Lesson 40

Two dimensional heat flow and Laplacian in polar form

In two dimensions, the heat equation is

$$\frac{\partial u}{\partial t} = c^2 \Delta u.$$

The steady state solution has $\partial u / \partial t = 0$, so we have only Laplace's equation

$$\Delta u = 0.$$

We shall study only this case.

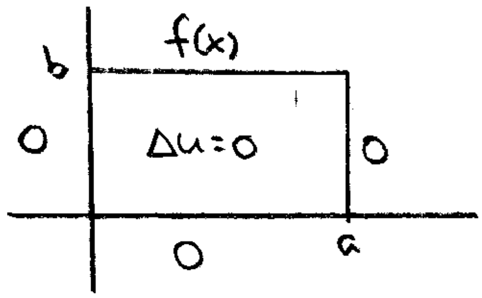
Consider first the boundary value problem for a rectangle

$$D: \quad 0 < x < a \qquad 0 < y < b$$

$$\Delta u(x,y) = 0 \qquad (x,y) \in D$$

boundary values
$$\begin{cases} u(0,y) = 0 \\ u(x,0) = 0 \\ u(a,y) = 0 \\ u(x,b) = f(x) \end{cases}$$

D



If the boundary values were function other than 0 on the sides, we could follow the same procedure as below for each side, then add solutions, since Laplace's equation is linear.

Using separation of variables $F(x)G(y)$ we get

$$\frac{F''}{F} = -\frac{G''}{G} = -k$$

For F we get the familiar S-L problem

$$F'' + kF = 0$$

$$F(0) = F(a) = 0$$

which gives

$$k = \frac{n^2\pi^2}{a^2}$$

and eigenfunctions

$$\sin \frac{n\pi}{a} x \qquad n = 1, 2, \ldots$$

From this we get

$$G'' - \frac{n^2\pi^2}{a^2} G = 0$$

which has solutions

$$A_n e^{\frac{n\pi}{a} y} + B_n e^{-\frac{n\pi}{a} y}$$

The condition $G(0) = 0$ implies

$$A_n + B_n = 0 \implies A_n = -B_n$$

so we write this as

$$A_n^* \sinh \frac{n\pi}{a} y.$$

Our building blocks are then

$$A_n^* \sinh \frac{n\pi}{a} y \sin \frac{n\pi x}{a}$$

and we write

$$u(x,y) = \sum_{n=1}^{\infty} A_n^* \sinh \frac{n\pi y}{a} \sin \frac{n\pi x}{a}$$

$$f(x) = u(x,b) = \sum_{n=1}^{\infty} A_n^* \sinh \frac{n\pi b}{a} \sin \frac{n\pi x}{a}.$$

Thus, $A_n^* \sinh \frac{n\pi b}{a}$ are the Fourier sine coefficients for $f(x)$, i.e.

$$A_n^* = \frac{2}{a \sinh \frac{n\pi b}{a}} \int_0^a f(x) \sin \frac{n\pi x}{a}\, dx$$

Notice that we could also replace some boundary conditions in this problem with insulated portions (mixed boundary value problem). For example, if the right side were insulated, we would get

$$\frac{\partial u}{\partial x}(a,y) = 0.$$

Suppose now that D is replaced by a disk $x^2+y^2<R^2$. Using polar form $x = r\cos\theta$ $y = r\sin\theta$, this is $r<R$.

Suppose we now compute the Laplacian in polar coordinates. We have to use chain rule.

We could try

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x}$$

With $r = \dfrac{x}{\cos\theta}$ we could say $\dfrac{\partial r}{\partial x} = \dfrac{1}{\cos\theta}$

but this would be <u>incorrect</u>. To see why, let's use "thermodynamics notation" for our bookkeeping:

Here $\frac{\partial r}{\partial x}$ means $\left(\frac{\partial r}{\partial x}\right)_y$ ← variable held fixed listed here.

If we use this notation, we see clearly that θ is not a variable held fixed. We can cure this by writing

$$r = \sqrt{x^2+y^2} \Rightarrow \left(\frac{\partial r}{\partial x}\right)_y = \frac{x}{\sqrt{x^2+y^2}} = \cos\theta$$

Also,
$$\theta = \arctan\frac{y}{x} \Rightarrow \left(\frac{\partial \theta}{\partial x}\right)_y = \frac{-\frac{y}{x^2}}{1+\left(\frac{y}{x}\right)^2}$$

$$= -\frac{\sin\theta}{r}$$

Thus,
$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\cos\theta - \frac{\partial u}{\partial \theta}\frac{\sin\theta}{r}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial r}\left(\frac{\partial u}{\partial x}\right)\frac{\partial r}{\partial x} + \frac{\partial}{\partial \theta}\left(\frac{\partial u}{\partial x}\right)\frac{\partial \theta}{\partial x}$$

$$= \left[\frac{\partial}{\partial r}\left(\frac{\partial u}{\partial r}\cos\theta - \frac{\partial u}{\partial \theta}\frac{\sin\theta}{r}\right)\right]\cos\theta - \left[\frac{\partial}{\partial \theta}\left(\frac{\partial u}{\partial r}\cos\theta - \frac{\partial u}{\partial \theta}\frac{\sin\theta}{r}\right)\right]\frac{\sin\theta}{r}$$

$$= \left[\frac{\partial^2 u}{\partial r^2}\cos\theta - \frac{\partial^2 u}{\partial r\partial\theta}\frac{\sin\theta}{r} + \frac{\partial u}{\partial\theta}\frac{\sin\theta}{r^2}\right]\cos\theta$$

$$- \left[\frac{\partial^2 u}{\partial\theta\partial r}\cos\theta - \frac{\partial u}{\partial r}\sin\theta - \frac{\partial^2 u}{\partial\theta^2}\frac{\sin\theta}{r} - \frac{\partial u}{\partial\theta}\frac{\cos\theta}{r}\right]\frac{\sin\theta}{r}$$

$$= \frac{\partial^2 u}{\partial r^2}\cos^2\theta - \left(\frac{2}{r}\frac{\partial^2 u}{\partial r\partial\theta} - \frac{1}{r^2}\frac{\partial u}{\partial\theta}\right)\sin\theta\cos\theta + \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}\sin^2\theta$$

$$+ \frac{\sin^2\theta}{r}\frac{\partial u}{\partial r}$$

Using the same procedure for u_{yy} and adding we get

$$\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}$$

Suppose for simplicity that we are working in the unit disk ($R = 1$) and have boundary function $f(\theta)$. $-\pi \leq \theta \leq \pi$

For homework you will verify that the functions (see also p. 10)

$$r^n \cos n\theta \; , \; r^n \sin n\theta \qquad n = 0, 1, \ldots$$

are harmonic. Taking combinations of these we get

$$u = a_0 + \sum_{n=1}^{\infty} a_n r^n \cos n\theta + b_n r^n \sin n\theta$$

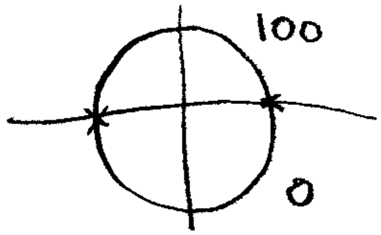
and when $r = 1$

$$f(\theta) = u(1, \theta) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\theta + b_n \sin n\theta,$$

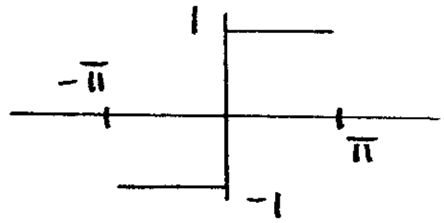
the Fourier series for $f(\theta)$.

NOTE: If we use complex notation $(x+iy)^n = (re^{i\theta})^n = r^n e^{in\theta} = r^n(\cos n\theta + i \sin n\theta) \Rightarrow r^n \cos n\theta = \mathrm{Re}\,(x+iy)^n$ and $r^n \sin n\theta = \mathrm{Im}\,(x+iy)^n$. If we use the binomial (p. 10 1010) expansion on $(x+iy)^n$ we can then get formulas for $r^n \cos n\theta$, $r^n \sin n\theta$ in terms of x and y.

Ex



In lesson 2C we did



$$\sum_{n=0}^{\infty} \frac{4}{(2n+1)\pi} \sin(2n+1)x$$

Add 1, then multiply by 50 and we get the desired $f(\theta)$.

The equilibrium temperature distribution is

$$u(r,\theta) = 50 + \frac{200}{\pi} \sum_{n=0}^{\infty} \frac{1}{2n+1} r^{2n+1} \sin(2n+1)\theta$$

$$\frac{\partial \, r^n \cos n\theta}{\partial r} = n r^{n-1} \cos n\theta$$

$$\frac{\partial^2}{\partial r^2} r^n \cos n\theta = n(n-1) r^{n-2} \cos n\theta$$

$$\frac{\partial^2}{\partial \theta^2} r^n \cos n\theta = -n^2 r^n \cos n\theta$$

$$\Delta(r^n \cos n\theta) = n(n-1) r^{n-1} \cos n\theta + \frac{n r^{n-1}}{r} \cos n\theta - \frac{n^2 r^n}{r^2} \cos n\theta$$

$$= 0$$

Note also in rectangular form

$$\frac{\partial^2}{\partial x^2} (x+iy)^n = n(n-1)(x+iy)^{n-2}$$

$$\frac{\partial^2}{\partial y^2} (x+iy)^n = -n(n-1)(x+iy)^{n-2}$$

so $\Delta (x+iy)^n = 0$. Since $r^n \cos n\theta = \mathrm{Re}\,(x+iy)^n$

this gives another proof that $\Delta r^n \cos n\theta = 0$

(using complex variables)