Lesson 43

Laplace's eqn. in cylindrical and spherical coordinates

Cylindrical coordinates $x = r\cos\theta$ $y = r\sin\theta$ $z = z$

require no more analysis

$$\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2}.$$

Spherical coordinates are more complicated

$x = \rho\cos\theta\sin\varphi$

$y = \rho\sin\theta\sin\varphi$

$z = \rho\cos\varphi$

$$\Delta u = \frac{\partial^2 u}{\partial \rho^2} + \frac{2}{\rho}\frac{\partial u}{\partial \rho} + \frac{1}{\rho^2\sin^2\varphi}\frac{\partial^2 u}{\partial \theta^2} + \frac{\cot\varphi}{\rho^2}\frac{\partial u}{\partial \varphi} + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial \varphi^2}$$

$$= \frac{1}{\rho^2}\left(\frac{\partial}{\partial \rho}\left(\rho^2 \partial u/\partial \rho\right) + \frac{1}{\sin\varphi}\frac{\partial}{\partial \varphi}\left(\sin\varphi\frac{\partial u}{\partial \varphi}\right) + \frac{1}{\sin^2\varphi}\frac{\partial^2 u}{\partial \theta^2}\right)$$

Suppose u is independent of θ. Then we can write $\Delta u = 0$ as

$$0 = \frac{\partial}{\partial \rho}\left(\rho^2 \frac{\partial u}{\partial \rho}\right) + \frac{1}{\sin\varphi}\frac{\partial}{\partial \varphi}\left(\sin\varphi \frac{\partial u}{\partial \varphi}\right)$$

In this case, separation of variables $u = G(\rho)H(\varphi)$ gives

$$\frac{1}{\sin\varphi}\frac{d}{d\varphi}\left(\sin\varphi \frac{dH}{d\varphi}\right) + kH = 0$$

$$\frac{1}{G}\frac{d}{d\rho}\left(\rho^2 \frac{dG}{d\rho}\right) = k.$$

If we write $k = n(n+1)$ and compute the second equation, we get

$$\rho^2 G'' + 2\rho G' - n(n+1)G = 0.$$

The reason for writing k this way is that now we can easily describe 2 solutions

$$G_n(\rho) = \rho^n \qquad G_n^*(\rho) = \frac{1}{\rho^{n+1}}.$$

The trick in dealing with the other equation is to let $w = \cos\varphi$. Then,

$$\sin^2\varphi = 1 - w^2$$

$$\frac{d}{d\varphi} = \frac{d}{dw}\frac{dw}{d\varphi} = -\sin\varphi\frac{d}{dw}$$

$$\Rightarrow \quad \frac{d}{dw}\left((1-w^2)\frac{dH}{dw}\right) + n(n+1)H = 0$$

or

$$(1-w^2)H'' - 2wH' + n(n+1)H = 0$$

which is Legendre's equation.

The solutions are the Legendre polynomials. $P_n(w)$. Since $w = \cos\varphi$, this becomes $P_n(\cos\varphi)$.

Building blocks are then

$$u_n(\rho,\varphi) = \rho^n P_n(\cos\varphi), \quad u_n^*(\rho,\varphi) = \frac{1}{\rho^{n+1}} P_n(\cos\varphi).$$

To solve problem on interior of the sphere we use the $u_n(\rho, \varphi)$'s and in the exterior of the sphere (assuming $u \to 0$ as $\rho \to \infty$) we use $u_n^*(\rho, \varphi)$'s.

Suppose we are solving an interior problem for $\Delta u = 0$ with boundary function $f(\varphi)$.

Then, taking $u(\rho, \varphi) = \sum_{n=0}^{\infty} A_n \rho^n P_n(\cos\varphi)$

with

$$f(\varphi) = u(R, \varphi) = \sum_{n=0}^{\infty} A_n R^n P_n(\cos\varphi)$$

we get

$$A_n = \frac{2n+1}{2R^n} \int_0^{\pi} f(\varphi) P_n(\cos\varphi) \sin\varphi \, d\varphi$$

For the exterior problem,

$$u(\rho, \varphi) = \sum_{n=0}^{\infty} \frac{B_n}{\rho^{n+1}} P_n(\cos\varphi)$$

$$B_n = \frac{2n+1}{2} R^{n+1} \int_0^{\pi} f(\varphi) P_n(\cos\varphi) \sin\varphi \, d\varphi.$$