Vibrating rectangular membrane

Let R be the rectangle $0<x<a$, $0<y<b$.

We consider the two dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \quad (x,y)\in R,\ t>0$$

boundary conditions
$$\begin{cases} u(0,y,t)=0 \\ u(a,y,t)=0 \\ u(x,0,t)=0 \\ u(x,b,t)=0 \end{cases}$$

initial conditions
$$\begin{cases} u(x,y,0)=f(x,y) \\ \frac{\partial u}{\partial t}(x,y,0)=g(x,y) \end{cases}$$

Separating variables

$$u = F(x,y)\, G(t)$$

$$F\ddot{G} = c^2 (F_{xx} G + F_{yy} G)$$

$$\frac{\ddot{G}}{c^2 G} = \frac{1}{F}(F_{xx} + F_{yy}) = \ell \; (= -\nu^2)$$

It turns out that with the boundary condition that the separation constant must be negative

To see this write

$$F_{xx} + F_{yy} - \ell F = 0$$

$$\Rightarrow \quad F(F_{xx} + F_{yy}) - \ell F^2 = 0$$

$$\Rightarrow \quad \int_0^b \int_0^a F(F_{xx} + F_{yy})\, dx\, dy = \ell \int_0^b \int_0^a F^2\, dx\, dy$$

int. by parts

$$\Rightarrow \quad \int_0^b \Big(\cancel{F F_x}\Big|_{x=0}^{x=a}{}^{0} - \int_0^a F_x^2\, dx \Big) dy + \int_0^a \Big(\cancel{F F_y}\Big|_{0}^{y=b}{}^{0} - \int_0^b F_y^2\, dy \Big) dx = \ell \int_0^b \int_0^a F^2\, dx\, dy$$

$u = F \quad dv = F_{xx} dx$

$du = F_x \quad v = F_x$

$u = F \quad dv = F_{yy} dy$

$du = F_y \quad v = F_y$

$$\Rightarrow \quad -\int_0^b\int_0^a F_x^2 + F_y^2 \, dx\,dy = \ell \int_0^b\int_0^a F^2 \, dx\,dy$$

So unless $F \equiv 0$ we must have $\ell < 0$.

Summarizing, we have

$$F_{xx} + F_{yy} + \nu^2 F = 0$$

$$\ddot{G} + c^2\nu^2 G = 0.$$

We now separate $F = H(x)Q(y)$ which gives

$$H_{xx} Q = -H\,Q_{yy} + \nu^2 HQ$$

$$\frac{1}{H} H_{xx} = -\frac{1}{Q}\left(Q_{yy} + \nu^2 Q\right) = -k^2$$

(Again the separation constant must be negative)

Thus,

$$H_{xx} + k^2 H = 0$$

$$Q_{yy} + p^2 Q = 0 \qquad p^2 = \nu^2 - k^2$$

This gives two S-L problems with solutions for H:

$$\sin \frac{m\pi x}{a} \qquad m = 1, 2, \ldots \quad \left(k = \frac{m\pi}{a}\right)$$

and for Q

$$\sin \frac{n\pi y}{b} \qquad n = 1, 2, \ldots \quad \left(p = \frac{n\pi}{b}\right)$$

It is important to note that each H has infinitely many Q's associated with it, and vice versa. Thus the $F(x,y)$ building blocks are

$$\sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \qquad \begin{matrix} m = 1, 2, \ldots \\ n = 1, 2, \ldots \end{matrix}$$

Returning to G we have solutions with $\cos c\nu t$ and $\sin c\nu t$. Since $\nu^2 = k^2 + p^2$ we get

$$B_{mn} \cos \lambda_{mn} t + B^*_{mn} \sin \lambda_{mn} t$$

where

$$\lambda_{mn} = c\nu = c\sqrt{\frac{m^2\pi^2}{a^2} + \frac{n^2\pi^2}{b^2}} = c\pi\sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}}$$

Thus

$$u(x,y,t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left(B_{mn} \cos \lambda_{mn} t + B^*_{mn} \sin \lambda_{mn} t\right) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}$$

and

$$f(x,y) = u(x,y,0) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} B_n \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}$$

The right hand side is a <u>double</u> <u>Fourier</u> <u>series</u>

$$B_{mn} = \frac{4}{ab}\int_0^b\int_0^a f(x,y) \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}\, dx\, dy.$$

Similarly,

$$B^*_{mn} = \frac{4}{ab\lambda_{mn}}\int_0^b\int_0^a g(x,y) \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}\, dx\, dy.$$

We finally get

$$u(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(B_{mn}\cos\lambda_{mn}t + B^*_{mn}\sin\lambda_{mn}t\right)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$$

Ex: $a=4 \quad b=2 \quad c=5$

$$f(x,y) = (4x-x^2)(2y-y^2)$$
$$g(x,y) = 0$$

$$B^*_{mn} = 0$$

$$B_{mn} = \frac{4}{4\cdot 2}\int_0^2\int_0^4 (4x-x^2)(2y-y^2)\sin\frac{m\pi x}{4}\sin\frac{n\pi y}{2}\,dx\,dy$$

$$= \begin{cases} \dfrac{4096}{\pi^6 m^3 n^3} & m, n \text{ both odd} \\ 0 & \text{otherwise} \end{cases}$$

$$u(x,y,t) = \frac{4096}{\pi^6}\sum_{\substack{m,n=1 \\ m,n \text{ both odd}}}^{\infty}\sum \frac{1}{m^3 n^3}\cos\left(\frac{5\pi}{4}\sqrt{m^2+4n^2}\right)t\,\sin\frac{m\pi x}{4}\sin\frac{n\pi y}{2}$$

If the membrane had been circular

$$x^2+y^2<R^2$$

we would use polar coordinates $u(r,\theta,t)$

and separation of variables $u = W(r)Q(\theta)G(t)$

would lead to

$$\ddot{G}+c^2k^2G=0$$

$$Q''+n^2Q=0$$

$$r^2W''+rW'+(k^2r^2-n^2)W=0$$

This last equation has the <u>Bessel functions</u> as its eigenfunctions.