}^{\cancel{m}\,n}$ of all solutions is a vector space called the <u>solution space</u> or <u>null space</u>.

<u>Check</u>:

If $v \in V$, then $Av = 0$ so if c is a number, $Acv = cAv = c0 = 0$. ✓

If $u, v \in V$, then $A(u+v) = Au + Av$
$= 0 + 0 = 0$. ✓

Notice that in the case of unique solution (no free variables) then $V = \{0\}$ which is a vector space (but not a very interesting one!)

The dimension of the solution space is called the nullity. It is easy to see from the echelon form that there is a relationship between the rank and nullity — called the <u>rank nullity theorem</u>:

$$\text{rank } A + \text{nullity } A = n.$$

There is a simple algorithm for finding a basis for the solution space of $Ax=0$.

1. Bring A to echelon form and identify the free variable

2. For each free variable, one at a time, set a free variable equal to 1 and others to 0, then back substitute. We then get one solution corresponding to each free variable. The solutions obtained this way form a basis for the solution space.

<u>Ex.</u> The example from a previous lesson is gives

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Going to echelon form:

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -4 & -8 & -12 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

x_3 and x_4 are free.

Take $x_3 = 1$ $x_4 = 0$. Back subst. gives

$$\begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix}$$

Now take $x_3 = 0$ $x_4 = 1$. We get $\begin{pmatrix} 2 \\ -3 \\ 0 \\ 1 \end{pmatrix}$.

Thus,

$$\begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 2 \\ -3 \\ 0 \\ 1 \end{pmatrix}$$

is a basis

Now we see that each free variable generates a basis element for the solution space, so the nullity A = # free variables

= # variable − # corner entries

= # columns − # nonzero rows

= n − rank A

Determinants.

$$\begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix}.$$

We only take determinants of square matrices.

Ex. 2×2 case

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

The ij minor M_{ij} is the determinant formed by deleting i^{th} row and j^{th} column.

Ex. $D = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$ $\qquad M_{23} = \begin{vmatrix} 1 & 2 \\ 7 & 8 \end{vmatrix} = -6$

The ij cofactor C_{ij} is $(-1)^{i+j} M_{ij}$. In the above example $C_{23} = (-1)^{2+3}(-6) = 6$

Formula for expansion by cofactors

$$D = \underbrace{\sum_{j=1}^{n} a_{ij} C_{ij}}_{\text{expanding on } i^{th} \text{ row}} = \underbrace{\sum_{i=1}^{n} a_{ij} C_{ij}}_{\text{expanding on } j^{th} \text{ column}}$$

expanding on i^{th} row $\qquad$ expanding on j^{th} column.

Ex

$$-\begin{vmatrix} 1 & 0 & 0 & 0 \\ 2 & -1 & 0 & 0 \\ 3 & 2 & 2 & 1 \\ 4 & 0 & -1 & 3 \end{vmatrix}$$

$$= -1\begin{vmatrix} 1 & 0 & 0 \\ 3 & 2 & 1 \\ 4 & -1 & 3 \end{vmatrix} - 2\begin{vmatrix} 1 & 0 & 0 \\ 2 & 0 & 0 \\ 4 & -1 & 3 \end{vmatrix} = (-1)(6+1) - 2\cdot 0$$

(expanding on column 2)

$$= -7$$

or

(expanding across row 1)

$$\begin{vmatrix} -1 & 0 & 0 \\ 2 & 2 & 1 \\ 0 & -1 & 3 \end{vmatrix} = -7 \quad \text{again}$$

Interaction with row operations:

1. Switch 2 rows changes sign of determinant.
2. Multiply row by c multiplies determinant by c
3. Add multiple of row to another row does not change determinant.

Odd facts: A $n \times n$

1. $\det(cA) = c^n \cancel{A} \det A$
2. $\det(A^T) = \det A$
3. $\det(AB) = \det A \det B$
4. $\det(A+B) \underset{\text{usually}}{\neq} \det A + \det B$
5. $\det A = 0 \iff \operatorname{rank} A < n$
6. If A is upper (or lower) triangular

   $\det A =$ product of diagonal entries

   $(\det I = 1)$
7. $Ax = 0$ has only trivial solution $\iff$ $\det A \neq 0$

Cramer's Rule

If $A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ & \vdots & \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$ has rank n

(Thus $Ax = b$ has unique solution $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$)

then

$$x_j = \frac{\begin{array}{c} \begin{array}{ccccc} & & j^{th}\text{ column} & & \end{array} \\ \begin{vmatrix} a_{11} & \cdots & b_1 & \cdots & a_{1n} \\ a_{21} & \cdots & b_2 & \cdots & a_{2n} \\ & & \vdots & & \\ a_{n1} & \cdots & b_n & \cdots & a_{nn} \end{vmatrix} \end{array}}{\det A}$$