Lesson 5

Inverses

Let A be an $n \times n$ matrix. If there exists a matrix B such that $AB = BA = I$, then B is called the <u>inverse</u> of A and denoted by A^{-1}.

Many matrices do not have inverses. Certainly the 0 matrix does not, but neither does

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}.$$

It is easy to see that the above A does not have an inverse from things we have done.

In Lesson 2 we saw that for this A $Ax = b$ did not have a solution for $b = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$.

However, if A did have an inverse we could find a solution $A^{-1} A x = A^{-1} b \Rightarrow I x = A^{-1} b$

$\Rightarrow x = A^{-1} b$.

This reasoning gives us a general principle.

<u>Theorem</u>. A $n \times n$ matrix has inverse $\Longleftrightarrow$ rank $A = n$.

A matrix is called <u>nonsingular</u> if it has an inverse; otherwise it is <u>singular</u>. With the above Theorem and observations we have previously made we can state many equivalent conditions for a matrix to be nonsingular:

A $n \times n$ is nonsingular $\Longleftrightarrow$

i) rank $A = n$
ii) echelon form of A has no zero rows
iii) rows of A are linearly independent
iv) columns of A are linearly independent
v) $Ax = b$ has unique solution for every $b \in \mathbb{R}^n$
vi) $Ax = 0$ has only trivial solution
vii) $\det A \neq 0$

Since $\det A \cdot \det A^{-1} = \det(AA^{-1}) = \det I = 1$, vii) can be refined to

$$\det A^{-1} = 1/\det A.$$

Suppose A is $n \times n$ nonsingular. Then,

$$A A^{-1} = I.$$

Now, $I^T = I$ so $I = (A A^{-1})^T = (A^{-1})^T A^T$

and $I = (A^{-1} A)^T = A^T (A^{-1})^T$. Thus,

$$(A^{-1})^T = (A^T)^{-1}.$$

Also, if A and B are $n \times n$ nonsingular then AB is nonsingular and $(AB)^{-1} = B^{-1} A^{-1}$

<u>Check</u>: $(AB)(B^{-1} A^{-1}) = A I A^{-1} = A A^{-1} = I$

$$(B^{-1} A^{-1})(AB) = B^{-1} I B = B^{-1} B = I.$$

The algorithm using row operations to find the inverse of a nonsingular matrix A is to first perform a sequence of row operations on A bringing it to I.

The same sequence applied to I gives A^{-1}.

The most efficient way to implement this algorithm is to put A and I side by side, and do the row operations on each simultaneously.

Ex $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{pmatrix}$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 4 & 5 & 6 & 0 & 1 & 0 \\ 7 & 8 & 10 & 0 & 0 & 1 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -3 & -6 & -4 & 1 & 0 \\ 0 & -6 & -11 & -7 & 0 & 1 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -3 & -6 & -4 & 1 & 0 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & -2 & 6 & -3 \\ 0 & -3 & 0 & 2 & -11 & 6 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & -2 & 6 & -3 \\ 0 & 1 & 0 & -2/3 & 11/3 & -2 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -2/3 & -4/3 & -1 \\ 0 & 1 & 0 & -2/3 & 11/3 & -2 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right)$$

There is another formula for the inverse of a non singular matrix A. Although it is somewhat useful for theory, its computation is lengthy.

As before, let C_{ij} denote the ij cofactor.

$$A^{-1} = \frac{1}{\det A}\begin{pmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & & \cdots & C_{n2} \\ & & \vdots & \\ C_{1n} & & \cdots & C_{nn} \end{pmatrix}$$

Note that the i^{th} row j^{th} col. entry is C_{ji} <u>not</u> C_{ij}.

Let's use row operations to help evaluate a determinant.

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -11 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 1 \end{vmatrix} = -3$$