Solving heat equation with Fourier transform.

We consider the one dimensional heat equation in an infinite bar, $-\infty < x < \infty$.

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

$$u(x,0) = f(x) \qquad -\infty < x < \infty.$$

Separation of variables $F(x)\,G(t)$ gives

$$F'' + k F = 0$$

$$\dot{G} - c^2 k G = 0.$$

Considering these as approximable by the previous case $-L<x<L$ with $L \to \infty$, we then have building blocks of the form

$$(A \cos px + B \sin px) e^{-c^2 p^2 t}$$

and the sum evolves into an integral

$$u(x,t) = \int_0^\infty (A(p)\cos px + B(p)\sin px)e^{-c^2p^2t}\,dp$$

In order to evaluate $A(p)$ and $B(p)$ we substitute the initial condition

$$f(x) = u(x,0) = \int_0^\infty A(p)\cos px + B(p)\sin px\,dp$$

which is the Fourier integral of f, so

$$A(p) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos pv\,dv$$

$$B(p) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin pv\,dv.$$

In Lesson 32 we used trig identities to rewrite the Fourier integral

$$u(x,0) = \frac{1}{\pi}\int_0^\infty\int_{-\infty}^{\infty} f(v)\cos(px-pv)\,dv\,dp$$

so that $\frac{1}{\pi}\int_0^\infty\int_{-\infty}^{\infty} f(v)(\cos pv\cos px + \sin pv\sin px)e^{-c^2p^2t}\,dv\,dp$

gives

$$u(x,t) = \frac{1}{\pi}\int_0^{\infty}\int_{-\infty}^{\infty} f(v)\cos(px-pv)e^{-c^2p^2t}\,dv\,dp$$

Reversing the orders of integration and using the identity

$$\int_0^{\infty} e^{-s^2}\cos 2bs\,ds = \frac{\sqrt{\pi}}{2}e^{-b^2}$$

we can derive the important formula

$$u(x,t) = \frac{1}{2c\sqrt{\pi t}}\int_{-\infty}^{\infty} f(v)e^{-\frac{(x-v)^2}{4c^2t}}\,dv$$

Another way to derive this formula is by Fourier transform and convolution. Taking Fourier transforms (w.r.t. x) on both sides of the heat eqn. gives

$$\mathcal{F}(u_t) = -c^2w^2\mathcal{F}(u)$$

so

$$\frac{\partial \hat{u}}{\partial t} = -c^2 w^2 \hat{u}$$

$\Rightarrow$

$$\hat{u}(w,t) = C(w) e^{-c^2 w^2 t}.$$

Using the initial condition $\hat{u}(w,0) = \hat{f}(w)$,

$$\hat{u}(w,t) = \hat{f}(w) e^{-c^2 w^2 t}$$

Taking inverse Fourier transforms,

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(w) e^{-c^2 w^2 t} e^{iwx} \, dw$$

Now, if $g(x) = \frac{\sqrt{2a}}{\sqrt{2\pi}} e^{-ax^2}$ $\left(a = \frac{1}{4c^2 t}\right)$

$\hat{g}(w) = \frac{1}{\sqrt{2\pi}} e^{-w^2/4a}$ so again

$$u(x,t) = \int_{-\infty}^{\infty} \hat{f}(w) \hat{g}(w) e^{iwx} \, dw = f * g(x).$$

$$= \frac{1}{\sqrt{2\pi}\, c\sqrt{2t}} \int_{-\infty}^{\infty} f(p) e^{-\frac{(x-p)^2}{4c^2 t}} \, dp$$

Ex.

$$f(x) = \begin{cases} 1 & |x|<1 \\ 0 & |x|>1 \end{cases}$$

$$u(x,t) = \frac{1}{2c\sqrt{\pi t}} \int_{-1}^{1} e^{-\frac{(x-v)^2}{4c^2t}} \, dv$$

Notice that $u(x,t) \to 0$ as $t \to \infty$.

Ex.

$$f(x) = \frac{\sin x}{x}$$

$$u(x,t) = \int_0^\infty \left(A(p)\cos px + B(p)\sin px\right) e^{-c^2p^2t} \, dp$$

$$A(p) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin v}{v} \cos pv \, dv$$

$$B(p) = \frac{1}{\pi}\int_{-\infty}^{\infty} \underbrace{\frac{\sin v}{v}}_{\text{even}} \underbrace{\sin pv}_{\text{odd}} \, dv = 0$$

$$A(p) = \frac{2}{\pi}\int_0^\infty \frac{\sin v}{v}\cos pv\,dv$$

Now, if $F(x) = \begin{cases} \pi/2 & 0 \le x < 1 \\ 0 & x > 1, \end{cases}$

$$\tilde{A}(w) = \frac{2}{\pi}\int_0^1 \frac{\pi}{2}\cos wv\,dv = \frac{1}{w}\sin w$$

Fourier cosine int for F

so $$F(x) = \int_0^\infty \frac{1}{w}\sin w\cos wx\,dw = \begin{cases} \frac{\pi}{2} & 0 \le x < 1 \\ 0 & x > 1 \end{cases}$$

Thus,

$$u(x,t) = \int_0^\infty A(p)\cos px\, e^{-c^2p^2t}\,dp$$

$$= \frac{2}{\pi}\int_0^1 \frac{\pi}{2}\cos px\, e^{-c^2p^2t}\,dp$$

$$= \int_0^1 \cos px\, e^{-c^2p^2t}\,dp$$

Ex. $f(x) = e^{-|x|}$

$$u(x,t) = \int_0^\infty (A(p)\cos px + B(p)\sin px)e^{-c^2p^2t}\,dp$$

$$B(p) = 0, \quad A(p) = \frac{2}{\pi}\int_0^\infty e^{-v}\cos pv\,dv$$

Now,

$$\int_0^\infty e^{-v}\cos pv\,dv = \frac{-1}{1+p^2}e^{-v}\left(-p\sin pv + \cos pv\right)\Big|_{v=0}^{v=\infty}$$

$$= \frac{1}{1+p^2}$$

Thus,

$$u(x,t) = \int_0^\infty \frac{2}{\pi}\frac{1}{1+p^2}\cos px\, e^{-c^2p^2t}\,dp$$